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Procedure to determine the optimal Roth IRA versus deductible IRA allocation

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Abstract

We offer a procedure to guide the Roth IRA versus deductible IRA (RVD) allocation decision. We require users to input 10 values to generate outputs that include contribution and withdrawal tax rates; maximum gain; and, optimal amount to allocate between the two major IRA types. By being at their optimal RVD allocation, we find that a modest earning couple can achieve a lifetime wealth gain amounting to about $180,000 in today's dollars. We provide figures and tables illustrating RVD outcomes when there are changes in key variables such as salary match, adjusted gross income, portfolio returns, and withdrawal years.

JEL classification: A20; C00; D14; G11; H21; K34

Keywords: Retirement planning; IRA; Tax rates

1. Introduction

In this article, we address broad goals in personal financial planning of promoting financial literacy and allocating lifetime income. Alhenawi and Elkhal (2013) indicate the need for public policies to encourage financial literacy and education, while Collins, Lam, and Stampfli (2015) note the sustainability of adequate lifetime income is a critical portfolio objective. Promotion of financial literacy and optimal lifetime investment cannot be realized unless educators and financial advisors have procedures to solve the most crucial financial

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decisions. In this article, we offer a procedure that provides a definitive solution to a critical financial decision involving the allocation of retirement contributions between the two major IRA types: a deductible IRA and a Roth IRA. A deductible IRA is synonymous to a traditional IRA.

We develop the first systematic procedure that renders a defined solution to the quest to discover an investor's optimal Roth IRA versus deductible IRA (RVD) allocation. We present new formulas and algorithms applicable to income levels for all marginal tax rates. To illustrate this applicability, we perform a detailed RVD illustration for a couple with an adjusted gross income (AGI) of $110,000 growing at 3% until retirement. The outcome of our illustration is an optimal range of percentages that should be allocated between a deductible IRA and a Roth IRA. By knowing this range, we demonstrate that our couple can achieve a maximum marginal gain of $28,496 a year during a 20-year retirement period. In today's dollars, the total lifetime gain in retirement wealth is around $180,000. However, there are variables where modifications in their values can cause the maximum gain to change while also shifting the optimal percentage allocated between a deductible IRA and a Roth IRA. For example, we show that greater employer matching funds will decrease the gain and increase the optimal percentage allocated to a Roth IRA.

We also provide results from scenario analysis illustrating how the optimal RVD allocation changes when modification are made to AGI, nominal rate of stock return, and number of withdrawal years. Our results are consistent with the research (Horan and Zaman, 2009; Shynkevich, 2013) that suggests investors should contribute more to a Roth IRA when larger retirement withdrawals are expected. In particular, our scenario tests illustrate that more should be put in a Roth IRA for situations where investors have greater matching funds, more aggressive investment strategies leading to greater returns, and shorter retirement periods. We cannot find evidence that greater increases in AGI during one's contribution years necessarily indicate that more should always be put in a Roth IRA.

Applied mathematical models to cover optimal retirement behavior (Ragsdale, Seila, and Little, 1994; Welch, 2008) discuss salient points involved in our article's RVD procedure. They also examine concerns beyond our RVD emphasis such as that by Welch (2015). Welch states that comparing the optimal retirement planner (ORP) model to other models that do not include progressive income taxes is impractical. Because of our focus on the critical RVD decision, our procedure is more simplified. For example, the ORP model can require dozens of inputs, while our procedure requires only 10 values to be inputted by the user. We keep our inputs down by supplying values for eight other variables (such as the expected inflation rate) although individual advisors can modify our supplied values.

Our focus on the RVD allocation choice is justified because of its central importance in retirement planning as the Roth IRA and deductible IRA are the two best choices for retirement investment in that they offer huge tax advantages compared with other alternatives. Thus, the main task for retirement planning is deciding what percentages of one's retirement funds should be allocated to these two IRA types. In attempting this task, we extend the RVD research in a pioneering fashion by giving a new procedure that precisely determines the optimal RVD allocation through innovative algorithms and new formulas that include the marginal cost-benefit notion. In addition, our procedure incorporates future tax tables that determine contribution and withdrawal tax rates as well as future and present value equations that generate the required future value lump sums and annuity withdrawal cash flows based on expected rates of return and cost of living increases. For simplicity, our withdrawals are annuities. Suarez, Suarez, and Walz (2015) develop withdrawal strategies for retirement portfolios. The use of such strategies could be used as a substitute to our annuity withdrawal choice to enable greater maximum gains.

Two consideration make our RVD procedure workable in achieving optimal lifetime income allocation. *First*, we can update inputted values periodically to ensure the investor is allocating their IRA contributions correctly. *Second*, we generate an optimal RVD allocation outcome that covers a flat range of optimal percentages determined by one's contribution tax rate (T_C) and withdrawal tax rate (T_W). The flat range that occurs around one's optimal RVD allocation allows for a margin of error for any values inputted in our procedure.

We offer three main outcomes when using our procedure to compute an investor's optimal RVD allocation. *First*, we determine the tax rate differential defined as $\mathbf{\Delta T} = T_C - T_W$ and use both average and marginal T_C and T_W values. *Second*, we detect the deductible IRA withdrawal (DIW) that maximizes the gain from attaining the optimal RVD allocation. The gain is a function of $\mathbf{\Delta T}$ and DIW. *Third*, we provide the percentage of IRA retirement funds that should be allocated between the deductible and Roth IRA types. The optimal deductible IRA withdrawal percentage (ODI%) is the optimal DIW as a percentage of the maximum DIW. While we focus on the role of T_C and T_W in making the RVD decision, there are other factors that could be applicable giving an advantage to either a Roth IRA (such as less restrictions on minimum distributions during retirement) or a deductible IRA (lowering one's taxable income to be eligible for educational tax credits). While these factors can be important, we believe they are minor for most investors when compared with the substantial economic value that results from the interplay of T_C and T_W.

We organize the remainder of our article as follows. In Section 2, we provide background for understanding the RVD allocation decision. Section 3 reviews retirement decision-making models and their equations while Section 4 introduces and explains the new variables needed to develop an innovative formula for computing the maximum marginal gain needed to pinpoint the optimal RVD allocation. In Section 5, we overview the key variables, provide tax bracket information, compute T_C using our contribution algorithm, and produce IRA cash flow information. Section 6 contains our withdrawal algorithm that generates outcomes such as T_W, maximum gain, and the optimal RVD allocation. In Section 7, we plot the optimal RVD withdrawal range and illustrate how the maximum gain changes based on various assumptions about the match and other income withdrawn during retirement. In Section 8, we perform scenario analysis that show how changes in key variables influence the optimal RVD allocation. Section 9 offers concluding remarks and a disclaimer.

2. Relevant literature on Roth IRA versus deductible IRA choice

A deductible IRA is an individual retirement account, established in the United States by the Employee Retirement Income Security Act of 1974. At the time of the introduction of the Roth IRA brought about by the passage of the Taxpayer Relief Act of 1997, researchers used mathematical frameworks, such as the Scholes and Wolfson or "SW" (1992), to choose among retirement savings alternatives available at that time. These alternatives included a deductible IRA, a nondeductible IRA, and a non-IRA investment. While the deductible IRA is viewed as superior because of its tax savings, there are complexities to consider before knowing if a nondeductible IRA is superior to a non-IRA investment. Major complexities involve unknown investment return rates and personal tax rates as well as different tax rates that are applicable to the various types of investments.

With the introduction of the Roth IRA in 1997, the complexities increased and the RVD research formally began (Adelman and Cross, 2010; Anderson and Hulse, 2013; Crain and Austin, 1997; Horan, Peterson, and McLeod, 1997; Horan and Peterson, 2001; Hrung, 2007; Hulse, 2003; Krishnan and Lawrence, 2001; Shynkevich, 2013; Sibley, 2002). The RVD research has evolved over time with the changes in legislation that have seen tax rates lowered, income limits to IRA contributions raised, and income caps suspended. These changes have not only made past research obsolete in terms of its specifics but also caused ongoing complications in dealing with how an individual determines their optimal RVD allocation.

Crain and Austin (1997) utilize the SW framework to analyze the RVD complexities including the choice between a non-IRA investment and a nondeductible IRA investment. They state the latter two choices are inferior to a deductible IRA and a Roth IRA. They note the RVD allocation choice favors a deductible IRA when the T_W is less than the T_C. Horan, Peterson, and McLeod (1997) point out the situations that favor a conversion of a deductible IRA into a Roth IRA. Regarding conversion, the decision is similar to Crain and Austin in that conversion to a Roth IRA should not be done when $T_C > T_W$ holds.

Horan and Peterson (2001) analyze the RVD decision when the tax savings from the deductible IRA is invested in a mutual fund with some inherent tax-deferral characteristics. They compare a Roth IRA with an employer-sponsored 401(k) plans that match some or all of an employee's contributions to a deductible IRA.[1] Like others, they reiterate that the RVD decision favors the deductible IRA if $T_C > T_W$. Horan and Peterson wrote in a time of greater restrictions governing employers and Roth IRA contributions but that would change over time, as restrictions would be loosened. While investors have been able to choose a Roth 401(k) for their own contribution since 2006, the employer's matching plan does not allow the employer's match to be put into a Roth IRA.

Hulse (2003) suggests the conversion option is valuable and investors should consider it in the RVD allocation choice. Because the Roth IRA does not have a conversion option, a deductible IRA is *ceteris paribus* favored over a Roth IRA. Hrung (2007) empirically examines the influence of tax and nontax factors on the RVD choice. He discovers taxpayer liquidity is often more important in choosing between IRA types than tax factors. For example, investors with children are more likely to contribute to a deductible IRA with the tax savings used for current expenditures or to increase one's current IRA contribution.

Adelman and Cross (2010) and Anderson and Hulse (2013) revisit the RVD dilemma by comparing the deductible and Roth IRA types. Their results indicate the RVD choice can be influenced by theoretical or practical assumptions an investor makes related to a tax bracket effect, required minimum distributions (RMDs), and impact of withdrawals on the amount of Social Security benefits taxed. Anderson and Hulse note that tax law changes (such as the American Taxpayer Relief Act of 2012) have relaxed restrictions making the Roth rollover option available to many more participants. They develop a framework for RVD decision-making to help an investor decide if a Roth rollover is advantageous.

As can be seen from our overview of general background information on the RVD choice, the research has been frequently directed towards decisions other than a strict Roth IRA versus deductible IRA choice. It regularly focuses on conversion from a deductible IRA to a Roth IRA reflecting financial advisors main task, which is dealing with that segment of the population over 50 years of age who hold most of the wealth. In this article, we have a different orientation as we focus on a younger age group who are beginning their retirement investing and have modest earnings. For these future retirees, they should be focusing on their optimal RVD choice. This article addresses this focus and demonstrates that sizeable gains in lifetime wealth can be achieved even for modest income earners.

3. Retirement decision alternatives and models

In this section, we first discuss the *proper* comparison between the Roth and deductible IRA types. We then overview four alternatives investments that can be used for retirement withdrawals. These four investments include two inferior alternatives (non-IRA and nondeductible IRA) and the two superior alternatives (deductible IRA and Roth IRA). Finally, we describe the need to create a procedure to determine a precise allocation between the two superior IRA alternatives. While we focus on a comparison between the two superior IRA types, other retirement income should not be ignored. As we will illustrate later in Figs. 1 and 2, withdrawals from an employer's match and other investments (OI) during retirement cause the optimal RVD decision to allocate more to a Roth IRA.

3.1. Retirement investment alternatives

Crain and Austin (1997) begin modern day RVD modeling. They point out the superiority of a deductible IRA and a Roth IRA to a nondeductible IRA or any non-IRA investment. Horan, Peterson, and McLeod (1997) borrow from the analysis of Crain and Austin and permit investors to fall into lower tax brackets upon withdrawal of retirement assets. This not only has a significant effect on the RVD choice but also the investor's choice between nondeductible IRA contributions and taxable mutual fund investments. They state that comparing future values of taxable investments and nondeductible IRA investments require establishing the formulas governing their after-tax accumulations.[2]

Seida and Stern (1998) use the SW framework to analyze the RVD dilemma. They argue that a Roth IRA investment made after taxes are paid cannot be directly compared with a before-tax investment in a deductible IRA because the latter costs less because of savings from a tax deduction. A Roth IRA investment also cannot be compared with an equal dollar amount of after-tax deductible IRA investment if the deductible IRA's initial balance would exceed statutory limitations. Seida and Stern argue that a *proper comparison* of a deductible IRA to a Roth IRA requires that a deductible IRA be coupled with a supplemental outlay equal to the tax savings. However, they were writing in an era with limited maximum contributions and restrictions on IRA contributions caused by ceilings on income that made the *proper comparison* hard to attain. In today's investment environment, most investors should have little problem in investing the tax savings in a deductible IRA without exceeding maximum IRA contributions. For example, IRS numbers as of 2016 indicate it is possible that a person can invest up to $76,500 (or $89,500 if over 50 years of age) in IRAs. Thus, an investor today should have little trouble in investing the tax savings in a deductible IRA.

In this article, we focus on the *proper comparison* that involves contrasting the dollar amount of a Roth IRA with that of a deductible IRA plus its tax savings. Except for unusual situations, it should be rare for this principle to be violated. However, if the *proper comparison* principle is violated, our procedure breakdowns because it assumes an investor's deductible IRA contribution includes all tax savings from the deductible IRA. Violation of this principle means that one could actually invest more retirement dollars by choosing a Roth IRA.

3.2. Nondeductible IRA and a non-IRA comparison

While we consider the nondeductible IRA and non-IRA alternatives inferior, they are still important, as some investors will be in a position to save beyond the two superior IRA choices. Funds withdrawn from the two inferior choices can influence the optimal RVD choice. This is because the Roth IRA becomes a more favorable choice over a deductible IRA when investors have other sources of income that increase their withdrawal tax rate.

