FINANCIAL SERVICES REVIEW

Financial Services Review 23 (2014) 189–206

The perpetual growth model and the cost of computational efficiency: Rounding errors or wild distortions?

Morris G. Danielson[a], Jean L. Heck[a,*]

[a]Department of Finance, Saint Joseph's University, 5600 City Avenue, Philadelphia, PA 19131, USA

Abstract

The constant growth model (Gordon, 1962) plays an important role in the stock selection process for individual investors, in part, because of its computational simplicity. However, value estimates from the model can be highly dependent on cash flows to be received in the distant future. If future events might constrain a firm's growth or lead to its demise, the unadjusted Gordon model can substantially overstate value. Because the model is less likely to misstate value for low-growth, high-payout firms, the ironic implication is that the model is most useful when its ability to value growth is needed least. © 2014 Academy of Financial Services. All rights reserved.

Jel classification: G30

Keywords: Constant growth model; Perpetual growth; Stock valuation

1. Introduction

"Nothing lasts forever," or so the old saying goes. When valuing assets, however, individual investors often ignore this adage and assume cash flows will grow at a constant rate in perpetuity. The Gordon (1962) growth model uses this assumption and is a first-cut tool for estimating stock prices and calculating terminal values in two-stage growth models, or discounted cash flow analyses. Bradley and Jarrell (2008) note that this model "… is taught in all top-tier business schools and used widely throughout the financial community. It is found in virtually all graduate-level corporate finance textbooks and valuation manuals."

* Corresponding author. Tel.: +1-610-660-3148; fax: +1-610-660-1986.
E-mail address: jean.heck@sju.edu (J. Heck)

Recent surveys (e.g., Block, 1999; Demirakos, Strong, and Walker, 2004; Dukes, Peng, and English, 2006; Imam, Barker, and Clubb, 2008) suggest that value estimates from dividend discount models (such as the Gordon model) are used to inform the recommendations of many analysts.

The constant-growth assumption greatly simplifies the calculation of the present value of an infinite dividend stream. Indeed, one of the main reasons why the Gordon model plays such a prominent role in valuation theory and practice is because it is so easy to use. However, value estimates from the perpetual-growth-model can rely heavily on cash flows to be received in the distant future. Shaffer (2006) notes that when the required return is 4% and the growth rate is 3%, 62% of the value estimate is accounted for by cash flows to be received more than 50 years in the future. Danielson, Heck, and Shaffer (2008) observe that almost 70% of an asset's value stems from cash flows to be received in years 11 to infinity if the discount rate is 8% and the growth rate is 4%.

From a mathematical perspective, a present value (today) of $20 is worth $20, regardless of whether the cash flow supporting this present value will be received in five years or 500 years. As long as cash flow estimates five and 500 years in the future are equally reliable, the timing of the future cash flows is not a cause for concern. From a practical standpoint though, it is decidedly easier to forecast future events (and cash flows) five years out, than to predict what will happen in 500 years. In the short term, consumer preferences are likely to evolve in subtle ways, current patents will remain in force, and technological innovations may already be in the works (e.g., new medications in the clinical trial process), making their implications at least somewhat identifiable. Over longer periods of time, market leaders can be replaced (e.g., Sears and K-Mart by Wal-Mart) and entire product markets can become obsolete (e.g., buggy whips and VCR machines).

In response to this concern, numerous valuation models allow the assumed growth rate to decrease (or change) over time. Examples include Miller and Modigliani (1961), Holt, (1962), Mao (1966), Fielitz and Muller (1985), Gordon and Gordon (1997), Danielson (1998), and O'Brien (2003). More recently, Shaffer (2006) extends the Gordon model to include a constant, annual probability of permanent failure. Both of these approaches (allowing for decreasing growth or permanent failure) effectively reduce the portion of an estimated present value accounted for by cash flows to be received far in the future. However, the Gordon model maintains computational advantages over even the simplest of these alternative models, accounting for its ongoing popularity.

Nevertheless, if future events might constrain a firm's growth rate, or lead to its eventual demise, the use of the unadjusted constant growth model can overstate an asset's value. The goal of this article is to quantify the size of these potential valuation errors. Are these errors modest (i.e., rounding differences), in which case the benefits of the model's computational simplicity would outweigh the potential costs created by its imprecise value estimates? Or, are the valuation errors large enough to materially distort investment decisions?

The analytical results in this article suggest that price estimates from the constant growth model can overstate a stock's intrinsic value by a sizeable amount—in some cases the valuation errors can be two or three times the underlying intrinsic value! The potential overstatement increases as the firm's dividend yield decreases, shifting a greater portion of the expected cash flow into later years. These results do not imply that the constant growth model should be abandoned as a valuation tool. Instead, the goal of this article is to promote a better understanding of the limitations of the model, and to place guardrails on its use. In particular, the constant growth model is most useful when the firm faces a low default probability, when only a small portion of the growth will be the result of positive net present value investments, and when the firm's dividend yield is sufficiently large. In all other cases, value estimates obtained from the constant growth model have the potential to significantly overstate value. The ironic implication is that the perpetual growth model is most useful when it is needed least. That is, when the model is used to value low-growth, high-payout firms.

2. The constant-growth model: an overview

In the constant-growth model, a firm's stock price (or an asset's value) is a function of the firm's future dividends (i.e., cash dividends or stock repurchases); the key assumption is that the dividends will increase at a constant annual rate, forever. To begin, we define each variable in real terms: D_1 is the dividend expected to be paid next year, stated in current dollars; r is the required (real) rate of return; and g is the (real) perpetual-growth rate. Using these definitions, the model can be written as Eq. (1).

$$P_0 = \frac{D_1}{r - g} \tag{1}$$

In future years, a firm's cash flows can grow as the result of new investments or inflation. Section 2.1 rewrites Eq. (1) to focus on growth from new investments when all inputs are stated in real terms. Section 2.2 extends the model further, to allow the firm's cash flow stream to increase with inflation.

2.1. Growth and new investments

Eq. (1) is often expanded to calculate a stock price as a function of the amount invested in new projects and the return on these investments (see, e.g., Brealey and Myers, 2003). To do this, the variable E_1 is defined as the perpetual earnings stream expected to be generated by the current operations (stated in current dollars), the plowback rate, b, is the portion of the earnings reinvested each year, and R_N is the economic (not accounting) real return on new investments. Using these additional definitions, Eq. (1) becomes Eq. (2).

$$P_0 = \frac{D_1}{r - g} = \frac{(1 - b)E_1}{r - bR_N} \tag{2}$$

When R_N exceeds r, each new project is expected to have a positive net present value and P_0 will increase with the reinvestment rate b.[1] If R_N equals r, Eq. (2) simplifies to $P_0 = E_1/r$ and value does not depend on the reinvestment rate b. In this case, the firm's dividend still increases each year at the rate $g = bR_N$. However, this dividend growth simply compensates investors for deferring dividends through the reinvestment process; the growth will not increase the estimated value P_0.

2.2. Growth and inflation

To convert Eq. (2) into nominal terms, the following definitions are used:

h = The annual inflation rate
R_N^* = The nominal return on new investments = $(1 + R_N)(1 + h) - 1$
r^* = The discount rate in nominal terms = $(1 + r)(1 + h) - 1$
E_1^* = Next year's nominal expected earnings = $(1 + h)E_1$
g^* = The nominal growth rate = $(1 - b)h + b\,R_N^*$

The nominal growth rate, g^*, has two terms on the right-hand side to allow the firm's entire cash flow stream (including amounts currently paid as dividends) to increase with inflation. The first term captures the growth (because of inflation) of the portion of the cash flow stream currently paid as a dividend. The second term is the growth created through the reinvestment process; the inflation rate h is embedded in the nominal investment return R_N^*. Bradley and Jarrell (2008) also define the nominal growth rate in this way (but with different notation).

When these definitions are substituted into the constant growth model, the inflation terms ultimately cancel out and the model simplifies to Eq. (2), with r, R_N, and g defined as real rates.[2] Eq. (3) outlines this process.

$$
\begin{aligned}
P_0 &= \frac{(1-b)E_1^*}{r^* - g^*} \\
&= \frac{(1-b)(1+h)E_1}{[(1+r)(1+h)-1] - [(1-b)h + b[(1+R_N)(1+h)-1]]} \\
&= \frac{(1-b)E_1}{r - bR_N} \qquad (3)
\end{aligned}
$$

Bradley and Jarrell (2008) show that when the constant growth model is not adjusted properly to incorporate nominal discount and growth rates, the model will produce incorrect value estimates. In particular, if the discount rate is stated in nominal terms (i.e., the discount rate is r^*), but the growth rate is defined simply as the product of the reinvestment rate b and R_N^*, rather than as g^* (as defined above), the constant growth model can understate an asset's value.[3] However, this result does not mean that the constant growth model produces "conservative" value estimates. Instead, the analysis in Bradley and Jarrell (2008) simply means that when the model is applied incorrectly (and inputs are not defined consistently in either nominal or real terms) the model's value estimates will be wrong (i.e., garbage in, garbage out).

Because the constant growth model produces identical value estimates using properly defined real or nominal inputs, and because the model's notation is simpler using real interest rates, the analysis in the remainder of this article defines all valuation inputs in real terms.

3. Value creation and the timing of future cash flows

As noted by Shaffer (2006), Danielson, Heck, and Shaffer (2008), and others, a substantial portion of the value estimates obtained from the constant growth model can be attributed to cash flows that may not be received for decades. To calculate the percentage of a perpetual-growth value estimate attributed to cash flows after year t, first note that the present value of a finite (t-period), growing dividend stream (i.e., the dividends will grow at a constant rate for t periods before dropping to \$0) can be written as follows (Welch, 2009):

$$P_0 = \frac{D_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^t\right] \tag{4}$$

Then, subtract Eq. (4) from Eq. (1) and divide the result by Eq. (1). This yields Eq. (5), which quantifies the portion of an asset's value created by cash flows to be received after year t.

$$PV(t+1 \text{ to } \infty)/PV(0 \text{ to } \infty) = \left(\frac{1+g}{1+r}\right)^t \tag{5}$$

Table 1 uses Eq. (5) to calculate the percentage of an asset's estimated value—using Eq. (1)—that is expected to be received after year t, for three discount rates (7, 10, and 13%) and a range of growth rates. For each combination of r and g, the table lists the portion of the value estimate created by cash flows received after years 10, 20, 30, 40, 50, and 100. In addition, the table lists the Macaulay (1938) duration for each combination of r and g.

Although r and g can be (and typically are) estimated individually, Payne and Finch (1999) point out that the key determinant of P_0 in Eq. (1) is the difference between r and g. For example, if D_1 is estimated to be \$1 and the difference between r and g is 5%, the estimated stock price is \$20 regardless of whether r = 7% and g = 2%, or r = 13% and g = 8%. In each case, the implied dividend yield (D/P) is 5%.[4]

Table 1 reveals that the portion of a value estimate attributable to cash flows after a specified year is also closely related to the difference r – g (= D/P), but this relation is not exact. For example, when r = 7% and g = 2%, 38.4% (9.1%) of the asset's value is created after year 20 (year 50); when r = 13% and g = 8%, 40.4% (10.4%) of the value is expected to be realized after year 20 (year 50).

As the difference between r and g becomes smaller, a progressively larger portion of the value estimate will depend on cash flows from the out years. For example, when r – g is four percentage points (D/P = 4%), over 20% of the estimated value will be realized after year 40. When r – g is two percentage points (D/P = 2%), more than twice as much (i.e., nearly 50%) of the asset's value will be received after year 40.

As r and g converge (and D/P decreases toward zero), value estimates from the Gordon model become highly dependent on cash flows to be received in the distant future. When r – g = 1%, almost 70% of the value will be realized after year 40, and ~40% will be received after year 100. The Macaulay duration in this case exceeds 100. When r – g = 0.5%, over 60% of the estimated value will be received after year 100 and the Macaulay duration exceeds 200!