In addition to deductible and Roth IRAs, the Horan, Peterson and McLeod, HPM, (1997) formulas (grounded in Seida and Stern) consider non-IRAs and nondeductible IRA. Given that earnings of a nondeductible IRA are taxed at the ordinary rate upon withdrawal while its actual principal contributed is not taxed on withdrawal, HPM state the after-tax future value of a nondeductible IRA dollar (FV_{ATN}) is

$$FV_{ATN} = 1 + ([1 + R_C]^{Yc} - 1)(1 - T_W) \tag{1}$$

where T_W = ordinary marginal tax rate upon withdrawal, R_C = expected rate of return on contribution, and Y_C = number of years until withdrawal. For a non-IRA taxable mutual fund investment (FV_{BTM}), HPM state that a dollar invested has, at withdrawal, a before-tax future value of

$$FV_{BTM} = (1 + R_C - R_C p_{oi} T_C - R_C p_{cg} T_{cg})^{Yc} \tag{2}$$

where p_{oi} = percent of annual return distributed as ordinary income, T_C = ordinary marginal tax rate,[3] p_{cg} = percent of annual return distributed as capital gains, and T_{cg} = intermediate marginal tax rate on capital gains.

HPM add that a capital gain tax is recognized is based on the before-tax future value at withdrawal of FV_{BTM} as given in (2) less the adjusted basis, composed of the initial investment and distributions less the income tax on distributions. At withdrawal, the after-tax future value of a taxable mutual fund investment (FV_{ATM}) is

$$FV_{ATM} = (1 + R_C - R_C p_{0i} T_C - r p_{cg} T_{cg})^{Yc}$$
$$- T_{cg} \left[\begin{array}{l} (1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc} - 1 \\ - R_C p_{0i}(1 - T_C)\dfrac{(1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc} - 1}{R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg}} \\ - R_C p_{cg}(1 - T_{cg})\dfrac{(1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc} - 1}{R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg}} \end{array} \right] \tag{3}$$

The term inside the brackets of (3) represents the before-tax accumulation at withdrawal less the adjusted basis. HPM state one may want to forgo the IRA tax deferral in exchange for paying lower capital gains tax.

When comparing a nondeductible IRA and a non-IRA, the objective is to determine the percentage of capital gains distribution that makes one indifferent between a taxable mutual fund investment and the same investment in a nondeductible IRA. HPM set (1) equal to (3) to get

$$1 + [(1 + R_C)^{Yc} - 1](1 - T_W) =$$
$$1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc}$$
$$- T_{cg} \left[\begin{array}{l} (1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc} - 1 \\ -R_C p_{0i}(1 - T_C)\dfrac{(1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc} - 1}{R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg}} \\ - R_C p_{0i}(1 - T_C)\dfrac{(1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc} - 1}{R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg}} \end{array} \right] \tag{4}$$

They use this expression to solve for the percentage of return distributed as capital gain (p_{cg}) subject to $0 \leq p_{cg} \leq 1$, $0 \leq p_{oi} \leq 1$, and $0 \leq p_{cg} + p_{oi} \leq 1$.

3.3. Roth IRA and deductible IRA comparison

While the Roth IRA allows for all future earnings and withdrawals to be free from tax, it is taxed at the ordinary rate when contributed. As such, an after-tax future value Roth IRA dollar is

$$FV_{Roth} = \$1(1 - T_C)(1 + R_C)^{Yc} \tag{5}$$

where T_C = ordinary marginal tax rate upon contribution. For a conversion into a Roth IRA, HPM assume all assets being converted are deductible contributions and earnings subject to tax. The after-tax future value of a dollar in an existing deductible IRA account is

$$FV_{ATD} = \$1(1 + R_C)^{Yc}(1 - T_W) \tag{6}$$

where T_W = ordinary marginal tax rate upon withdrawal.

At the time they were writing, HPM stated that investors with AGIs of no more than \$100,000 might convert existing deductible IRAs to Roth IRAs. If the tax liability is paid from the IRA assets in the year of conversion then investors should convert when $FV_{Roth} >$ FV_{ATD} or when we have

$$\frac{FV_{Roth}}{FV_{ATD}} = \frac{(1 - T_C)(1 + R_C)^{Yc}}{(1 - T_W)(1 + R_C)^{Yc}} > 1. \tag{7}$$

Equation (7), in essence, states to convert if $T_W > T_C$.

HPM argue that paying the tax liability from the IRA assets is suboptimal if the tax liability paid out of the assets being converted decreases the principal in the new Roth IRA. In this case, the future value of a converted Roth IRA is simply $(1 - T_C)(1 + R_C)Yc$. Alternatively, the tax liability can be paid from assets that would not qualify for tax-deferred status, leaving the principal in the new Roth IRA unchanged from the deductible IRA. In this case, the future value of a converted IRA dollar equals the future value of the new Roth IRA dollar less the after-tax future value of conversion tax. We have

$$FV_{Roth} = (1 + R_C)^{Yc} - T_C(FV_{ATM}) \tag{8}$$

where FV_{ATM} is the after-tax future value of a taxable mutual fund investment from Equation (3).

The first term in (8) represents the future value of a dollar in the new Roth IRA. The second term represents the lost future value of the T_C dollars used to pay the conversion tax. Substituting Equation (3) into Equation (8) gives

$$FV_{Roth} = (1 + R_C)^{Yc} - T_C\left[(1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc} - T_{cg}\left(\begin{array}{l} (1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc} - 1 \\ - R_C p_{0i}(1 - T_C)\dfrac{(1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc} - 1}{R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg}} \\ - R_C p_{0i}(1 - T_C)\dfrac{(1 + R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg})^{Yc} - 1}{R_C - R_C p_{0i} T_C - R_C p_{cg} T_{cg}} \end{array}\right)\right] \tag{9}$$

Given the above, HPM adds that conversion should be made when

$$\frac{FV_{Roth}}{FV_{ATD}} = \frac{(1 + R_C)^{Yc} - T_C(FV_{ATM})}{(1 - T_W)(1 + R_C)^{Yc}} > 1. \tag{10}$$

3.4. Extending prior RVD research

The prior RVD research points out the interplay of T_W and T_C be it converting from a deductible IRA to a Roth IRA or trying to determine which IRA type should be put in one's retirement funds. While financial advisors assumedly know this interplay and can attempt to guide their clients towards an optimal IRA allocation, they do not have a rigorous procedure to guide them with scientific precision. We extend the prior RVD research by addressing this issue through our RVD procedure that provides definitive guidance to financial advisors.

Keying on the prior research finding that shows a Roth is favorable when $T_W > T_C$, our RVD procedure takes this finding to its logical conclusion by computing both marginal and average T_C and T_W values for use in achieving maximum gain in retirement wealth. With our algorithms and formulas in hand, we determine the maximum gain from all possible deductible IRA withdrawals (DIWs). The point where the gain is maximized establishes the percentage of one's total IRA contribution that should be allocated to a deductible IRA with the remainder allocated to a Roth IRA.

4. Approaches and alternatives to computing optimal RVD allocation

In this section, we discuss two approaches to determine the optimal RVD allocation: the "year-by-year" approach and the "mean" approach. We then clarify "marginal" versus "average" use of tax rates. We also present our two gain formulas. The average gain formula uses average tax rates and the marginal gain formula uses marginal tax rates. The key to a marginal gain formula lies in identifying discovery points (DPs) as they signify the DIW dollar that causes the taxable income to jump to a higher marginal tax rate. DPs are uniquely determined by different retirement withdrawals that cause different taxable income. Besides a DIW, a retiree's taxable income is influenced by tax deductions, Social Security benefits (SSB), employer salary match, and other investments (OI). When used together, we refer to the latter two items as match/OI.

4.1. The two RVD allocation approaches

We develop two approaches to determine an optimal RVD allocation. We first develop a "year-by-year" RVD allocation approach that looks at each year separately. With this approach, we consider each year's contribution and tax savings. We then, if necessary, separate out tax savings made at different marginal tax rates. We then compute the future value of each individual tax savings. This allows earlier contributions to grow more and have a greater weight. We next partition future values for all years based on savings gotten from their specific marginal tax rates. The partitions form the annual withdrawals that are designed to pay taxes below the tax savings. Each withdrawal year has its own gain calculation and other RVD outcomes. While this approach should be more accurate, it has the disadvantage of requiring one to generate a large number of yearly values causing complications including cumbersome complexities when presenting its RVD procedure. Thus, we reserve the year-by-year approach for future research until the basic principles of this article's RVD procedure are better known making its complex presentation more workable.

Second, we develop a "mean" RVD allocation approach that creates factors generating mean contribution and withdrawal values to replace annual values. This approach has the advantage of quickly creating a plethora of withdrawals from which the maximum gain can be identified while also showing the cost of straying from the optimal. By using the mean approach, we can more easily generate results including those from scenario analysis. For these reasons, this article uses the mean approach as the beginning point in creating research interest on discovering the optimal RVD allocation.

4.2. The two tax rate alternatives

Within any optimal RVD allocation approach are two alternative uses of tax rates, namely, the use of either the "average" tax rate or the "marginal" tax rate. We define our average contribution tax rate as

average T_C = tax savings from deductible IRA contributions / deductible IRA contributions.

Our contribution algorithm in Section 5.3 computes the average T_C. We define our average withdrawal tax rate as

average T_W = taxes paid on all retirement withdrawals / taxable income on all retirement withdrawals.

Our withdrawal algorithm in Section 6 computes the average T_W.

While the marginal T_C involves the tax rate(s) corresponding to the highest taxable income bracket(s) for which the last dollars earned are taxed, the marginal T_W covers tax rate(s) corresponding to the highest taxable income bracket(s) for which the last dollars withdrawn are taxed. We are able to avoid the average versus marginal usage decision for T_C because the average T_C and marginal T_C are similar for our illustration and most other situations. However, similarity in the average T_W and marginal T_W rarely occurs.[4]

In this article, we will focus on the use of marginal tax rates. This usage is consistent with marginal analysis that focuses on the incremental benefits and costs of an activity and considers the opportunity cost of the next best alternative. When looking at the incremental benefit of using a dollar of deductible IRA, the next best IRA alternative would be putting the dollar in a Roth IRA. When an investor contributes to a deductible IRA account, the incremental benefits of tax savings from dollars contributed are viewed as based on the last dollars contributed. These last dollars will be in the highest tax bracket(s). Similarly, when an investor withdraws deductible IRA contributions, we should view the incremental costs of the last dollars withdrawn and assign them the highest tax bracket(s). Thus, if investors have the match/OI included in their retirement withdrawals, these withdrawals are assumed to absorb the lowest marginal tax bracket(s) with the DIW partitions absorbing the highest bracket(s).

Given the above discussion, we will focus on the mean approach and the marginal use of tax rates. Nonetheless, we think there is merit in knowing average tax rates. Because the average and marginal T_C values are the same for this article's RVD illustration, we focus on what extent, if any, the use of marginal T_W disagrees with the use of the average T_W. Thus, we will report results using both marginal and average T_W values. In the process, we can identify situations for which the use of average tax rates render similar results to the use of marginal tax rates.

4.3. Discovery points and eight tax rate differentials

When using average tax rates, we use an average gain formula. this formula is

$$\text{average gain} = (T_C - T_W)(DIW) \tag{11}$$

where T_C and T_W are average tax rates as defined above and DIW is the before-tax deductible IRA withdrawal. Determining a marginal gain formula is more difficult because we require partitioning of DIW based on applicable tax brackets. We use these eight brackets: 0, 0.12, 0.18, 0.3, 0.336, 0.396, 0.42, and 0.471.[5] The general expression for the marginal gain is $\sum_{j=1}^{8} \Delta T_j\,(\text{Partition}_j)$ where $\Delta T_j = (T_C - T_W)_j$ and each ΔT is aligned with its corresponding partition.

Identifying taxable income levels for each ΔT is a prerequisite for developing a gain formula using marginal tax rates. To perform this task, we first ascertain what we call discovery points (DPs). We define a DP as:

discovery point = the taxable income dollar that leads to a jump to a higher marginal tax rate.

DPs enable us to determine the partitions that correspond to unique ΔT values. DPs for each investor's DIW has to be independently found. This is because an investor's taxable income is influenced by the factors that can be different, namely, DIW, tax deduction, SSB, and match/OI.

To illustrate DPs, we assume tax deduction = \$50,000; taxable proportion of SSB = \$40,000; match/OI = \$0; DIW = \$15,000; and, $T_C = 0.18$. Because tax deduction > SSB + match/OI, we have \$50,000 − (\$40,000 + \$0) = \$10,000 for which $T_W = 0$ as it is not taxed. We refer to this \$10,000 as the slack and define it as:

slack = DIW partition that is not taxed where slack > 0 if tax deduction > SSB+match/OI.

If there is no slack then the lowest marginal T_W that can occur is 0.12. In our illustration, the \$10,001st deductible IRA dollar will get the marginal tax rate of $T_W = 0.12$. The \$10,001st dollar is DP1. Before this, $T_W = 0$ and so marginal gain$_1$ = $\Delta T_1(\text{DIW}_1) = (T_C - T_W)\$10{,}000 = (0.18 - 0)\$10{,}000 = \$1{,}800$. There remains \$15,000 − \$10,000 = \$5,000 in a deductible IRA, and so we have: marginal gain$_2$ = $\Delta T_2(\text{DIW}_2) = (T_C - T_W)\$5{,}000 = (0.18 - 0.12)\$5{,}000 = \$300$. The total marginal gain is \$2,100. If DP2 is \$39,001 because we jump to a higher tax bracket with the \$39,001st dollar, then the partition between DP1 and DP2 would be \$39,001 − \$10,001 = \$29,000 and a DIW greater than \$15,000 could have a marginal gain$_2$ as large as 0.06(\$29,000) = \$1,740.

In Table 1, we give DPs, changes in DPs, marginal T_W values, and ΔT values. Because there are seven federal tax brackets including the possibility of slack where part of DIW can be taxed at zero percentage, we have eight DPs. They are important because they reveal where tax rates jump to a higher marginal level. In Panel A, we label these eight DPs from DP1 to DP8. The eight DPs are identified by our withdrawal algorithm presented in Section 6. They correspond to eight ΔT values that cover the changes in T_W from 0 to 0.471. When using a marginal analysis, identifying DPs are vital in computing the marginal gain from the IRA allocation between the two major IRA types.

In Panel B, we report the eight ΔT ranges with the ranges created by subtracting DPs. These differences signify maximum partitions for which a corresponding T_W is applicable. In Panel C, we give eight T_W values corresponding to the eight DP partitions. The beginning of each DP partition represents a jump to a higher T_W value. For example, the beginning of the partition of DP3-DP2 is DP2 = \$69,111 and the \$69,111th DIW dollar achieves a jump

Table 1 Discovery points, ΔT ranges, marginal tax rates, and tax rate differentials

Panel A. Eight discovery points determining maximum partitions applicable for a marginal T_W							
DP1 $46,397	DP2 $69,111	DP3 $156,512	DP4 $307,879	DP5 $465,076	DP6 $824,525	DP7 $890,921	DP8 Unlimited
Panel B. Eight tax rate differentials (ΔTs) ranges created by subtracting discovery points (DPs)							
DP1-0 $46,397	DP2-DP1 $22,714	DP3-DP2 $87,401	DP4-DP3 $151,367	DP5-DP4 $157,197	DP6-DP5 $359,449	DP7-DP6 $66,396	DP8-DP7 Unlimited
Panel C. Eight marginal T_W values corresponding to the eight ΔT ranges							
0.000	0.120	0.180	0.300	0.336	0.396	0.420	0.471
Panel D. Eight tax rate differentials (ΔT) when T_C is 0.30							
ΔT_1 0.300	ΔT_2 0.180	ΔT_3 0.120	ΔT_4 0.000	ΔT_5 −0.036	ΔT_6 −0.096	ΔT_7 −0.120	ΔT_8 −0.171

Note: Panel A contains the eight discovery points (DPs) applicable to this article's RVD illustration. DP is defined as the taxable income dollar that leads to a jump to a higher marginal T_W. DPs are identified by our withdrawal algorithm presented in Section 6. Panel B covers the eight ranges determined by adjoining DPs with larger adjoining DPs often representing the tax rate bracket ranges given in Table 3. Panel C provides the eight marginal T_W values that correspond to the eight ranges in Panel B. Panel D gives the eight tax rate differentials (ΔT) corresponding to the eight ranges in Panel B where $\Delta T = T_C - T_W$ with the marginal $T_C = 0.30$ (as computed in Table 4) used for all DPs.

in T_W from 0.12 to 0.18. In Panel D, we list the eight tax rate differentials (ΔTs). Each ΔT is computed using the marginal T_C (i.e., 0.30 as determined by our contribution algorithm in Section 5) and one of the eight T_W values.

Of interest, the changes in larger adjoining DPs become the tax bracket ranges that can be gathered from information in the last row of Table 2. For example, beginning with DP4-DP3 = $151,367, we see that this is the tax bracket range ending with $300,166 for 0.30 and $457,363 for 0.336 that covers $457,363 − $300,166 = $151,367. If there were no SSB, no match/OI, and no tax deductions, DPs would be determined solely by tax brackets.

4.4. Five-step procedure to compute the marginal gain

Given the above preliminary information, we can now create our marginal gain formula. It is

$$\text{marginal gain} = \Delta T_1 DP1 + \Delta T_2(DP2 - DP1) + \ldots + \Delta T_8(DP8 - DP7). \qquad (12)$$

Given (12), we describe our five-step procedure to compute the marginal gain. *First*, we identify as many DPs as needed to form partitions to cover DIW. *Second*, we associate each partition with its corresponding marginal T_W. Using our marginal $T_C = 0.30$, we compute all applicable ΔT values so that each ΔT is now associated with its corresponding partition. *Third*, we compute the average and marginal gains given in (11) and (12). To illustrate (12) when the match/OI = 0, consider a DIW of $69,111, which is DP2. Using the information in Table 1, we have: marginal gain = ΔT_1DP1 + ΔT_2(DP2-DP1) = (0.30 − 0)$46,397 + (0.30 − 0.12)($69,111 − $46,397) = 0.30($46,397) + 0.18($22,714) = $13,919 + $4,089 = $18,008.

Table 2 Values used in RVD procedure

Panel A. Ten values supplied by a couple filing jointly	
Birth year	1986
Age	30
Retirement year	2050
Portfolio mix during contribution years (proportion in stock)	0.95
Portfolio mix during withdrawal year (proportion in stock)	0.60
Adjusted gross income (AGI)	$110,000
Salary	$106,000
IRA contribution (without the match/OI and without the reinvested tax savings)	$14,661.61
IRA employee salary match (3% of combined salary): 0.03($106,000) = $3,180	$3,180
Other investments (OI): funds invested that are source of retirement withdrawals	$3,180
Note: Together the match and OI (called match/OI) = $3,180 + $3,180 = $6,360	
Panel B. Eight values supplied by our procedure that can be modified	
Y_W (years of withdrawal from IRA based on life expectancy)	20
Annual inflation rate for remainder of lifetime	1.5%
Nominal annual rate of return for stocks	8.3743%
Premium of stocks over non-stock investments	3.5%
Annual growth rate in AGI/salary/match/OI	3%
Current tax deduction (does not include the tax deduction from the deductible IRA)	$25,600
Social Security benefits (SSB) worksheet, line 8 (current value, married joint filer)	$32,000
Social Security benefits (SSB) worksheet, line 10 (current value, married joint filer)	$12,000
Panel C. Computed rates of return	
R_C (nominal annual contribution rate) = 0.95(8.3743%) + (1−0.95)(8.3743%−3.5%)	8.1993%
R_W (Nominal annual withdrawal rate) = 0.60(8.3743%) + (1−0.60)(8.3743%−3.5%)	6.9743%
Panel D. Four factors	
Contribution factor for years 1–35 (based on 1.5% growth)	1.302631083
AGI/salary/match/OI factor for years 1–35 (based on 3% growth)	1.727488052
Withdrawal factor for years 36–55 (based on 1.5% growth)	1.976078686
IRA factor for years 1–35 used with annual rates of return (based on 3% growth)	1.383340574
Panel E. Mean values	
AGI (years 1–35) = AGI/salary/match/OI factor(current AGI) = 1.727488052($110,000)	$190,024
IRA contribution (years 1–35) = IRA factor(maximum IRA contribution) = 1.383340574($14,661.61)	$20,282
Match/OI (years 1–35) = IRA factor(current match/OI) = 1.383340574($6,360)	$8,798
Tax deduction (years 1–35) = Contribution factor(current tax deduction) = 1.302631083($25,600)	$33,347
Tax deduction (years 36–55) = Withdrawal factor(current tax deduction) = 1.976078686($25,600)	$50,588
SSB worksheet, line 8 (years 36–55) = Withdrawal factor(current value) = 1.976078686($32,000)	$63,235
SSB worksheet, line 10 (years 36–55) = Withdrawal factor(current value) = 1.976078686($12,000)	$23,713
Future estimated annual SSB during retirement (estimated based on AGI)	$50,441

Note: Panel A contains 10 current values inputted by our couple who are assumed to have the same age and birth retirement years. An 11th value is the sum of the salary match and other investments (OI). OI can include nondeductible IRA, inheritance, cash value life insurance policies, non-IRA mutual fund investments, or rental income used for withdrawal during retirement but does not include Social Security benefits, employer salary match, or deductible and Roth IRAs. The dollar values are annual values. Panel B contains eight values supplied by our procedure. Panel C computes annual returns for contribution and withdrawal periods. Panel D provides the four factors used to simplify analysis and create mean values used to simplify our RVD procedure. Panel E gives mean values over time for current values based on the factors in Panel D.

Table 3 Tax information from 2016 to 2070

Marginal tax rate	0.120	0.180	0.300	0.336	0.396	0.420	0.471
Current year (2016)	\$0-\$18,550	\$18,551-\$75,300	\$75,301-\$151,900	\$151,901-\$231,450	\$231,451-\$413,350	\$413,351-\$466,950	\$466,951 and greater
Contribution years (2016–2050)	\$0-\$24,164	\$24,165-\$98,088	\$98,089-\$197,870	\$197,871-\$301,494	\$301,495-\$538,443	\$538,444-\$608,264	\$608,265 and greater
Retirement year (2050)	\$0-\$30,774	\$30,775-\$124,992	\$124,993-\$252,002	\$252,003-\$383,975	\$383,976-\$685,746	\$685,747-\$741,488	\$741,489 and greater
Withdrawal years (2051–2070)	\$0-\$36,656	\$36,657-\$148,799	\$148,800-\$300,166	\$300,167-\$457,363	\$457,364-\$816,812	\$816,813- \$922,731	\$922,730 and greater

Note: The first row reports marginal tax rates for periods used in our RVD illustration. The marginal tax rate combines the 2016 federal income marginal tax rate information and 2016 state income marginal tax information that is estimated by averaging the mean statutory state tax rates for all fifty states and matching them as best as possible with the federal taxable income brackets. The seven federal tax rates are 0.10, 0.15, 0.25, 0.28, 0.33, 0.35, and 0.396. We only report information relevant to our illustration, which is the "Married Joint Filers" taxable income bracket ranges but information for Single Filers, Married Separate Filers, and Head of Household Filers could be similarly gotten.

Fourth, if the last DIW dollar is between two adjacent DPs, we substitute it for the larger of the two DPs in (12). To illustrate, if the last DIW dollar is \$50,000 instead of \$69,111, we substitute \$50,000 for \$69,111 and get marginal gain = ΔT_1DP1 + ΔT_2(\$50,000-DP1) = 0.30(\$46,397) + 0.18(\$50,000 − \$46,397) = \$13,919 + \$649 = \$14,568. If the withdrawal includes both DIW and match/OI, we replace DIW with this withdrawal and use its last dollar. *Fifth*, if we have a match and/or OI withdrawal that lies between two adjacent DPs, then we use this withdrawal for the smaller of the two DPs and any component where all DPs are less than the match and/or OI exits (1). To illustrate, if DP2 = \$69,111, DP3 = \$156,513, and match = \$74,655, then we use \$74,655 for DP2 and get marginal gain = ΔT_3(DP3-match) = (0.30−0.18)(DP3-match) = 0.12(\$156,513 − \$74,655) = \$9,823 where the first two components of (1), ΔT_1DP1 + ΔT_2(DP2-DP1), exit the equation. DIW for this illustration is DP3 − match = \$156,513 − \$74,655 = \$81,858. Thus, whereas the annuity withdrawal is \$156,513, only \$81,858 is from a deductible IRA.

5. Inputs, tax brackets, T_C, and T_W

In this section, we introduce the variables and values for our couple. We next provide tax brackets applicable to our couple over their lifetime followed by our contribution algorithm that computes T_C. Lastly, we generate future values for contributions to deductible and Roth IRAs with these future values supplying retirement withdrawals.

5.1. Inputs and values

Table 3 contains values used in our RVD procedure. In Panel A, we display 10 variables and their values supplied by our couple. We assume the same values for birth year, current age and retirement year for our couple. The portfolio mixes are consistent with investors with a long-term, aggressive strategy. The AGI, salary, employer's salary match, other investments (OI), and IRA contribution are values that occur at the end of the current year. The IRA contribution of $14,661.61 is the maximum amount our couple would invest in an IRA if all of their investment went into a Roth IRA so that there are no tax savings from a deductible IRA that could be used for investing in an IRA. The match and OI are each $3,180 and together the match/OI is $6,360. To the extent OI consists of dividends and long-term capital gains, OI does not influence T_W but can still be a factor in determining one's optimal RVD allocation. This is because larger DIWs can increase the long-term capital gains and dividend tax rates by bumping them from 0% in the two lowest tax brackets to either 15% in the intermediate tax brackets or 20% in the two highest tax brackets.

In Panel B, we provide our supplied values used in our RVD procedure. Consistent with the Social Security life expectancy calculator, we assume a 20-year retirement period. The inflation rate of 1.5% is based on recent periods such as given by http://www.multpl.com/inflation/table. The latter reports annual inflation rates based on the 12-month change in the Consumer Price Index (data courtesy of the U.S. Bureau of Labor Statistics and Robert Shiller). We supply a nominal rate of stock return of 8.3743% based on the average of the S&P 500 and NASDAQ Composite returns since 1971. Nonstock investments can be construed as fixed-income investments assumed to earn 3.5% less than stocks. The 3% annual growth rate is consistent with the historical growth in the U.S. GDP over the past 35 years as given by the World Bank. We expect the 3% growth rate for the salary match to hold for our couple as this growth will prevent them from reaching the IRA contribution limits. This is because they are currently well under the current limits of $47,000 with these limits increasing at age 50 to $61,000. The current tax deduction of $25,600 includes personal exemptions, standard deductions and factors in itemized deductions and tax credits. It does not include tax deductions from a deductible IRA as that is determined later once the optimal DIW is known. The two Social Security values of $32,000 and $12,000 are current dollars that grow over time at our annual inflation rate of 1.5%.