Table 1 Percent of value created after year t

Panel A (r = 7%)

	g = 0	0.02	0.03	0.04	0.05	0.06	0.065
t = 0	100%	100%	100%	100%	100%	100%	100%
10	50.8%	62.0%	68.3%	75.2%	82.8%	91.0%	95.4%
20	25.8%	38.4%	46.7%	56.6%	68.6%	82.9%	91.1%
30	13.1%	23.8%	31.9%	42.6%	56.8%	75.5%	86.9%
40	6.7%	14.7%	21.8%	32.1%	47.0%	68.7%	82.9%
50	3.4%	9.1%	14.9%	24.1%	38.9%	62.5%	79.1%
100	0.1%	0.8%	2.2%	5.8%	15.2%	39.1%	62.6%
Duration	15.3	21.4	26.8	35.7	53.5	107.0	214.0

Panel B (r = 10%)

	g = 0	0.05	0.06	0.07	0.08	0.09	0.095
t = 0	100%	100%	100%	100%	100%	100%	100%
10	38.6%	62.8%	69.0%	75.8%	83.2%	91.3%	95.5%
20	14.9%	39.4%	47.7%	57.5%	69.3%	83.3%	91.3%
30	5.7%	24.8%	32.9%	43.6%	57.7%	76.0%	87.2%
40	2.2%	15.6%	22.7%	33.1%	48.0%	69.4%	83.3%
50	0.9%	9.8%	15.7%	25.1%	40.0%	63.3%	79.6%
100	0.0%	1.0%	2.5%	6.3%	16.0%	40.1%	63.4%
Duration	11.0	22.0	27.5	36.7	55.0	110.0	220.0

Panel C (r = 13%)

	g = 0	0.03	0.08	0.09	0.11	0.12	0.125
t = 0	100%	100%	100%	100%	100%	100%	100%
10	29.5%	39.6%	63.8%	69.7%	83.6%	91.5%	95.7%
20	8.7%	15.7%	40.4%	48.6%	70.0%	83.7%	91.5%
30	2.6%	6.2%	25.7%	33.9%	58.5%	76.6%	87.5%
40	0.8%	2.5%	16.4%	23.7%	49.0%	70.1%	83.7%
50	0.2%	1.0%	10.4%	16.5%	40.9%	64.1%	80.1%
100	0.0%	0.0%	1.1%	2.7%	16.8%	41.1%	64.2%
Duration	8.7	11.3	22.6	28.3	56.5	113.0	226.0

Notes. This table lists the percentage of a perpetual-growth model value estimate attributed to cash flows received after year t (where t = 0, 10, 20, and so forth) calculated using Eq. (5). The calculations use required returns of 7% (Panel A), 10% (Panel B) and 13% (Panel C), and a range of assumed growth rates, g. For each combination or r and g, the table also lists the Macaulay duration of the value estimate.

Only when the dividend yield is large, can the majority of the estimated value be linked to cash flows within a foreseeable time horizon. For example, assume that g = 0 and D/P = r. If r = 13% (r – g = D/P = 0.13), over 90% of the value will be received over the next 20 years; if r = 10% (r – g = D/P = 0.10), over 90% of the value will be received during the next 30 years; and when r = 7% (r – g = D/P = 0.07), over 90% of the value will be received in the next 40 years.

4. By how much can the perpetual growth model overstate value?

Table 1 demonstrates that perpetual-growth-model value estimates can be highly dependent on the present value of cash flows that will not be received for decades (i.e., 20, 50, 100, or more years in the future). This fact would not be a cause for concern if the future cash flows were certain to be paid. However, economic conditions can change dramatically over the course of any 20, 50, or 100 year period; the survival of no firm is guaranteed.

Indeed, the experience of the past 100 years highlights the challenges individual firms face to remain in business (let alone to maintain a positive growth rate) over an extended period. During the past century, the economy has been transformed by two world wars, a global depression, and innovations in the transportation, communication, and information systems industries. Orville and Wilbur Wright's first flight was in 1903, the Ford Model T debuted in 1908, AT&T completed the first transcontinental telephone line in 1915, the first electrical, binary, programmable computer was invented during the 1930s, and the construction of the interstate highway system commenced in 1956. For an individual firm to have maintained a constant (or even positive) growth rate across the past century, the firm would have been required to reinvent itself multiple times to adjust to these (and other) events.

Simply maintaining a competitive advantage (and a constant, positive growth rate) over a 25 year period can be a challenge. As noted by Sheth (2009), many of the 62 firms highlighted in the 1982 book "In Search of Excellence" (including Sears, Xerox, Kodak, Dana, and Digital Computer Corp.) experienced financial hardships in the ensuing 25 years. Thus, investors should be wary of purchasing any asset at a price that can only be justified by cash flows to be received in the unforeseeable, distant future.

Previous studies identify two alternative ways to minimize the importance of far distant cash flows in the valuation process. First, Shaffer (2006) extends the Gordon model to include a constant, annual probability of permanent failure. Second, numerous valuation models allow the assumed growth rate to decrease over time (e.g., Miller and Modigliani, 1961; Holt, 1962; Mao, 1966; Fielitz and Muller, 1985; Gordon and Gordon, 1997; Danielson, 1998; O'Brien, 2003). This section reports on potential valuation errors embedded in perpetual growth model price estimates if future cash flows will be constrained by default, finite growth, or both.

4.1. Valuation models: default and finite growth

Shaffer (2006) extends the perpetual growth model to allow for the possibility that the firm will irreversibly fail (and the cash flow stream will end) in the future. Assuming that the annual failure probability (p) is constant over time, Eq. (1) can be rewritten as Eq. (6), and Eq. (2) can be rewritten as Eq. (7).

$$P_0 = \frac{D_1(1 - p)}{r + p - g(1 - p)} \tag{6}$$

$$P_0 = \frac{E_1(1 - b)(1 - p)}{r + p - bR_N(1 - p)} \tag{7}$$

Shaffer (2006) separately modifies the perpetual growth model to allow the growth rate to permanently decrease—from a supernormal level (i.e., $R_N > r$) down to the required return—in any future year. Again, Shaffer assumes that this event occurs with a constant annual probability. However, Shaffer does not develop a form of the perpetual growth model that allows for both default (where the cash flow stream ends) and finite growth (where the firm's growth rate permanently decreases, but the cash flow stream continues).

To develop such a model, we simplify the assumptions, and stipulate that the firm's return on new investments, R_N, be equal to the required return, r, starting in the specific future year t + 1. As derived in Appendix 1, the adjusted perpetual growth model, allowing for default and finite growth, can be written as Eq. (8).

$$P_0 = \frac{E_1(1-b)(1-p)}{r+p-bR_N(1-p)}\left(1-\left(\frac{(1-p)(1+bR_N)}{(1+r)}\right)^t\right) + \left(\frac{(1-p)(1+bR_N)}{(1+r)}\right)^t\left(\frac{E_1(1-b)(1-p)}{r+p-br(1-p)}\right) \tag{8}$$

If the default probability is zero ($p = 0$), Eq. (8) can be written as Eq. (9). This is the traditional finite growth model from Miller and Modigliani (1961) and Gordon and Gordon (1997).

$$P_0 = \frac{E_1(1-b)}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^t\right] + \frac{E_1(1+g)^t}{r(1+r)^t} \tag{9}$$

4.2. Estimation errors with default (but perpetual growth)

In this section, we compare value estimates from Eq. (7) to those obtained using Eq. (2). If the intrinsic value of the firm is defined by Eq. (7) (because the firm might fail in the future) by how much will the perpetual growth model overstate firm value? The percentage overstatement embedded in a perpetual growth value estimate in this case (positive default probability; perpetual growth) is Eq. (10). Appendix 2 derives this equation.

$$\%\ \text{Overstatement (p} > 0\text{; perpetual growth)} = \frac{p(1+r)}{(r-g)(1-p)} \tag{10}$$

Table 2 reports on the potential valuation errors—calculated using Eq. (10)—that can arise when the perpetual growth model is used to value a firm that might fail in the future. Panel A lists the assumptions used in this exercise, and calculates firm value using the unadjusted perpetual growth model. In particular, the firm is expected to have earnings per share of \$1 next year, the required return (stated as a real interest rate) is 7%, and the return on new investments (again stated as a real interest rate) is 7%. The unadjusted perpetual growth model, Eq. (2), estimates the current stock price as \$14.29 regardless of whether the reinvestment rate is 0%, 50%, or 99%. The results in Table 2, Panel A illustrate this for eight different reinvestment rates, b. Although each reinvestment rate produces the same value estimate when plugged into the perpetual growth model, they do not create identical challenges for the firm's managers, or impose equal risks on the firm's shareholders.

Mathematically, value is preserved if retained earnings are reinvested in projects earning a return equal to the cost of capital, and if the default probability is zero. However,

Table 2 Estimation errors–default

Panel A: Assumptions and unadjusted value estimates (default probability = 0)

Scenario	1	2	3	4	5	6	7	8
r =	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
R =	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
b =	0.0000	0.1429	0.2857	0.4286	0.5714	0.7143	0.8571	0.9286
g = bR =	0	0.01	0.02	0.03	0.04	0.05	0.06	0.065
r − g =	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.005
P_{PGM} =	14.286	14.286	14.286	14.286	14.286	14.286	14.286	14.286

Panel B: Default probability = 0.0025

Scenario	1	2	3	4	5	6	7	8
P_{adj} =	13.759	13.675	13.559	13.388	13.113	12.597	11.265	9.299
%OS =	3.8%	4.5%	5.4%	6.7%	8.9%	13.4%	26.8%	53.6%

Panel C: Default probability = 0.005

Scenario	1	2	3	4	5	6	7	8
P_{adj} =	13.267	13.111	12.899	12.593	12.114	11.259	9.290	6.883
%OS =	7.7%	9.0%	10.8%	13.4%	17.9%	26.9%	53.8%	107.5%

Panel D: Default probability = 0.0075

Scenario	1	2	3	4	5	6	7	8
P_{adj} =	12.806	12.589	12.297	11.884	11.253	10.173	7.899	5.459
%OS =	11.6%	13.5%	16.2%	20.2%	27.0%	40.4%	80.9%	161.7%

Panel E: Default probability = 0.01

Scenario	1	2	3	4	5	6	7	8
P_{adj} =	12.375	12.105	11.747	11.247	10.502	9.274	6.865	4.518
%OS =	15.4%	18.0%	21.6%	27.0%	36.0%	54.0%	108.1%	216.2%

Notes. This table compares price estimates from the unadjusted perpetual growth model, Eq. (2), to price estimates from the perpetual growth model with default, Eq. (7), and reports the percentage overstatement of value embedded in the unadjusted perpetual growth model price estimate. The firm's estimated earnings next year, E_1, is assumed to be \$1, the required return, r, is 7%, and the return on new investments, R, is 7%. Panel A lists combinations of reinvestment rate b and growth rate g that produce a stock price of \$14.286 using Eq. (2). Panels B through E list the adjusted price estimates from Eq. (7), P_{adj}, for each scenario using default probabilities of 0.0025 (Panel B), 0.005 (Panel C), 0.0075 (Panel D), and 0.01 (Panel E), and report the % difference between the unadjusted and adjusted value estimates: $\%OS = (P_{PGM} - P_{adj})/P_{adj}$. The values reported in the %OS rows can also be calculated using Eq. (10).

identifying projects that can earn a return equal to the cost of capital is not a trivial task in a competitive economy. For example, if the economy is growing at a real rate of 2% per year, but the firm must earn 7% (i.e., the cost of capital) on new investments to preserve value, the process of deferring dividends into the future might require the firm to grow faster than the economy as a whole, simply to maintain its value.[5] It is possible that the aggressive policies required to achieve this growth could ultimately cause the firm to "grow broke," contributing to (or magnifying) the firm's default probability.

Table 2, Panels B to E, report potential overstatements embedded in perpetual-growth value estimates when the default probability is positive. The table reports results for failure probabilities ranging from 0.25% to 1%. We use this range of failure probabilities because Shaffer (2006) reports that the average annual failure rate for U.S. businesses during the 1955 to 1995 period was 0.6%. The Table 2 results reveal that the potential overstatement is directly related to the timing of the implied future cash flows. If r – g is 0.5 percentage points—that is, D/P is 0.5% (Scenario 8) and the bulk of the estimated value is created by cash flows from distant years—the perpetual growth model can overstate value by almost 54% (over 216%) if the firm faces a constant, annual, failure probability of 0.25% (1%). In contrast, if r – g is seven percentage points (Scenario 1), the potential overstatement implied by the perpetual growth model is only 3.8% (15.4%) when the annual failure probability is 0.25% (1%).