Panel C computes the nominal annual rates of return during contribution and withdrawal years as 8.1993% and 6.9743%, respectively. As can be seen from their calculations, these returns are determined by their portfolio mixes. Panel D gives four factors used to create mean values for use in our RVD procedure that uses our mean approach. The use of these factors enable us to avoid computing 35 values for variables used during our contribution period and 20 values for variables used during our withdrawal period. Below we explain these factors.

The general formula used to compute our first three factors is:

$$\text{factor} = (1 + x)^{t-1} \frac{(1 + x)^0 + (1 + x)^1 + \ldots + (1 + x)^{n-2} + (1 + x)^{n-1})}{n} \tag{13}$$

where x is the assumed rate of increase or growth per year; t is the year the first cash flow begins; and, n is the number of years in the period being considered. Using (13) with x = 1.5%, t = 1 and n = 35, we get our contribution factor. We have:

$$\text{contribution factor} = (1.015^{1-1})\frac{(1.015^{0} + 1.015^{1} + \ldots + 1.015^{33} + 1.015^{34})}{35} =$$

$$(1.015^{0})\frac{45.59208789}{35} = (1)1.302631083 = 1.302631083.$$

A current value is multiplied by this factor to get the mean value for the 35-year contribution period. Our 35-year AGI/salary/match/OI factor of 1.727488052 is computed in the same manner, except we use 3% in (13) instead of 1.5%.

For our withdrawal factor where t = 35 and n = 20, we have to modify (13) by increasing all exponents by plus one. After doing this, we have:

$$\text{withdrawal factor} = (1.015)^{35}\frac{(1.015^{1} + 1.015^{2} + \ldots + 1.015^{18} + 1.015^{20})}{20} =$$

$$(1.015^{35})\frac{23.47052211}{20} = (1.68388132)1.17352611 = 1.976078686.$$

To illustrate, a value of \$18,550 in the current tax bracket for a married jointly filer becomes 1.976078686(\$18,550) = \$36,656 for the 20-year withdrawal period of 2051–2070. The value of \$36,656 is the same value wrought from computing the 20 future values for \$18,550 from 2051 through 2070 and then averaging them.[6]

The process to compute our IRA factor in Panel D is as follows. *First*, we calculate the future value of the initial investment of \$14,661.61 if it is growing annually at 3% and achieves an annual nominal rate of 8.1993%. We have:

$$\$14{,}661.61(1.03)^{0}(1.081993)^{34} + \ldots + \$14{,}661.61(1.03)^{34}(1.081993)^{0} = \$3{,}653{,}762.$$

Second, we divide \$3,653,762 by the future value annuity factor of $\frac{(1 + 0.081993)^{35} - 1}{0.081993} =$ 180.14799987 to generate the equivalent annual annuity cash flow of \$20,282. Dividing this cash flow by \$14,661.61 gives our IRA factor of 1.383340574. Creating this factor has the advantage of only requiring the advisee to identify their current IRA contribution of \$14,661.61. Because this factor is dependent on the number of contribution years, growth in AGI and nominal annual contribution rate, it has to be generated separately when any of these values change. In a spreadsheet format, our factors are recomputed automatically when the values changed.

Panel E gives mean values for contribution and withdrawal periods by multiplying a current value by its applicable factor. To illustrate, we use the AGI/salary/match/OI factor to calculate a mean AGI of 1.727488052(\$110,000) = \$190,024 for the contribution years 1–35 (that for our illustration are years 2016–2050). In regards to the \$20,282 discussed previously, we can show that this IRA contribution is achievable for our couple. *First*, for investors working for companies with a 401(k) plan, the current allowable IRA contribution is \$18,000. *Second*, besides the 401(k), their income levels allow them to each qualify for a non-401(k) IRA of \$5,500 that can be invested in an IRA. Under current law, they can invest \$18,000 + \$5,500 = \$23,500 in either a deductible IRA or a Roth IRA per person or \$47,000 per couple. Given our contribution factor, the mean for the next 35 years can be represented by 1.302631083(\$47,000) = \$61,224. This amount is not only much greater than the couple's maximum Roth IRA contribution of \$20,282 but will also more than cover the maximum they can put in a deductible IRA where the mean is \$20,282/(1-T_C) for our couple's contribution years.

The final value in Panel E is our couple's future retirement estimate for SSB. Estimating SSB is difficult because of uncertainties surrounding Social Security's future solvency. We consulted the Social Security online benefit calculator to derive our estimate of \$50,441.[7] Our estimate takes into account the current payout for retiring at today's normal age of 66 and adjusting it downward because our couple will retire at the age of 64. We then use our withdrawal factor and further adjust the number downwards based on current projections about social security payouts.

5.2. Four tax bracket tables

In Table 2, we present tax information for seven taxable income brackets. We only report information relevant to our illustration, which is the Married Joint Filers taxable income bracket ranges. Information for Single Filers, Married Separate Filers, and Head of Household Filers could be similarly gotten. The total marginal tax rate is the federal marginal tax rate plus the marginal state tax rate. We estimate the state tax rate by averaging the mean statutory state tax rates for all fifty states and match them as best as possible to the federal taxable income brackets when computing the total marginal tax rate.

Row one has tax information for 2016. Row two has projected tax information for the 35 contribution years from 2016 to 2050 computed by multiplying each taxable income value in row one times our contribution factor of 1.302631083 given in Table 3. Row three provides tax information for the retirement year of 2050 by multiplying 1.015^{34} = 1.658996373 times each taxable income value in row one. The last row has projected tax information for the 20 withdrawal years from 2051 to 2070 computed by multiplying each taxable income value in row one by our withdrawal factor of 1.976078686 given in Table 3.

5.3. Computing the contribution tax rate (T_C)

Given the information from Tables 2 and 3, we compute T_C using our contribution algorithm in Table 4. If there is an overlap between taxable income ranges, then T_C is computed using two tax rates. If there is no overlap, the average T_C is the marginal T_C. In Table 4, we input our couple's maximum Roth IRA of \$20,282 (line 1), AGI of \$190,024 (line 2), and tax deduction of \$33,347 (line 3). These values are given in Panel E of Table 3. The tax deduction does not include the deduction from the optimal deductible IRA contribution, which is determined later after we introduce our withdrawal algorithm. While not inputting the total tax deduction can overestimate T_C such is not the case for our couple because we still get T_C = 0.30 (line 100) even if we were to input \$33,347 plus the

Table 4 Contribution algorithm to determine the contribution tax rate (T_C)

1. Enter maximum amount available to invest in a Roth IRA.	$20,282
2. Enter adjusted gross income (AGI).	$190,024
3. Enter tax deduction (for now we exclude the tax deduction when investing in a deductible IRA because we have not determined the optimal RVD allocation).	$33,347
4. Line 2 minus line 3. This is the taxable income.	$156,677
5. Enter taxable income that is first taxed at 0.471.	$582,212
6. Enter larger between (line 4 minus line 5) and zero. This is the taxable income for those who qualify for the 0.471 tax bracket.	$0
7. Enter 0 if line 6 is $0; else 1. Equals 1 if qualifies at least in part for 0.471 tax savings.	0
8. Enter 0.471. This is the marginal tax rate for those with taxable income given on line 5 and greater.	0.471
9. Divide line 1 by (one minus line 8). This is maximum deductible IRA for 0.471.	$38,340
10. Enter smaller of line 7 and (line 6 divided by line 9). This is the percentage of the maximum tax savings for those getting 0.471.	0.00%
11. One minus line 10. This is the percentage qualifying for 0.42 overlap if between 0% and 100%.	100.00%
12. Multiply lines 9 and 10. This amount qualifies to lower the taxable income using 0.471 tax rate.	$0
13. Multiply lines 7, 11, and 21. This overlap amount qualifies to lower the taxable income using 0.42 tax rate.	$0
14. Enter 0 if line 13 is $0; else 1. Eliminate all but the overlap at 0.42 tax rate.	0
15. Enter 0 if line 11 is 0%; else 1. Eliminate those who qualified for tax savings at 0.471.	1
16. Multiply lines 8, 9 and 10. Dollar tax savings at 0.471. This is also total tax savings to date.	$0
17. Enter taxable income that is first taxed at 0.42.	$538,444
18. Multiply line 15 by larger of (line 4 minus line 17) and zero. The taxable income for those who qualify in 0.42 bracket.	$0
19. Enter zero if line 18 equals $0; else 1. Equals 1 if qualifies at least in part for 0.42 tax savings.	0
20. Enter 0.42. This is marginal tax rate for those with taxable income falling in the range given by lines 5 and 17.	0.42
21. Divide line 1 by (one minus line 20). This is maximum deductible IRA for 0.42.	$34,969
22. Enter smaller of line 19 and (line 18 divided by line 21). This is the percentage of the maximum tax savings for those getting 0.42.	0.00%
23. Enter one minus line 22. This is the percentage qualifying for 0.396 overlap if between 0% and 100%.	100.00%
24. Enter 1 if line 14 equals 0; else 0.	1
25. Multiply lines 21, 22, and 24. This amount qualifies to lower the taxable income using 0.42 tax rate.	$0
26. Multiply lines 19, 23, and 35. This overlap amount qualifies to lower the taxable income using 0.396 tax rate.	$0
27. Enter 0 if line 23 equals 0%; else 1. Eliminate those that maxed out tax savings at 0.42.	1
28. Enter smaller of lines 27 and 15. We have now taken out those who have maxed out to date.	1
29. Multiply lines 20, 21, and 22. This is dollar tax savings if getting 0.42.	$0
30. Multiply lines 11 and 29. This adjusts for overlap and gives new tax savings to add in.	$0
31. Enter taxable income that is first taxed at 0.396.	$301,495
32. Multiply line 28 by the larger of (line 4 minus line 31) and 0. The taxable income for those who qualify in 0.396 bracket.	$0

Table 4 (Continued)

33. Enter 0 if line 32 equals \$0; else 1. Equals 1 if qualifies at least in part for 0.396 tax savings.	0
34. Enter 0.396. This is the marginal tax rate for those with taxable income falling in the range given by lines 17 and 31.	0.396
35. Divide line 1 by (one minus line 34). This is maximum deductible IRA for 0.396.	\$33,579
36. Enter smaller of line 33 and (line 32 divided by line 35). This is the percentage of the maximum tax savings for those getting 0.396.	0.00%
37. Enter one minus line 36. This is the percentage qualifying for 0.336 overlap if between 0% and 100%.	100.00%
38. Enter 1 if line 29 equals 0; else 0.	1
39. Multiply lines 35, 36, and 38. This amount qualifies to lower the taxable income using 0.396 tax rate.	\$0
40. Multiply lines 33, 37, and 49. This overlap amount qualifies to lower the taxable income using 0.336 tax rate.	\$0
41. Enter 0 if line 37 equals 0%; else 1. Eliminate those that maxed out tax savings at 0.396.	1
42. Enter smaller of lines 41 and 28. We have now taken out those who have maxed out to date.	1
43. Multiply lines 34, 35, and 36. This is dollar tax savings if getting 0.396.	\$0
44. Multiply lines 23 and 43. This adjusts for overlap and gives new tax savings to add in.	\$0
45. Enter taxable income that is first taxed at 0.336.	\$197,871
46. Multiply line 42 by the larger of (line 4 minus line 45) and 0. The taxable income for those who qualify in 0.336 bracket.	\$0
47. Enter 0 if line 46 equals \$0; else 1. Equals 1 if qualifies at least in part for 0.336 tax savings.	0
48. Enter 0.336. This is the marginal tax rate for those with taxable income falling in the range given by lines 31 and 48.	0.336
49. Divide line 1 by (one minus line 48). This is maximum deductible IRA for 0.336.	\$30,545
50. Enter smaller of line 47 and (line 46 divided by line 49). This is the percentage of the maximum tax savings for those getting 0.336.	0.00%
51. Enter one minus line 50. This is the percentage qualifying for 0.30 overlap if between 0% and 100%.	100.00%
52. Enter 1 if line 43 equals 0; else 0.	1
53. Multiply lines 49, 50, and 52. This amount qualifies to lower the taxable income using 0.336 tax rate.	\$0
54. Multiply lines 47, 51, and 63. This overlap amount qualifies to lower the taxable income using 0.30 tax rate.	\$0
55. Enter 0 if line 51 equals 0%; else 1. Eliminate those that maxed out tax savings at 0.336.	1
56. Enter smaller of line 55 and line 42. We have now taken out those who have maxed out to date.	1
57. Multiply lines 48, 49, and 50. This is dollar tax savings if getting 0.336.	\$0
58. Multiply lines 37 and 57. This adjusts for overlap and gives new tax savings to add in.	\$0
59. Enter taxable income that is first taxed at 0.30.	\$98,089
60. Multiply line 56 by the larger of (line 4 minus line 59) and 0. The taxable income for those who qualify in 0.30 bracket.	\$58,588
61. Enter 0 if line 60 equals \$0; else 1. Equals 1 if qualifies at least in part for 0.30 tax savings.	1
62. Enter 0.30. This is the marginal tax rate for those with taxable income falling in the range given by lines 48 and 59.	0.30

Table 4 (Continued)