If a firm faces a positive default probability, the process of deferring dividends creates additional risk (i.e., default risk) for shareholders. Thus, for a firm to maintain value after deferring dividends, the return on investment earned by the new projects must exceed the cost of capital in those outcomes in which the firm does not fail. That is, the project must produce an "ex-post" positive NPV to compensate investors for the firm's default risk. Eq. (1), which is derived in Appendix 3, calculates the return on new investments required to preserve value for a firm that retains and reinvests earnings, if the firm faces an annual default probability of p.

$$R_N(\text{breakeven}) = \frac{r + p}{1 - p} \tag{11}$$

Thus, if a firm's annual default probability is 1% and its required return is 7%, new projects must earn an 8.08% return in those outcomes in which the firm is successful for the investments to preserve value.

4.3. Estimation errors with finite growth (no default)

Because the Gordon model assumes growth will continue forever, the model can overstate the value of an asset if $R_N > r$, and if competition will limit the period of time positive net present value projects will be available (Stigler, 1963). In this section, we assume that the default probability is zero and focus directly on the value overstatement created by the use of a growth phase that is too long (i.e., perpetual vs. finite growth period).

If Eq. (9) defines an asset's intrinsic value, the use of the perpetual growth model, Eq. (2), will produce a value estimate that is too high. To calculate this valuation error as a percentage of the intrinsic value, subtract Eq. (9) from Eq. (2), and divide the difference by Eq. (9). After much algebra, this yields Eq. (12).

$$\%\text{Overstatement (p = 0; finite growth)} = \frac{1}{1 - \left(\frac{1 + \mathrm{g}}{1 + \mathrm{r}}\right)^{t}\left(\frac{b}{1 - b}\right)\left(\frac{R_N}{r} - 1\right)} - 1 \tag{12}$$

Table 3 Estimation errors–finite growth

Scenario	1	2	3	4
$r =$	0.07	0.07	0.07	0.07
$R =$	0.4	0.125	0.0857	0.0765
$b =$	0.1	0.4	0.7	0.85
$g = bR =$	0.04	0.05	0.06	0.065
$r - g =$	0.03	0.02	0.01	0.005
$P_{PGM} =$	30	30	30	30
$t = 10$:				
$P_{adj} =$	18.175	16.988	15.694	15.005
%OS =	65.1%	76.6%	91.2%	99.9%
$t = 20$:				
$P_{adj} =$	21.102	19.225	16.976	15.691
%OS =	42.2%	56.0%	76.7%	91.2%
$t = 50$:				
$P_{adj} =$	26.209	23.883	20.173	17.567
%OS =	14.5%	25.6%	48.7%	70.8%
$t = 100$:				
$P_{adj} =$	29.085	27.619	23.855	20.163
%OS =	3.1%	8.6%	25.8%	48.8%

Notes. This table compares value estimates from the unadjusted perpetual growth model, Eq. (2), to value estimates from the extended model allowing finite growth (but no default), Eq. (9), and reports the percentage overstatement embedded in the unadjusted perpetual growth model value estimate. The firm's estimated earnings next year, E_1, is $1, the required return, r, is 7%, and the firm's P/E ratio is 30. The table lists four combinations of reinvestment rate b, return on new investments R, and growth rate g that produce a stock price estimate of $30 (P/E = 30) using Eq. (2). For each combination, the table reports the adjusted value estimate from Eq. (9), P_{adj}, and the % difference between the unadjusted and adjusted value estimates: $\%OS = (P_{PGM} - P_{adj})/P_{adj}$. The values reported in the %OS rows can also be calculated using Eq. (12). This information is reported assuming the period in which the firm can invest in projects where $R > r$ will last 10, 20, 50, and 100 years.

Table 3 reports potential valuation errors using Eq. (12). The inputs include four combinations of r, g, b, and R_N (with $p = 0$) and competitive advantage periods ranging from 10 to 100 years. Each combination of r, g, b, and R_N produces a P/E ratio of 30 using the unadjusted perpetual growth model.[6]

If competition will limit a firm's growth phase to 10 years (but the default probability is zero) Table 3 shows that the unadjusted perpetual growth model will overstate firm value by over 65% when the dividend yield is 3% (Scenario 1), and by almost 100% when D/P is 0.5% (Scenario 4).[7] If the firm's growth phase can be maintained for 20 years, Eq. (2) will still overstate firm value by over 42% when D/P is 3% and by over 91% when D/P is 0.5%.[8] Even if the true growth period is 100 years, the unadjusted perpetual growth model will overstate value by almost 49% when D/P is 0.5%. In contrast, the unadjusted perpetual growth model only overstates value by 3% when the growth period equals 100 years and the dividend yield is 3%. These results suggest that the perpetual growth model is an appropriate first-cut valuation tool for a fairly limited set of firms: high-dividend firms with very strong competitive advantages (i.e., 50 or more years).

Table 4 Estimation errors–default and finite growth
Panel A: Assumptions and perpetual growth model value estimate

Scenario	1	2	3	4
$r =$	0.07	0.07	0.07	0.07
$R =$	0.4	0.125	0.0857	0.0765
$b =$	0.1	0.4	0.7	0.85
$g = bR =$	0.04	0.05	0.06	0.065
$r - g =$	0.03	0.02	0.01	0.005
$P_{PGM} =$	30	30	30	30

Panel B: Default probability = 0.0025

Scenario	1	2	3	4
t = 10:				
$P_{adj} =$	17.384	15.935	13.900	11.944
%OS =	72.6%	88.3%	115.8%	151.2%
t = 20:				
$P_{adj} =$	20.087	17.959	14.994	12.470
%OS =	49.4%	67.1%	100.1%	140.6%
t = 50:				
$P_{adj} =$	24.593	21.9791	17.593	13.839
%OS =	22.0%	36.5%	70.5%	116.8%
t = 100:				
$P_{adj}\ p =$	26.911	24.916	20.311	15.556
%OS =	11.5%	20.4%	47.7%	92.9%

Panel C: Default Probability = 0.005

Scenario	1	2	3	4
t = 10:				
$P_{adj} =$	16.653	14.997	12.465	9.910
%OS =	80.1%	100.0%	140.7%	202.7%
t = 20:				
$P_{adj} =$	19.151	16.834	13.410	10.330
%OS =	56.6%	78.2%	123.7%	190.4%
t = 50:				
$P_{adj} =$	23.135	20.317	15.550	11.372
%OS =	29.7%	47.7%	92.9%	163.8%
t = 100:				
$P_{adj} =$	25.007	22.636	17.583	12.556
%OS =	20.0%	32.5%	70.6%	138.9%

(continued on next page)

4.4. Estimation errors with default and finite growth

Table 4 compares value estimates from the perpetual growth model, Eq. (2), to value estimates from Eq. (8), which adjusts the perpetual growth model for both finite growth and potential default. The examples in Table 4 calculate potential valuation errors, for competitive advantage periods ranging from 10 to 100 years, using the same four combinations of r, g, b, and R_N as in Table 3. Panel A lists the assumptions used in each scenario. Panels B through E list potential value overstatements—embedded in value estimates from Eq. (2)—when firms face positive default probabilities ranging from 0.25% to 1%.

Table 4 *(continued)*

Panel D: Default probability = 0.0075

Scenario	1	2	3	4
t = 10:				
P_{adj} =	15.975	14.157	11.291	8.461
%OS =	87.8%	111.9%	165.7%	254.6%
t = 20:				
P_{adj} =	18.288	15.828	12.115	8.806
%OS =	64.0%	89.5%	147.6%	240.7%
t = 50:				
P_{adj} =	21.815	18.856	13.895	9.621
%OS =	37.5%	59.1%	115.9%	211.8%
t = 100:				
P_{adj} =	23.330	20.693	15.432	10.463
%OS =	28.6%	45.0%	94.4%	186.7%

Panel E: Default probability = 0.01

Scenario	1	2	3	4
t = 10:				
P_{adj} =	15.344	13.399	10.312	7.375
%OS =	95.5%	123.9%	190.9%	306.8%
t = 20:				
P_{adj} =	17.488	14.925	11.038	7.665
%OS =	71.5%	101.0%	171.8%	291.4%
t = 50:				
P_{adj} =	20.615	17.564	12.531	8.316
%OS =	45.5%	70.8%	139.4%	260.7%
t = 100:				
P_{adj} =	21.844	19.025	13.704	8.928
%OS =	37.3%	57.7%	118.9%	236.0%

Notes. This table compares value estimates from the unadjusted perpetual growth model, Eq. (2), to value estimates from the extended model allowing both default and finite growth, Eq. (8), and reports the percentage overstatement embedded in the unadjusted perpetual growth model value estimate. The firm's estimated earnings next year, E_1, is \$1, the required return, r, is 7%, and the firm's P/E ratio is 30. Panel A lists combinations of reinvestment rate b, return on new investments R, and growth rate g that produce a stock price estimate of \$30 (P/E = 30) using Eq. (2). Panels B through E list the adjusted value estimates from Eq. (8), P_{adj}, for each scenario using default probabilities of 0.0025 (Panel B), 0.005 (Panel C), 0.0075 (Panel D), and 0.01 (Panel E), and report the % difference between the unadjusted and adjusted value estimates: %OS = $(P_{PGM} - P_{adj})/P_{adj}$. Within each panel, this information is reported assuming the period in which the firm can invest in projects where R > r will last 10, 20, 50, and 100 years.

The results in Table 4 show that the introduction of a positive default probability can magnify the potential overstatements reported in Table 3 (where the default probability is 0). If the firm's dividend yield is 3% (Scenario 1), Table 4, Panel B reveals that the percentage overstatement in a perpetual growth model value estimate is 72.6% when $p = 0.25\%$ and t = 10 years (vs. 65.1% in Table 3, $p = 0$), and is 11.5% when $p = 0.25\%$ and t = 100 (vs. 3.1% in Table 3, $p = 0$). If the firm's dividend yield is only 0.5% (Scenario 4), the differences between the percentage overstatements in Table 4, Panel B and Table 3 become more pronounced. The percentage overstatement in a perpetual growth model value estimate is 151.2% when $p = 0.25\%$ and t = 10 years (vs. 99.9% in Table 3, $p = 0$), and is 92.9% when $p = 0.25\%$ and t = 100 (vs. 48.8% in Table 3, $p = 0$).

As the estimated default rate increases from Panel B to Panel E, the potential overstatement embedded in perpetual-growth-model value estimates continues to grow. If the firm's dividend yield is 3% (Scenario 1), Table 4, Panel E reveals that the percentage overstatement in a perpetual-growth-model value estimate is 95.5% when $p = 1\%$ and t = 10 years (vs. 65.1% in Table 3, $p = 0$), and is 37.3% when $p = 1\%$ and t = 100 (vs. 3.1% in Table 3, $p = 0$). If the firm's dividend yield is only 0.5% (Scenario 4), the percentage overstatement is 306.8% when $p = 1\%$ and t = 10 years (vs. 99.9% in Table 3, $p = 0$), and is 236.0% when $p = 1\%$ and t = 100 (vs. 48.8% in Table 3, $p = 0$).

The results in Table 4 further limit the population of firms for which the perpetual growth model is an appropriate valuation tool. In addition to having a high dividend yield and a strong competitive advantage, the firm should also be financially strong, with a very low probability of default.

5. Conclusions

The perpetual-growth model provides investors with a simple way to calculate the present value of a perpetual dividend stream. The model is a very useful teaching tool, as it clearly illustrates the links between reinvestment policies, reinvestment returns, and stock prices. In addition, the economic interpretation of valuation ratios (such as the price-to-earnings ratio) is often explained within the framework of the constant growth model (e.g., Leibowitz and Kogelman, 1990). Thus, it is not surprising that the model "… is taught in all top-tier business schools …" as noted by Bradley and Jarrell (2008).