63. Divide line 1 by (one minus line 62). This is maximum deductible IRA for 0.30.	$28,974
64. Enter smaller of line 61 and (line 60 divided by line 63). This is the percentage of the maximum tax savings for those getting 0.30.	100.00%
65. Enter one minus line 64. This is the percentage qualifying for 0.18 overlap if between 0% and 100%.	0.00%
66. Enter 1 if line 57 equals 0; else 0.	1
67. Multiply lines 63, 64, and 66. This amount qualifies to lower the taxable income using 0.30 tax rate.	$28,974
68. Multiply lines 61, 65, and 77. This overlap amount qualifies to lower the taxable income using 0.18 tax rate.	$0
69. Enter 0 if line 65 equals 0%; else 1. Eliminate those that maxed out tax savings at 0.30.	0
70. Enter smaller of line 69 and line 56. We have now taken out those who have maxed out to date.	0
71. Multiply lines 62, 63, and 64. This is dollar tax savings if getting 0.30.	$8,692
72. Multiply lines 51 and 71. This adjusts for overlap and gives new tax savings to add in.	$8,692
73. Enter taxable income that is first taxed at 0.18.	$24,165
74. Multiply line 70 by the larger of (line 4 minus line 73) and 0. The taxable income for those who qualify in 0.18 bracket.	$0
75. Enter 0 if line 74 equals $0; else 1. Equals 1 if qualifies at least in part for 0.18 tax savings.	0
76. Enter 0.18. This is the marginal tax rate for those with taxable income falling in the range given by lines 59 and 73.	0.18
77. Divide line 1 by (one minus line 76). This is maximum deductible IRA for 0.18.	$24,734
78. Enter smaller of line 75 and (line 74 divided by line 77). This is the percentage of the maximum tax savings for those getting 0.18.	0.00%
79. Enter one minus line 78. This is the percentage qualifying for 0.12 overlap if between 0% and 100%.	100.00%
80. Enter 1 if line 71 equals 1; else 0.	0
81. Multiply lines 77, 78, and 80. This amount qualifies to lower the taxable income using 0.18 tax rate.	$0
82. Multiply lines 75, 79, and 91. This overlap amount qualifies to lower the taxable income using 0.12 tax rate.	$0
83. Enter 0 if line 79 equals 0%; else 1. Eliminate those that maxed out tax savings at 0.18.	1
84. Enter smaller of line 83 and line 70. We have now taken out those who have maxed out to date.	0
85. Multiply lines 76, 77, and 78. This is dollar tax savings if getting 0.18.	$0
86. Multiply lines 65 and 85. This adjusts for overlap and gives new tax savings to add in.	$0
87. Enter taxable income that is first taxed at 0.12.	$0
88. Multiply line 84 by the larger of (line 4 minus line 87) and 0. The taxable income for those who qualify in 0.12 bracket.	$0
89. Enter 0 if line 88 equals $0; else 1. Equals 1 if qualifies at least in part for 0.12 tax savings.	0
90. Enter 0.12. This is the marginal tax rate for those with taxable income falling in the range given by lines 73 and 90.	0.12
91. Divide line 1 by (one minus line 90). This is maximum deductible IRA for 0.12.	$23,048
92. Enter smaller of line 89 and (line 88 divided by line 91). This is the percentage of the maximum tax savings for those getting 0.12.	0.00%
93. Enter one minus line 92. This is the percentage qualifying for 0% overlap if between 0% and 100%.	100.00%

Table 4 (Continued)

94. Enter 1 if line 85 equals 0; else 0.	1
95. Multiply lines 91, 92, and 94. This amount qualifies to lower the taxable income using 0.12 tax rate.	\$0
96. Multiply lines 90, 91, and 92. This is dollar tax savings if getting 0.12.	\$0
97. Multiply lines 79 and 96. This adjusts for overlap and gives new tax savings to add in.	\$0
98. Add lines 16, 30, 44, 58, 72, 86, and 97. This is the total tax saving if maximum deductible used.	\$8,692
99. Enter larger of 0.01 and addition of lines 12, 13, 25, 26, 39, 40, 53, 54, 67, 68, 81, 82, and 95. Amount taxable deduction if maximum deductible IRA taken. This amount times the optimal RVD allocation once determined will be added to line 3.	\$28,974
100. Divide line 98 by line 99. T_C = (tax savings from deductible IRA contributions)/(deductible IRA contributions).	0.3000

maximum IRA deduction of \$20,282/(1−0.3) = \$28,974 so that the total tax deduction is \$62,321.

While $T_C = 0.30$ is our best guess, we still need further scrutiny. Because our couple has a 3% annual increase in earnings compared to only a 1.5% annual increase in the tax brackets, they will be paying taxes at a lower marginal tax rate earlier in their working lives. Thus, their deductible IRA contribution over time can have an increasingly higher T_C. When we test all 35 contribution years after adjusting each year for increases, we find that by the 2nd year, our couple gets 57% of its maximum contributions at the 0.30 tax savings level and by the 9th year, they get 100% at the 0.30 level. By the 34th year, they are getting all of their tax savings at 0.336. Thus, it is highly unlikely that they would ever have to contribute to an IRA and get only 0.18 savings on the dollar especially given that their optimal contribution is almost certainly less than their maximum contribution. We also know they are likely to get withdrawals with tax savings greater than 0.30. Thus, there is evidence that a marginal T_C of 0.30 is a minimum estimation of their true tax savings. Finally, for situations where an optimal deductible IRA is lower and the growth in earnings is greater than inflation, our couple should wait to contribute to a deductible IRA to get the highest possible marginal T_C.

5.4. Computing future withdrawal cash flow values for IRA

Having computed T_C, the next step in the development of our RVD procedure is to supply future withdrawal cash flow values for IRAs. In Table 5, we supply this information. We begin in Panel A by computing the maximum deductible IRA contribution for the current year of \$28,974. Maximum refers to the fact our couple would have to put all of their maximum Roth IRA investment of \$20,282 in a deductible IRA and then use the tax savings of 0.3(\$28,974) = \$8,692. Thus, they could invest a maximum of \$20,282 + \$8,692 = \$28,974 in a deductible IRA. As seen in Panel A, if we add in the match/OI of \$8,798, then we get \$37,772.

In Panel B, we calculate the maximum future value for the Roth IRA (MFV_{Roth}) as \$3,653,762 and it is not taxable. The future value for the deductible IRA is taxable and so has a maximum before-tax future value deductible IRA (MFV_{BTDed}) computed as

Table 5 Definitions and equations to compute maximum withdrawal cash flows

Panel A. Maximum annual contributions during working years: Means for 2016–2050

MAC_{Roth} (maximum annual contribution Roth IRA) = \$20,282 (computed in Panel E of Table 2)

MAC_{BFDed} (maximum before-tax annual contribution deductible IRA) = $AAC_{Roth}/(1-T_C)$ = \$20,282/(1−0.3) = \$28,974

AC_{BFMOI} (before-tax annual contribution match/OI) = \$8,798 (computed in Panel E of Table 2)

$MAC_{BFDedBFMOI}$ (maximum before-tax annual contribution deductible IRA and match/OI) = $MAC_{BFDed} + AC_{BFMOI}$ = \$28,974 + \$8,798 = \$37,772

Panel B. Maximum before-tax lump sum future values at retirement: Means for end of 2050

MFV_{Roth} (maximum future value Roth IRA at retirement for $Y_C = 35$ and $R_C = 8.1993\%$) =

$$MAC_{Roth}(FVAF_{R_C,Y_C}) = MAC_{Roth}\left(\frac{(1+R_C)^{Y_C}-1}{R_C}\right) = \$20{,}282\left(\frac{(1+0.0811993)^{35}-1}{0.081993}\right) =$$

\$20,282(180.14799987) = \$3,653,762

MFV_{BTDed} (maximum before-tax future value deductible IRA) = $MAC_{BFDed}(FVAF_{R_C,Y_C})$ = \$28,974(180.14799987) = \$5,219,608

FV_{BTMOI} (before-tax future value match/OI) = $AC_{BFMOI}(FVAF_{R_C,Y_C})$ = \$8,798(180.14799987) = \$1,584,942

$MFV_{BTDedBTMOI}$ (maximum before-tax future value deductible IRA and match/OI) = MFV_{BTDed} + FV_{BTMOI} = \$5,219,608 + \$1,584,942 = \$6,804,550

Panel C. Maximum after-tax lump sum future values if $T_W = 0.30$ at retirement: Means for end of 2050

MFV_{ATDed} (maximum after-tax future value deductible IRA) = $(1-T_W)(MFV_{BTDed})$ = (1−0.30)(\$5,219,608) = \$3,653,762

FV_{ATMOI} (after-tax future value match/OI) = $(1-T_W)FV_{BTMOI}$ = (1−0.30)\$1,584,942 = \$1,109,459

$MFV_{ATDedMOI}$ (maximum after-tax future values for deductible IRA and match/OI) = MFV_{ATDed} + FV_{ATMOI} = \$3,653,762 + \$1,109,459 = \$4,763,221

Panel D. Maximum before-tax annual withdrawals: Means for 2051–2070

MAW_{Roth} (max annual withdrawal from Roth IRA for $Y_W = 20$ and $R_W = 6.9743\%$) =

$$MFV_{Roth}(1/PVAF_{r_W,Y_W}) = FV_{Roth}\left[R_W/\left(1-\frac{1}{(1+R_W)^{Y_C}}\right)\right] = \$3{,}653{,}762\left[0.069743/\left(1-\frac{1}{(1+0.069743)^{20}}\right)\right] =$$

\$3,653,762(0.094204462) = \$344,201

MAW_{BTDed} (maximum before-tax annual withdrawal deductible IRA) = $MFV_{BTDed}(1/PVAF_{r_w,Y_w})$ = \$5,219,608(0.094204462) = \$491,710

AW_{BTMOI} (before-tax annual withdrawal match/OI) = $FV_{BTMOI}(1/PVAF_{r_w,Y_w})$ = \$1,584,942(0.094204462) = \$149,309

$MAW_{BTDedMOI}$ (maximum before-tax annual withdrawals for deductible IRA and match/OI) = $MAW_{BTDed} + AW_{BTMOI}$ = \$491,710 + \$149,309 = \$641,019

Panel E. Maximum after-tax annual withdrawals if $T_W = 0.30$: Means for 2051–2070

MAW_{ATDed} (maximum after-tax annual withdrawal deductible IRA) = $(1-0.3)MAW_{BTDed}$ = (1−0.3)\$491,710 = \$344,201

AW_{ATMOI} (after-tax annual withdrawal match/OI) = $(1-0.3)AW_{BTMOI}$ = (1−0.3)\$149,309 = \$104,516

$MAW_{ATDedMOI}$ (maximum after-tax annual withdrawals for deductible IRA and match/OI) = $MAW_{ATDed} + AW_{ATMOI}$ = \$344,201 + \$104,516 = \$448,717

\$5,219,608 in Panel B and a maximum after-tax future value deductible IRA (MFV_{ATDed}) computed as \$3,653,762 in Panel C if $T_W = 0.30$. Thus, when $T_C = T_W = 0.30$, we see that $MFV_{Roth} = MFV_{ATDed} =$ \$3,653,762. There are two conditions for this equality to hold. *First*, the tax savings from the deductible IRA must be invested in the deductible IRA along with the same amount invested in the Roth IRA. *Second*, T_C must equal T_W. We provide a formal proof in Appendix 1.

With a Roth IRA, the IRS collects its taxes upfront. With a deductible IRA, the IRS collect its taxes on the principal and earnings when withdrawn. In regards to the earnings on a deductible IRA, the IRS holds the right to get a proportion of the earnings and that proportion is T_W. The future value of the annuity tax savings from the deductible IRA explains the difference of \$1,565,846 between the before-tax lump sum future values of the deductible IRA of \$5,219,608 and the Roth IRA of \$3,653,762. To illustrate using our future value annuity factor of 180.14799987 in Panel B, the lump sum future value of this annuity tax savings is 0.3(\$28,974)(180.14799987) = \$1,565,846. This is the same value if the deductible IRA is withdrawn with taxes paid at $T_W = 0.30$ as the value of the taxes paid is 0.30(\$5,219,608) = \$1,565,846.[8] When $T_C = T_W$, the value of the tax savings is the same as the value of the taxes paid.

Panel D of Table 5 provides the before-tax annual withdrawals for 20 years generated from the lump sum future values in Panel C. The value of \$149,309 for the before-tax annual withdrawal from the match/OI is important because it raises the taxable income increasing T_W. When using a marginal analysis, the match/OI is assumed to have been withdrawn before DIW. Finally, Panel E gives after-tax withdrawal values where we see that the maximum Roth annual withdrawal of \$344,201 in Panel D is equal to the after-tax value of the maximum annual DIW of \$344,201 in Panel E. Once again, the equality results because of the two conditions described above.

6. Procedure to determine the optimal RVD solution

In this section, we introduce our withdrawal algorithm that computes the optimal RVD allocation. We present introductory illustrations in Table 6 that use values given earlier such as our couple's Social Security Benefits (SSB) and tax deductions. In Table 7, we determine our couple's optimal RVD range. Among the outcomes of our withdrawal algorithm are the average T_W and the marginal T_W. The latter is determined from lines that contain them, for example, line 27 has $T_W = 0.12$, line 31 has $T_W = 0.18$, and so forth, for every subsequent fourth line until we reach line 51 where $T_W = 0.471$.

6.1. First discovery point

In Table 6, we begin filling in our withdrawal algorithm by inputting our couple's SSB of \$50,441 (line 1). For Example 1, we input a DIW of \$0 (line 3) indicating all contributions were put in a Roth IRA. The percentage of SSB subject to taxes is 0% (line 21). While the zero taxes on SSB is one benefit of a Roth IRA, the marginal gain of zero (line 58) reflects a lost opportunity by not investing in a deductible IRA.

Table 6 Withdrawal algorithm: Determines first discovery point (DP1)

	Example 1	Example 2	Example 3
1. Enter Social Security benefit (found on Form 1040, line 20a).	$50,441	$50,441	$50,441
2. Enter one-half of line 1.	$25,221	$25,221	$25,221
3. Enter Total Income on Form 1040, line 22 minus SSB line 20a. (All withdrawals are DIW; match/OI not yet considered.)	$0.00	$46,396.83	$46,396.84
4. Enter total of any exclusions/adjustments (typically not applicable so enter $0).	$0	$0	$0
5. Add lines 2, 3, and 4.	$25,221	$71,617	$71,617
6. Add lines 23 through 35 from Form 1040 (these can lower your AGI for most years; enter $0).	$0	$0	$0
7. Subtract line 6 from line 5.	$25,221	$71,617	$71,617
8. Enter $63,235 since married filing jointly.	$63,235	$63,235	$63,235
9. Subtract line 8 from line 7. If zero or less, enter $0. If line 9 is more than zero, go to line 10.	$0	$8,382	$8,382
10. Enter $23,713 since married filing jointly.	$23,713	$23,713	$23,713
11. Subtract line 10 from line 9. If zero or less, enter $0.	$0	$0	$0
12. Enter smaller of line 9 or line 10.	$0	$8,382	$8,382
13. Enter one-half of line 12.	$0	$4,191	$4,191
14. Enter smaller of line 2 or line 13.	$0	$4,191	$4,191
15. Multiply line 11 by 85% (.85). If line 11 is zero, enter $0.	$0	$0	$0
16. Add lines 14 and 15.	$0	$4,191	$4,191
17. Multiply line 1 by 85% (0.85).	$42,875	$42,875	$42,875
18. Enter smaller of line 16 or line 17.	$0	$4,191	$4,191
19. Enter amount from line 20 of the lump sum SS Worksheet. Not applicable so enter $0.	$0	$0	$0
20. Enter smaller of line 18 or line 19. This is the taxable SSB.	$0	$4,191	$4,191
21. Percent of taxable SSB subject to taxes. Divide line 20 by line 1. Put in percentage form. Put in 0% if no SSB.	0%	8.3090%	8.3091%
22. Multiply line 20 by line 56. (Have to compute T_W later in line 56.) This is taxes paid on SS if we use T_W and if SSB are actually taxed.	$0	$0	$503
23. Subtract line 1 from line 20. This is the nontaxable SSB.	$50,441	$46,250	$46,250
24. Adjusted gross income (AGI). Add line 3 and line 20.	$0	$50,588	$50,588
25. Enter $50,588. This is the total tax deduction.	$50,588	$50,588	$50,588
26. Subtract line 25 from line 24. This is the annual taxable income during retirement withdrawal.	$0.00	$0.00	$0.01
27. Enter 0.12. This is the marginal tax rate for $36,656 of taxable income.	0.12	0.12	0.12
28. Enter $36,656. This is maximum taxable income for 0.12 marginal tax rate.	$36,656	$36,656	$36,656
29. Enter smaller of line 26 or line 28. If zero or less, enter $0. This is applicable taxable income for 0.12 marginal tax rate.	$0.00	$0.00	$0.01
30. Multiply line 27 by line 29. Taxes paid at 0.12 marginal tax rate.	$0	$0	$0
31. Enter 0.18. This is the marginal tax rate for the next $112,143 of taxable income.	0.18	0.18	0.18
32. Enter $112,143. This is maximum taxable income for 0.18 marginal tax rate.	$112,143	$112,143	$112,143
33. Enter smaller of line 32 or (line 26 minus line 28). If zero or less, enter $0. This is applicable taxable income for 0.18 marginal tax rate.	$0	$0	$0

Table 6 (Continued)

	Example 1	Example 2	Example 3
34. Multiply line 31 by line 33. Taxes paid at 0.18 marginal tax rate.	$0	$0	$0
35. Enter 0.30. This is the marginal tax rate for $151,367 of taxable income.	0.30	0.30	0.30
36. Enter $151,367. This is maximum taxable income for 0.30 marginal tax rate.	$151,367	$151,367	$151,367
37. Enter smaller of line 36 or (line 26 minus lines 28 and 32). If zero or less, enter $0. This is applicable taxable income for 0.30 marginal tax rate.	$0	$0	$0
38. Multiply line 35 by line 37. Taxes paid at 0.30 marginal tax rate.	$0	$0	$0
39. Enter 0.336. This is the marginal tax rate for $157,197 of taxable income.	0.336	0.336	0.336
40. Enter $157,197. This is maximum taxable income for 0.336 marginal tax rate.	$157,197	$157,197	$157,197
41. Enter smaller of line 40 or (line 26 minus lines 28, 32, and 36). If zero or less, enter $0. This is applicable taxable income for 0.336 marginal tax rate.	$0	$0	$0
42. Multiply line 39 by line 41. Taxes paid at 0.336 marginal tax rate.	$0	$0	$0
43. Enter 0.396. This is the marginal tax rate for $359,449 of taxable income.	0.396	0.396	0.396
44. Enter $359,449. This is maximum taxable income for 0.396 marginal tax rate.	$359,449	$359,449	$359,449
45. Enter smaller of line 44 or (line 26 minus lines 28, 32, 36, and 40). If zero or less, enter $0. This is applicable taxable income for 0.396 marginal tax rate.	$0	$0	$0
46. Multiply line 43 by line 45. Taxes paid at 0.396 marginal tax rate.	$0	$0	$0
47. Enter 0.42. This is the marginal tax rate for $66,396 of taxable income.	0.42	0.42	0.42
48. Enter $66,396. This is maximum taxable income for 0.42 marginal tax rate.	$66,396	$66,396	$66,396
49. Enter smaller of line 48 or (line 26 minus lines 28, 32, 36, 40, and 44). If zero or less, enter $0. This is applicable taxable income for 0.42 marginal tax rate.	$0	$0	$0
50. Multiply line 47 by line 49. Taxes paid at 0.42 marginal tax rate.	$0	$0	$0
51. Enter 0.471. This is marginal tax rate for unlimited amount of taxable income.	0.471	0.471	0.471
52. Enter $9,999,999 as proxy for unlimited for the maximum taxable income for 0.471 marginal tax rate.	$9,999,999	$9,999,999	$9,999,999
53. Enter smaller of line 52 or (line 26 minus lines 28, 32, 36, 40, 44, and 48). If zero or less, enter $0. This is applicable taxable income for 0.471 marginal tax rate.	$0	$0	$0
54. Multiply line 51 by line 53. Taxes paid at 0.471 marginal tax rate.	$0	$0	$0
55. Add lines 30, 34, 38, 42, 46, 50, and 54. Total taxes paid (on retirement withdrawal).	$0	$0	$0

Table 6 (Continued)

	Example 1	Example 2	Example 3
56. Divide line 55 by line 26. This is average T_W = taxes paid/taxable income.	0.00	0.00	0.12
57. Enter 0.30. This is T_C computed in Table 4.	0.30	0.30	0.30
58. Compute the marginal gain (using marginal tax rates) = $\Delta T_1 DP1 + \Delta T_2(DP2-DP1) + ... + \Delta T_8(DP8-DP7)$ replacing DPs and dropping components as prescribed by the five-step procedure.	\$0.000	\$13,919.049	\$13,919.051
59. Enter \$491,710. This is the maximum future value of the before-tax annual annuity withdrawal from deductible IRA.	\$491,710	\$491,710	\$491,710
60. Divide line 3 by line 59 and put in percentage form. Percent withdrawn from maximum possible.	0.00%	9.44%	9.44%
61. Enter \$344,201. This is the maximum future value of the before-tax annual annuity withdrawal from Roth IRA.	\$344,201	\$344,201	\$344,201
62. Subtract line 60 from 100% and multiply by line 61 to get the dollar amount of the Roth IRA.	\$344,201	\$311,723	\$311,723

We input a DIW of \$46,396.83 for Example 2. If there is taxable income, SSB will be taxed at a rate of 8.309% (line 21) implying the taxable SSB is 0.08309(\$50,441) = \$4,191 (line 20). However, the tax deduction of \$50,588 (line 25) is greater than \$4,191 causing the taxable income to be \$0 (line 26). Thus, T_W is zero. The latter is verified in line 29, where we see taxes are not paid at a 0.12 marginal tax rate. Given a contribution tax rate of 0.30, the first tax rate differential is $\mathbf{\Delta T_1} = (T_C - T_W) = (0.30 - 0) = 0.30$. To the nearest dollar, we saw in Table 1 that \$46,397 was DP1. Using (12) with \$46,396.83 for DP1, we have: marginal gain = ΔT_1DP1 = (0.30 − 0)\$46,396.83 = \$13,919.049 (line 58).

In Example 3, we see that to the nearest penny DP1 is \$46,396.84 (line 3) because, at this point, the marginal T_W goes from 0 to 0.12 as seen by \$0.01 (line 26) that indicates taxable income is no longer zero. At the precise point of \$46,396.84, the taxable income jumps to a marginal T_W of 0.12 so that the next \$36,656 (line 28) of taxable income would be taxed at $T_W = 0.12$ with $\Delta T_2 = (T_C - T_W) = (0.30 - 0.12) = 0.18$. Thus, the extra \$0.01 in taxable income produces 0.18(\$0.01) = \$0.02 in extra gain and we have a gain of \$13,919.051 (line 58). The value of 9.44% (line 60) tells us what percentage of \$46,396.84 (line 3) is of \$491,710 (line 59) where the latter was given in Table 5 as the maximum future value of the before-tax annual annuity withdrawal from a deductible IRA. The percentage put in a Roth IRA would be 100% to 9.44% = 90.56%. Multiplying 90.56% times the maximum Roth IRA of \$344,201 (line 61) tells us that \$311,723 (line 62) will be withdrawn from a Roth IRA.

6.2. Subsequent discovery points

In Table 7, we do not repeat the instructions given in Table 6 in the first column but only provide the line numbers of prior instructions given in Table 6. In line 33 of Table 7, we find that the taxable income for second marginal tax rate is \$0.00 for Example 1 and \$0.01 for Example 2. This indicates that the DIW of \$69,111.29 (line 3) in Example 2 is DP2 to the nearest penny. This also tells us that the marginal T_W of 0.18 (line 31) kicks in during the

Table 7 Withdrawal algorithm to determine T_W, optimal RVD, and DP2, DP3 and DP4

Line no.	Example 1	Example 2	Example 3	Example 4	Example 5	Example 6	Example 7
Line 1	$50,441	$50,441	$50,441	$50,441	$50,441	$50,441	$50,441
Line 2	$25,221	$25,221	$25,221	$25,221	$25,221	$25,221	$25,221
Line 3	$69,111.28	$69,111.29	$98,220	$156,512.00	$156,512.16	$307,879.00	$307,879.16
Line 4	$0	$0	$0	$0	$0	$0	$0
Line 5	$94,332	$94,332	$123,441	$181,733	$181,733	$333,100	$333,100
Line 6	$0	$0	$0	$0	$0	$0	$0
Line 7	$94,332	$94,332	$123,441	$181,733	$181,733	$333,100	$333,100
Line 8	$63,235	$63,235	$63,235	$63,235	$63,235	$63,235	$63,235
Line 9	$31,097	$31,097	$60,206	$118,498	$118,498	$269,865	$269,865
Line 10	$23,713	$23,713	$23,713	$23,713	$23,713	$23,713	$23,713
Line 11	$7,384	$7,384	$36,493	$94,785	$94,785	$246,152	$246,152
Line 12	$23,713	$23,713	$23,713	$23,713	$23,713	$23,713	$23,713
Line 13	$11,857	$11,857	$11,857	$11,857	$11,857	$11,857	$11,857
Line 14	$11,857	$11,857	$11,857	$11,857	$11,857	$11,857	$11,857
Line 15	$6,276	$6,276	$31,019	$80,567	$80,567	$209,229	$209,229
Line 16	$18,133	$18,133	$42,875	$92,423	$92,423	$221,085	$221,085
Line 17	$42,875	$42,875	$42,875	$42,875	$42,875	$42,875	$42,875
Line 18	$18,133	$18,133	$42,875	$42,875	$42,875	$42,875	$42,875
Line 19	$0	$0	$0	$0	$0	$0	$0
Line 20	$18,133	$18,133	$42,875	$42,875	$42,875	$42,875	$42,875
Line 21	35.95%	35.95%	85.00%	85.000%	85.00%	85.000%	85.00%
Line 22	$2,176	$2,176	$6,676	$7,084	$7,084	$9,998	$9,998
Line 23	$32,308	$32,308	$7,566	$7,566	$7,566	$7,566	$7,566
Line 24	$87,244	$87,244	$141,095	$199,387	$199,387	$350,754	$350,754
Line 25	$50,588	$50,588	$50,588	$50,588	$50,588	$50,588	$50,588
Line 26	$36,656	$36,656	$90,507	$148,799	$148,799	$300,166	$300,166
Line 27	0.12	0.12	0.12	0.12	0.12	0.12	0.12
Line 28	$36,656	$36,656	$36,656	$36,656	$36,656	$36,656	$36,656
Line 29	$36,656	$36,656	$36,656	$36,656	$36,656	$36,656	$36,656
Line 30	$4,399	$4,399	$4,399	$4,399	$4,399	$4,399	$4,399
Line 31	0.18	0.18	0.18	0.18	0.18	0.18	0.18
Line 32	$112,143	$112,143	$112,143	$112,143	$112,143	$112,143	$112,143
Line 33	$0.00	$0.01	$53,851	$112,143	$112,143	$112,143	$112,143
Line 34	$0	$0	$9,693	$20,186	$20,186	$20,186	$20,186
Line 35	0.30	0.30	0.30	0.30	0.30	0.30	0.30
Line 36	$151,367	$151,367	$151,367	$151,367	$151,367	$151,367	$151,367
Line 37	$0	$0	$0	$0.00	$0.01	$151,367	$151,367
Line 38	$0	$0	$0	$0	$0	$45,410	$45,410
Line 39	0.336	0.336	0.336	0.336	0.336	0.336	0.336
Line 40	$157,197	$157,197	$157,197	$157,197	$157,197	$157,197	$157,197
Line 41	$0	$0	$0	$0	$0	$0.00	$0.01
Line 42	$0	$0	$0	$0	$0	$0.00	$0.00
Line 43	0.396	0.396	0.396	0.396	0.396	0.396	0.396
Line 44	$359,449	$359,449	$359,449	$359,449	$359,449	$359,449	$359,449
Line 45	$0	$0	$0	$0	$0	$0	$0
Line 46	$0	$0	$0	$0	$0	$0	$0
Line 47	0.42	0.42	0.42	0.42	0.42	0.42	0.42
Line 48	$66,396	$66,396	$66,396	$66,396	$66,396	$66,396	$66,396
Line 49	$0	$0	$0	$0	$0	$0	$0
Line 50	$0	$0	$0	$0	$0	$0	$0
Line 51	0.471	0.471	0.471	0.471	0.471	0.471	0.471
Line 52	$9,999,999	$9,999,999	$9,999,999	$9,999,999	$9,999,999	$9,999,999	$9,999,999
Line 53	$0	$0	$0	$0	$0	$0	$0