Moving from the classroom to the investing world, however, the usefulness of the model can be called into question. To exploit the computational elegance of the model, one must assume that cash flows will grow at a constant rate forever. This assumption is of dubious validity in a competitive economy, where a firm's growth can be derailed by internal mistakes, innovations by other firms, or macroeconomic shocks.

The analytical results in this article suggest that the perpetual-growth model should be applied in a very limited set of circumstances: when the firm faces a low default probability, when only a small portion of the growth will be the result of positive net present value investments, and when the firm's dividend yield is sufficiently large. Along these lines, Foerster and Sapp (2005) show that the constant-growth model produced reasonable value estimates for the Bank of Montreal (e.g., value estimates that approximated historical stock prices), a large, mature, dividend-paying company. For firms that do not fit this profile, individual investors, financial analysts, and financial planners should use the perpetual-growth model with caution.

Notes

1 We acknowledge that this statement is true (for the perpetual growth model) only over the range of reinvestment rates, b, in which the implied growth rate g (= bR_N) is less than the required return r.

2 Danielson and Scott (2000) and Weston, Chung, and Siu (1998) use a similar approach to reconcile the constant growth model in nominal terms to the constant growth model in real terms.

3 For example, assume that b = 0.4, E_1 = \$1, r = 8%, R_N = 12%, and h = 2%. Thus, next year's nominal earnings are \$1.02, the nominal required return is 10.16%, and the nominal growth rate is 6.896% (= (1 – 0.4)(2%) + 0.4(14.24%)). Using real inputs, the stock price is \$18.75 (= [\$1(1 – 0.4)]/[0.08 – 0.4(0.12)]. The stock price using Eq. (3) and nominal inputs is also \$18.75 (= [\$1.02(1 – 0.4)]/[0.1016 – 0.06896]. In contrast, if the nominal growth rate is defined simply as the retention rate times the nominal return on investments, 5.696% = 0.4(14.24%), the estimated stock price using this (incomplete) nominal growth rate, the nominal discount rate, and the nominal earnings is \$13.71 (= [\$1.02(1 – 0.4)]/[0.1016 –0.05696]. Thus, the incorrect application of the constant growth model will understate value by almost 27%.

4 According to the Gordon model, if the expected dividend next year is D_1, the required return is r, and the perpetual growth rate is g, then the estimated stock price today is $P_0 = D_1/(r - g)$. This expression can be rearranged to write the dividend yield as $D_1/P_0 = (r - g)$.

5 The firm's growth rate will depend on the reinvestment rate and the return on new investments. For example, if the retention rate b is 90% and the return on new investments is 7%, the firm's growth rate is 6.3%. For an extreme illustration of the challenge facing firms, assume that a firm currently has a market share of 2%. If the firm's product market is growing at an annual rate of 2%, and the firm grows at an annual rate of 6.3%, the firm's revenues will become larger than those of its industry in 95 years. However, the examples in Table 1 show that a sizeable portion of firm value must be realized after year 100 when the difference between r and g is small.

6 The P/E ratio can be derived from the constant growth model by dividing each side of Eq. (2) by E_1. This yields $P/E = (1 - b)/(r - bR_N)$.

7 Empirical evidence in Fuller, Huberts, and Levinson (1993) and Lakonishok, Shleifer and Vishney (1994) reveals that high P/E firms (e.g., P/E = 30) experience higher future growth rates than low-P/E firms. However, the superior growth of these firms typically lasts for less than 10 years.

8 The traditional finite growth model, and Eqs. (8) and (9), make the unrealistic simplifying assumption that a firm's competitive advantage period will end abruptly at a specified future date, and that the return on new investments will at that point fall immediately to the cost of capital. It is perhaps more likely that the return on new investments will decrease gradually over some future period. For example, if the return on new investments remains equal to R_N for the next 10 years, and then gradually decreases to r over a 10 year period, the true valuation errors will fall between the overstatements reported in the t = 10 and t = 20 rows of Table 3.

Appendix 1

This appendix derives the perpetual growth model adjusted to allow for both permanent failure and finite growth. To begin, note that the perpetual growth model, allowing for permanent failure, is defined by Eq. (7). This expression is repeated here as Eq. (A1).

$$P_0 = \frac{E_1(1-b)(1-p)}{r+p-bR_N(1-p)} \tag{A1}$$

If the return on new investments remains R_N forever, the stock price at the end of year t can be written as Eq. (A2). If the return on new investments changes to R^* starting in year t + 1, whereas the plowback rate (b) and the failure probability (p) remain constant, the stock price at the end of year t (P_t^*) can be written as Eq. (A3).

$$P_t = \frac{E_{t+1}(1-b)(1-p)}{r+p-bR_N(1-p)} \tag{A2}$$

$$P_t^* = \frac{E_{t+1}(1-b)(1-p)}{r+p-bR^*(1-p)} \tag{A3}$$

Given that the annual failure probability is p, there is a $(1 - p)^t$ chance that the firm will survive past year t. In addition, only the discounted values of Eqs. (A2) and (A3) impact the stock price today; the future stock prices must be divided by $(1 + r)^t$ to calculate their present values. Thus, the stock price today, allowing for the shift in the investment return, can be written as Eq. (A4).

$$P_0 = \frac{E_1(1-b)(1-p)}{r+p-bR_N(1-p)} + \left(\frac{(1-p)}{(1+r)}\right)^t (P_t^* - P_t) \tag{A4}$$

In each of years 1 through t, the firm's earnings will grow at the annual rate bR_N. Thus, the firm's earnings in year t + 1 can be written as Eq. (A5).

$$E_{t+1} = E_1(1+bR_N)^t \tag{A5}$$

If the firm cannot invest in positive net present value projects after year t (that is the typical assumption in finite growth models), R^* equals r. Using this assumption, and substituting Eq. (A5) into (A4), produces Eq. (A6), which is Eq. (8) in the text.

$$P_0 = \frac{E_1(1-b)(1-p)}{r+p-bR_N(1-p)}\left(1 - \left(\frac{(1-p)(1+bR_N)}{(1+r)}\right)^t\right) + \left(\frac{(1-p)(1+bR_N)}{(1+r)}\right)^t \left(\frac{E_1(1-b)(1-p)}{r+p-br(1-p)}\right) \tag{A6}$$