Table 7 (Continued)

Line no.	Example 1	Example 2	Example 3	Example 4	Example 5	Example 6	Example 7
Line 54	$0	$0	$0	$0	$0	$0	$0
Line 55	$4,399	$4,399	$14,092	$24,584	$24,584	$69,995	$69,995
Line 56	0.1200	0.1200	0.1557	0.1652	0.1652	0.2332	0.2332
Line 57	0.30	0.30	0.30	0.30	0.30	0.30	0.30
Line 58	$18,008	$18,008	$21,501	$28,495.62	$28,495.62	$28,495.62	$28,495.62
Line 59	$491,710	$491,710	$491,710	$491,710	$491,710	$491,710	$491,710
Line 60	14.055%	14.055%	19.98%	31.83%	31.83%	62.61%	62.61%
Line 61	$344,201	$344,201	$344,201	$344,201	$344,201	$344,201	$344,201
Line 62	$295,823	$295,823	$275,446	$234,641	$234,641	$128,683	$128,683

69,111th withdrawal dollar causing $\Delta T_3 = (T_C - T_W) = 0.30 - 0.18 = 0.12$ to also kick in. Using (12), we have: marginal gain = ΔT_1DP1 + ΔT_2(DP2-DP1) = (0.30 − 0)$46,397 + (0.30 − 0.12)($69,111 − $46,397) = $18,008 (line 58).

In Example 3 of Table 7, we input DIW = $98,220 (line 3). Doing this gives 85% (line 21). If we were to input $98,219 (line 3), we would get 84.999% (line 21), while withdrawals greater than $98,220 would still get 85% as this is the maximum percentage at which SSB can be taxed. For a withdrawal of $98,220, taxes paid on SSB is 0.85($50,441) = $42,875 (line 20). The first $36,656 of taxable income (line 28) is taxed at 0.12 and creates 0.12($36,656) = $4,399 in taxes (line 30). The next $53,851 (line 33) is taxed at 0.18 (line 31) and creates 0.18($53,851) = $9,693 in taxes (line 34). Because the taxable income range for 0.18 is $112,143 (line 32), we see that there will be no taxable income beyond the marginal tax rate of 0.18 for a withdrawal of $98,220. The total taxes paid are $4,399 + $9,693 = $14,092 (line 55). Dividing the total taxes paid by the taxable income of $90,507 (line 26) yields an average T_W of $14,092/$90,507 = 0.1557 (line 56). Using (12) and noting that DP2<$98,220<DP3, we substitute $98,220 for DP3 (as described in Section 4.4 in our five-step procedure), to get marginal gain = ΔT_1DP1 + ΔT_2(DP2-DP1) + ΔT_3(withdrawal-DP2) = (0.30 − 0)$46,397 + (0.30 − 0.12)($69,111 − $46,397) + (0.30 − 0.18)($98,220 − $69,111) = $21,501 (line 58). The percentage allocated to a deductible IRA is 19.98% (line 60). This means our couple puts 81.02% in a Roth generating an annuity withdrawal from a Roth IRA of $275,446 (line 62).

Suppose SSB is zero. For this situation, the taxable SSB falls from $42,875 to $0 and the taxable income falls to $90,507 − $42,875 = $47,632. Our new DP1 is $98,220 − $47,632 = $50,588 (that is the slack) and gets a marginal T_W of 0 yielding marginal gain$_1$ = (0.30 − 0)($50,588) = $15,176.40. The remaining withdrawal is $98,220 − $50,588 = $42,875. The first $36,656 is taxed at 0.12. Because a zero SSB has eliminated the peculiarities of the way withdrawals are taxed, our new DP2 is simply DP1 plus the marginal tax bracket range for 0.12. The latter tax bracket range is $36,656 as can be seen from row four of Table 2. Thus, DP2 = $50,588 + $36,656 = $87,244. Our second incremental gain is marginal gain$_2$ = (0.30 − 0.12)$36,656 = $6,598.08. For the remaining $42,875 − $36,656 = $6,219 that is taxed at 0.18, we get marginal gain$_3$ = (0.30 − 0.18)$6,219 = $746.28. The sum of these marginal gains is $22,521.[9] A gain of $22,521 is greater than $21,501 in Example 3 showing that the effect of SSB imposes a cost of $22,521 − $21,501 = $1,020.

Suppose SSB and tax deduction are both zero. Without SSB and tax deductions, and keeping match/OI at zero, all DPs would be easily identifiable because DP1 = 0 and each subsequent DP would be the amount of a taxable income bracket range. For example, DP2 = \$36,656, DP3 = \$112,143, and so forth. To illustrate, if we input \$0 for SSB (line 1) and \$0 for the tax deduction (line 25) while keeping our current assumption of match/OI = \$0, we get: marginal gain = ΔT_1(DP1) + ΔT_2(DP2-DP1) + ΔT_3(withdrawal-DP2) = (0.30 − 0.0)\$0 + (0.30 − 0.12)\$36,656 + (0.30 − 0.18)(\$98,220 − \$36,656) = \$13,986. Without the positive effect of a tax deduction, we see that the gain is significantly lowered.

As seen in Example 3, the SSB's taxable income is \$42,875. On a marginal basis where SSB is taxed first and DIW is large enough to cover the tax deduction, the impact is 0.12(\$42,875) = \$5,145 in taxes given that SSB would not be taxed if the withdrawal was zero because all IRA funds were put in a Roth IRA. Thus, on a marginal comparative basis, one could argue that the marginal gain is not \$21,501 but \$21,501 − \$5,145 = \$16,356. By not being taxable, the Roth IRA does not impose this opportunity cost of \$5,145. However, the Roth IRA also does not have the gross advantage of \$21,501 given by the deductible IRA when using our marginal gain formula.

6.3. Identifying the maximum marginal gain range

In Example 4, we input DIW = \$156,512 (line 3) and taxable income = \$0 (line 37). DP3 is achieved during the withdrawal of the \$156,512th dollar as seen in Example 5 when DIW is increased by \$0.16 to \$156,512.16 (line 3) because at this point the taxable income is no longer \$0 but \$0.01 (line 37). Using (12), we have: marginal gain = ΔT_1(DP1) + ΔT_2(DP2-DP1) + ΔT_3(DP3-DP2) = (0.30–0.0)\$46,397 + (0.30 − 0.12)(\$69,111 − \$46,397) + (0.30 − 0.18)(\$156,512 − \$69,111) = \$28,495.62 (line 58). This is the same as the marginal gain for a \$156,512 because \$156,512 is a discovery point where the next ΔT is zero as we have $\Delta T_4 = (T_C - T_W) = (0.30 - 0.30) = 0$. We can also see that the applicable taxable income for 0.30 marginal tax rate (line 35) is the same as T_C. To illustrate to the nearest penny, we would add the following component to our computation: ΔT_4(withdrawal-DP3) = (0.30 − 0.30)(\$156,512.16 − \$156,512) = 0(\$0.16) = \$0. Thus, the marginal gain remains at \$28,495.62 until we reach DP4 at which point the gain will fall because $\Delta T_5 = 0.336 - 0.300 = -0.036$. The lower bound optimal RVD allocation involves a DIW of \$156,512. The optimal DIW as a percentage of the maximum DIW (called ODI%) is 31.83% (line 60). It is our lower bound ODI% because our maximum marginal gain first occurs at 31.83%. Noting that our maximum marginal gain is a 20-year annuity, we can discount its total value to get this value in today's dollar. In doing this, we get a lifetime wealth gain amounting to \$179,637 or about \$180,000. Adjusting for our inflation rate of 1.5%, we get \$302,487 in future value dollars at the time of retirement.

What happens if an investor withdraws enough to jump to the tax rate of 0.336? This is illustrated in Examples 6 and 7 of Table 7 where there begins a decrease in the marginal gain as the withdrawal increases past our next discovery point of DP4 = \$307,879 to the nearest dollar. The applicable taxable income for the marginal tax rate of 0.336 (line 39) is \$0 (line 41) in Example 6 but becomes positive at \$0.01 (line 41) in Example 7 when DIW = \$307,879.16 indicating we have reached DP4 to the nearest penny. If we were to increase

DIW to $307,880, we would find a fall in the gain on line 58 from $28,495.62 to $28,495.59. The three cents fall is explain by the additional component in our gain formula of ΔT_5(withdrawal-DP4) = (0.30 − 0.336)($307,880 − $307,879.16) = −$0.03. The fall continues with greater withdrawals since our couple's contribution to a deductible IRA gets only $T_C = 0.30$ and the marginal T_W beginning with DP4 is 0.336 and covers the DP5-DP4 range. Finally, as seen in Table 7, the ODI% is 62.61% (line 60) and this is our upper bound ODI%.

6.4. Our couple's optimal RVD outcomes

As we just saw, marginal gains are maximized between DP3 and DP4. The incremental increase in gain is zero from the 156,512th withdrawal dollar to the 307,879th withdrawal dollar. This is because $T_C = T_W = 0.30$ for this range. During the withdrawal of 307,879th dollar, our couple jumps to a higher tax bracket and begin to be penalized since the marginal T_W is greater than T_C. The midpoint of the optimal withdrawal range is ($156,512 + $307,879)/2 = $232,196. On a marginal gain basis, there is a broad ODI% range from 31.83% to 62.61%.

Unless our couple has factors other than the tax rate differential to consider that would favor either a deductible IRA or a Roth IRA, an argument can be made that the safest RVD allocation would be ODI% = (31.83% + 62.61%)/2 = 47.22%. By choosing this midpoint percentage, we allow leeway on both sides if any inputs are less than precise. Thus, a recommendation would be 47.22% of the annual maximum deductible IRA contribution and 52.78% of the annual maximum Roth IRA contribution. Given the maximum mean annual contribution of $28,974 for a deductible IRA as computed in Table 5, we have: 0.4722214($28,974) = $13,682 allocated to a deductible IRA. Similarly, given the maximum Roth IRA contribution of $20,282, we have 0.5277786($20,282) = $10,704 allocated to a Roth IRA. The corresponding optimal annual withdrawals generated at retirement would be 0.4722214($491,710) = $232,196 from a deductible IRA and 0.5277786($344,201) = $181,662 from a Roth IRA.

7. Plotting the optimal RVD range

In this section, we plot the annual gains and annual withdrawals. The extent to which the annual match/OI influence the gain depends on assumptions about when the match/OI is taxed and how much of it is taxed.[10] Fig. 1 assumes DIW is withdrawn first and the match/OI is withdrawn last. Fig. 2 assumes the match is withdrawn first while OI is withdrawn last. Both figures show average and marginal gains.

7.1. An optimal RVD allocation without the match/OI

For Fig. 1, we choose the following 18 annual withdrawals: $0 (all Roth), $46,397 (DP1), $69,111 (DP2), $112,812, $156,512 (DP3), $194,354, $232,196 (midpoint of optimal range), $270,037, $307,879 (DP4), $347,178, $386,478, $425,777, $465,076 (DP5), $491,710 (maximum DIW), $529,037, $566,365 (maximum DIW plus match), $603,692, and $641,019 (maximum DIW plus match/OI). Fig. 1 has a flat optimal range covering DIWs from