Appendix 2

This appendix derives an equation that quantifies the percentage overstatement embedded in perpetual growth model value estimates for firms that may fail in the future, with an annual failure probability of p, but will grow at the perpetual annual rate g if they do not fail. The valuation error is stated as a percentage of the true stock price, calculated by Eq. (7). To do this, subtract Eq. (7) from Eq. (2), and divide the difference by Eq. (7). This step yields Eq. (A7), which simplifies to Eq. (A8).

$$\%\ \text{Overstatement (p} > 0\text{; perpetual growth)} = \frac{\dfrac{D_1}{r-g} - \dfrac{D_1(1-p)}{r+p-g(1-p)}}{\dfrac{D_1(1-p)}{r+p-g(1-p)}} \tag{A7}$$

$$\%\ \text{Overstatement (p} > 0\text{; perpetual growth)} = \frac{r+p-g(1-p)}{(r-g)(1-p)} - 1 \tag{A8}$$

After rewriting the integer 1 on the right-hand side of Eq. (A8) as 1 = [(r – g)(1 – p)]/[(r – g)(1 – p)], the equation further simplifies to Eq. (A9), which is Eq. (10) in the text.

$$\%\ \text{Overstatement (p} > 0\text{; perpetual growth)} = \frac{p(1+r)}{(r-g)(1-p)} \tag{A9}$$

Appendix 3

This appendix derives the breakeven return on new investments for a firm with a positive probability of failure. If a firm's default probability is zero, the firm's breakeven reinvestment return will equal the required return. If a firm faces a positive default probability, the process of deferring dividends through the reinvestment process creates additional default risk for shareholders. For a firm to maintain value after increasing its plowback ratio, the expected return on new investments must exceed the cost of capital. The breakeven return on new investments is the return a firm must earn, in outcomes in which the firm does not default, to maintain value if the firm reduces its payout ratio.

To illustrate this, assume that Eq. (A1) defines firm value before it changes its plowback rate b. If the firm increases its plowback rate to b + Δ, firm value can be written as Eq. (A10).

$$P_0^* = \frac{E_1(1-b-\Delta)(1-p)}{r+p-(b+\Delta)R_N(1-p)} \tag{A10}$$

To solve for the breakeven return on new investments, set Eq. (A1) equal to (A10), and solve algebraically for R_N. This expression simplifies to Eq. (A11), which is Eq. (11) in the text.

$$R_N(\text{breakeven}) = \frac{r+p}{1-p} \tag{A11}$$

References

Block, S. (1999). A study of financial analysts: practice and theory. *Financial Analysts Journal, 55,* 86–95.

Bradley, M., & Jarrell, G. (2008). Expected inflation and the constant-growth valuation model. *Journal of Applied Corporate Finance, 20,* 66–78.

Brealey, R. A., & Myers, S. C. (2003). *Principles of Corporate Finance*. New York: McGraw-Hill/Irwin.

Danielson, M. (1998). A simple valuation model and growth expectations. *Financial Analysts Journal, 54,* 50–57.

Danielson, M., Heck, J., & Shaffer, D. (2008). Shareholder theory: How opponents and proponents both get it wrong. *Journal of Applied Finance, 18,* 62–66.

Danielson, M., & Scott, J. (2000). Expanding the scope of valuation analysis. *Financial Practice and Education, 10,* 52–61.

Demirakos, E. G., Strong, N. C., & Walker, M. (2004). What valuation models do analysts use? *Accounting Horizons, 18,* 221–240.

Dukes, W., Peng, Z., & English, P. (2006). How do practitioners value common stock? *Journal of Investing, 15,* 90–104.

Fielitz, B., & Muller, F. (1985). A simplified approach to common stock valuation. *Financial Analysts Journal, 41,* 35–41.

Foerster S., & Sapp, S. (2005). The dividend discount model in the long-run: A clinical study. *Journal of Applied Finance, 15,* 55–75.

Fuller, R., Huberts, L., & Levinson, M. (1993). Returns to E/P strategies, higgledy-piggledy growth, analysts' forecast errors, and omitted risk factors. *Journal of Portfolio Management, 19,* 13–24.

Gordon, M. (1962). *The Investment, Financing, and Valuation of the Corporation.* Homewood, IL: Irwin.

Gordon, J., & Gordon, M. (1997). The finite horizon expected return model. *Financial Analysts Journal, 53,* 52–61.

Holt, C. (1962). The influence of growth duration on share prices. *Journal of Finance, 17,* 465–475.

Imam, S., Barker, R., & Clubb, C. (2008). The use of valuation models by UK investment analysts. *European Accounting Review, 17,* 503–535.

Lakonishok, J., Shleifer, A., & Vishny, R. (1994). Contrarian investment, extrapolation, and risk. *Journal of Finance, 49,* 1541–1578.

Leibowitz, M., & Kogelman, S. (1990). Inside the P/E ratio: The franchise factor. *Financial Analysts Journal, 46,* 17–35.

Macaulay, F. (1938). *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States Since 1856*. New York: Columbia University Press.

Mao, J. (1966). The valuation of growth stocks: The investment opportunities approach. *Journal of Finance, 21,* 95–102.

Miller, M., & Modigliani, F. (1961). Dividend policy, growth, and the valuation of shares. *Journal of Business, 34,* 411–433.

O'Brien, T. (2003). A simple and flexible DCF valuation model. *Journal of Applied Finance, 13,* 54–62.

Payne, T., & Finch, J. H. (1999). Effective teaching and use of the constant growth dividend discount model. *Financial Services Review, 8,* 283–291.

Shaffer, S. (2006). Corporate failure and equity valuation. *Financial Analysts Journal, 62,* 71–80.

Sheth, J. (2009). *The Self-Destructive Habits of Good Companies.* Philadelphia: Wharton School Publishing.

Stigler, G. J. (1963). *Capital and Rates of Return in Manufacturing Industry.* Princeton, NJ: University Press.

Weston, J. F., Chang, S. C., & Siu, J. (1998). *Takeovers, Restructuring, and Corporate Governance*. Upper Saddle River, NJ: Prentice Hall.

Welch, I. (2009). *Corporate Finance*. New York: Prentice Hall.