Figure 1

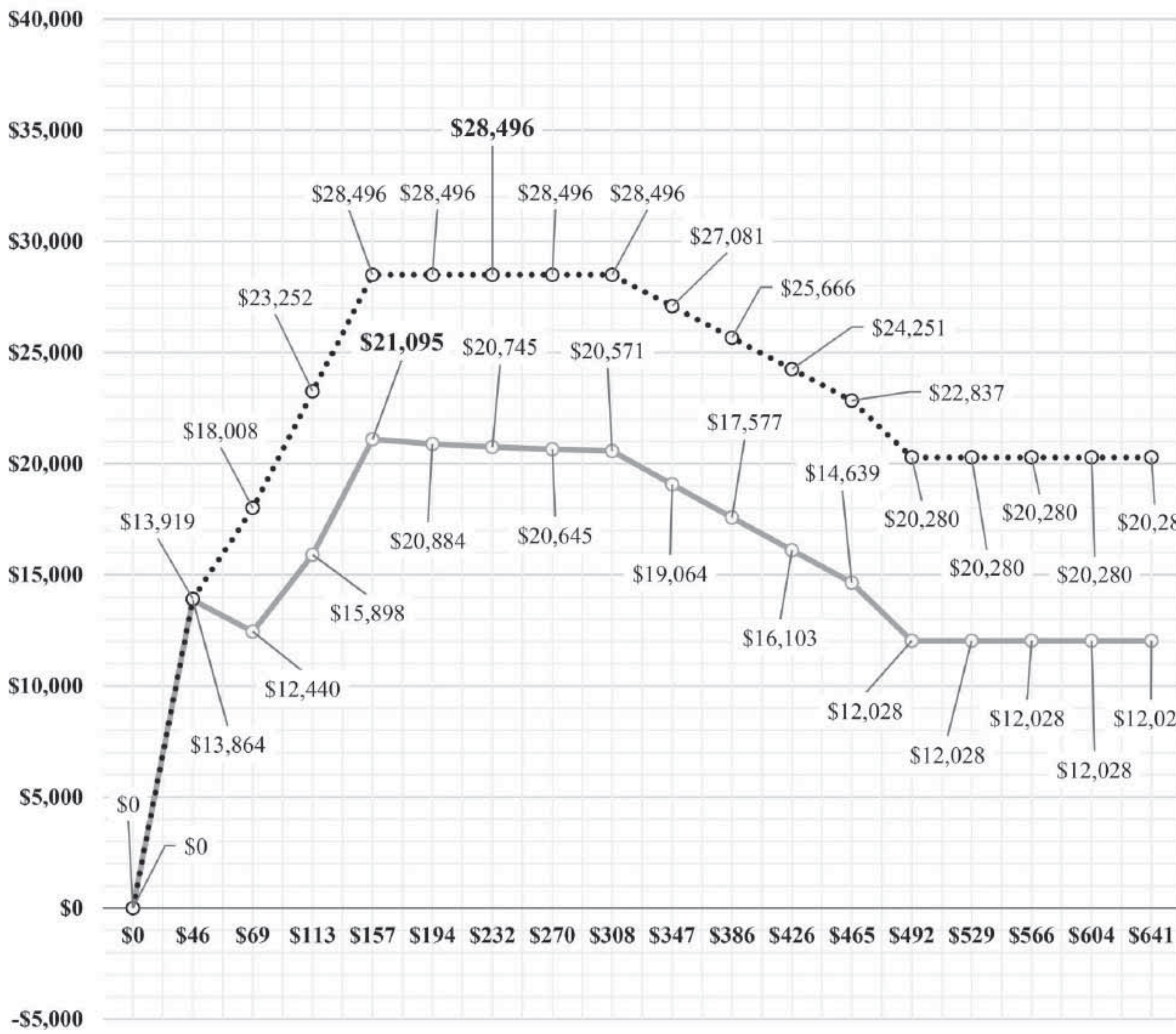

Fig. 1. Marginal Gain (dotted line) and Average Gain as a Function of Withdrawals (in units of $1,000)–Without Match and Other Investments (OI).

$156,512 (DP3) to $307,879 (DP4) that maximize IRA retirement wealth by providing a marginal gain of $28,496. The midpoint of this optimal range is $232,196. As seen in Fig. 1, the average gain for this range is from $21,095 to $20,571 with the average gain falling throughout the range. The maximum average gain of $21,095 occurs at DP3.

After the maximum DIW of $491,710 is reached, Fig. 1 shows further gain does not occur because the assumption is that the match/OI is withdrawn last with no impact on the gain. Fig. 1 reflects this latter assumption by having a flat range at the end where the marginal gain is fixed at $20,280 once the maximum DIW of $491,710 is reached.

In comparing the average and marginal gains, we see they are equivalent up to DP1 = $46,397. This is because the taxable income is zero before reaching DP1 so the average and marginal T_W are both zero. We can also notice the average gain reaches a relative maximum around DP1 with a steady decline after this point until DP2 = $69,111 is reached. We attribute this relative maximum to the interplay of an increasing tax on SSB in combination with the progressive nature of the U.S. tax system. Using the average T_W renders an average gain equal to or proportional to the marginal gain for all points except the short withdrawal span where the relative maximum occurs.

7.2. The optimal RVD allocation with only the match considered

As seen in Table 5, the match/OI of \$8,798 added to the maximum deductible IRA contribution of \$28,974 increases the total annual IRA contribution to \$37,772. As reported there, the maximum before-tax annuity withdrawal increases from \$491,710 without the match/OI to \$641,019 with the match/OI, which is an increase of \$149,309. Because the match and OI are equal, the match is half of the \$149,309 or \$75,654.50. In Fig. 2, the match of \$75,654.50 is withdrawn first and absorbs lower T_W values.

In Fig. 2, the withdrawals are somewhat different from Fig. 1 because the first withdrawal of \$74,654.50 is the match where DP1 and DP2 are not possible as they involve amounts below the match. Other than DP1 and DP2, we keep all other key withdrawals the same for both figures. By the time the match is withdrawn, one is already at the 0.18 marginal tax rate and it will stay this way until there is a jump to the 0.30 tax bracket at which point the marginal gain cannot increase. For the match withdrawal of \$74,654.50, the taxable income is \$46,911. Thus, the match covers the \$36,656 tax range for 0.12 and causes us to be in the 0.18 tax rate range for \$46,911 − \$36,656 = \$10,255 withdrawal dollars. Thus, by the time the match is withdrawn, we have $\Delta T_3 = 0.12$. Using (12) and the beginning of the optimal range at DP3, we have marginal gain = ΔT_3(DP3-withdrawal) = (0.30 − 0.18)(\$156,512 − \$74,655) = (0.12)(\$81,857) = \$9,823.

The lower marginal gains in Fig. 2 compared with Fig. 1 is because the match is now assumed to absorb the lower T_W values so that the first \$74,655 withdrawn produces no gain because it does not get a tax savings during the contribution years. However, the match is valuable for two reasons. *First*, it was given free by the employer and no taxes were paid by the employee. *Second*, by assuming the match absorbs the lower tax rate brackets, the match is more valuable on an after-tax basis than other subsequent taxable withdrawals because it causes less to be paid in taxes. Thus, the value of the match is enhanced by having a greater after-tax value. If this enhancement is a marginal benefit, then it is possible that \$9,823 underestimates the gain and we need to analyze a situation where neither the match nor DIW absorbs the lowest tax rates. This particular situation requires investigating the average T_W where absorption for any withdrawal does not occur first but all withdrawals are treated as equal. In our investigation, we find total taxes paid on the match are \$6,246 and the taxable income is \$46,911 by the time the first dollar of DIW kicks in. Thus, we get \$6,246/\$46,911 = 0.1331 for the average T_W. At the midpoint of our optimal gain (that is the same for both figures), T_W is 0.2701 but the match has not been taxed at any T_W above 0.1331. In conclusion, the maximum marginal gain of \$9,823 only holds if we disregard any positive marginal effect the match usurps from the marginal gain by absorbing the lower T_W values.

Unlike Fig. 1, Fig. 2 reveals that the average gain values are now greater than the marginal gain values. By considering the match, there is still a defined optimal marginal range in Fig. 2 as was found in Fig. 1. The same optimal marginal range of withdrawal dollars are found

Figure 2

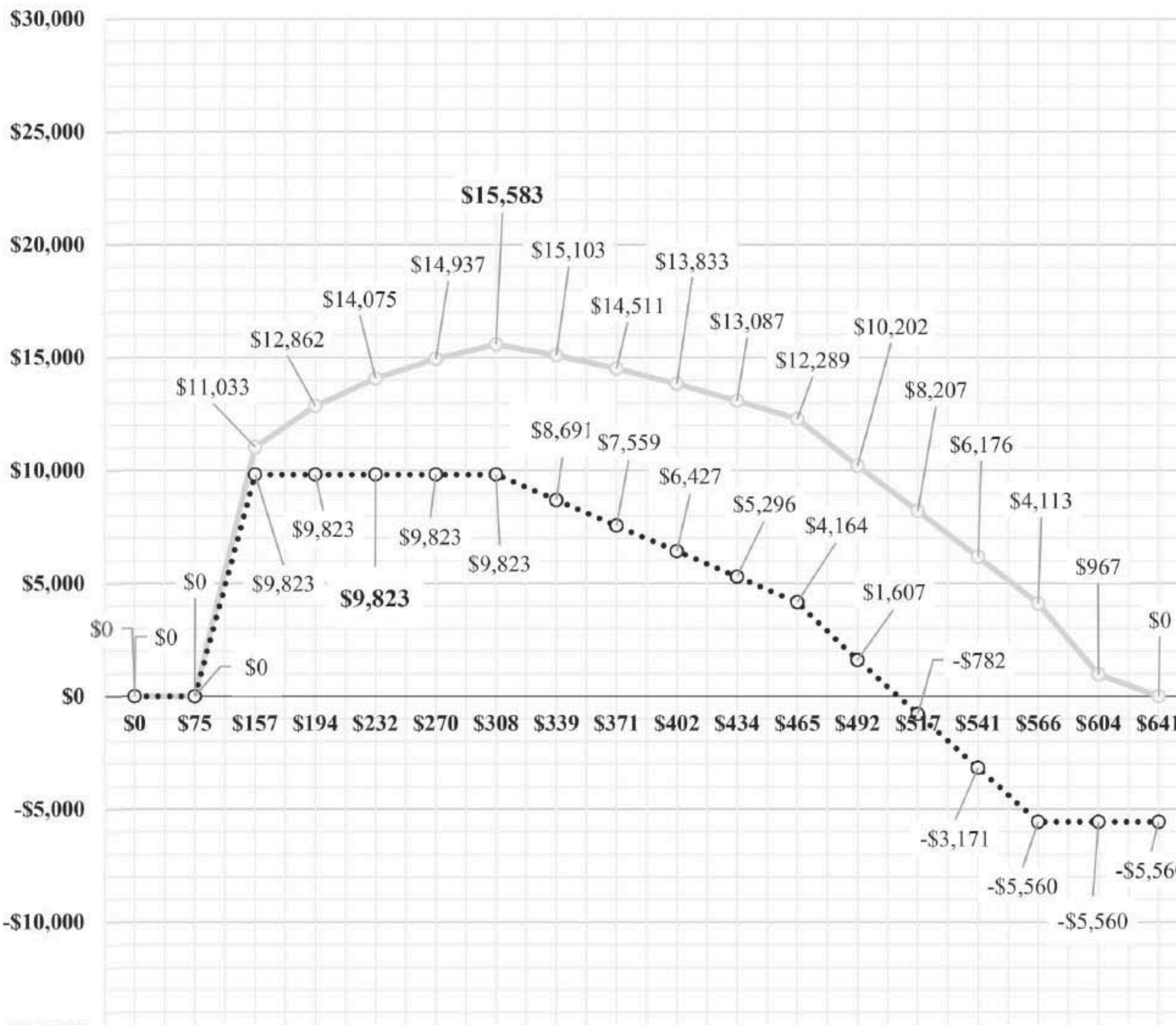

Fig. 2. Marginal Gain (dotted line) and Average Gain as a Function of Withdrawals (in units of $1,000)–With Match and Without Other Investments (OI).

except they now include the match so that in terms of the deductible IRA dollars, the range is $81,858 − $233,225. For both figures, we find the marginal gain values tapering off rapidly before the optimal range. After the range ends, values gradually taper off. Using the average T_W values, we can also find flat ranges that contain the maximum average gain. The peak of $15,583 for the average gain in Fig. 2 occurs where DIW is $307,879, which is DP4. Of importance, even when using the average T_W, we still find there can be a margin of error because of a somewhat flat range around its optimal average gain of $15,583.

In summary, by considering the match and focusing on a marginal analysis, the optimal midpoint withdrawal of $232,196 in Fig. 1 did not change in Fig. 2, while ODI% fell from $232,196/$491,710 = 0.4722 or 47.22% to ($232,196 − $74,655)/$491,710 = 0.3204 or 32.04%. Thus, the match causes a fall that is about one-third of Fig. 1's ODI%. When the match is withdrawn before the deductible IRA, we find a decrease in the marginal gain of \$28,496 − \$9,823 = \$18,673. This represents a large fall of about two-thirds from that found in Fig. 1. Although not shown, we repeated the figures when assuming both the match and OI are withdrawn first and taxed at the ordinary rate. As expected, we still found the same flatness with the gains and ODI% both lowered.

8. Scenario analysis results when AGIs, returns, withdrawal years, and match change

In the prior section, we found the optimal RVD allocation using the average gain was within the optimal range given by the marginal gain. Furthermore, the two gains have correlation coefficients of 0.95 and 0.865 for Figs. 1 and 2, respectively. Thus, the average gain parallels the marginal gain. Given this knowledge and the ease in computing the average gain, we use it for our scenario analysis that requires a large number of computations to make general conclusions about how key variables influence the optimal RVD allocation. Whereas many scenario and outcome variables could be chosen, for brevity's sake, we limit the number of variables. Regardless, these illustrations demonstrate how our RVD procedure can help financial advisors provide RVD guidance to their clientele.

For our scenario analysis, we will change three variables that we call scenario variables. They are adjusted gross income (AGI), nominal stock returns (returns), and number of withdrawal years during retirement (years). AGI changes in increments of \$30,000, returns change in increments of 2% (nonstock premium remains at 3.5%), and years change in increments of five years. We investigate 10 other variables for outcome changes when our scenario variables change. We call these variables by the name of outcome variables and their abbreviations and definitions are:

MDI_{BT} = maximum annual before-tax DIW
MRI = maximum annual Roth IRA withdrawal
Match = employer's salary match
T_C = average contribution tax rate
T_W = average withdrawal tax rate
ODI_{BT} = optimal annual before-tax DIW
Gain = $(T_C - T_W)ODI_{BT}$ (Gain is the maximum gain)

ODI% is ODI_{BT}/MDI_{BT} and put in percentage form

ODI_{AT} = optimal annual after-tax DIW = $(1 - T_W)ODI_{BT}$
ORI = optimal annual Roth IRA withdrawal

Each time a scenario variable changes, we identify the maximum gain from all withdrawals. We report the values for all other outcome variables that occur at the maximum point and report them in Table 8. In Panels A, B, and C of Table 8, we assume no match/OI. In Panels D, E, and F, we allow a 3% salary match and assume no OI. The only changes in variables from those used previously are the following. AGI is \$140,000 (unless used as a scenario

Table 8 Scenario analysis

Panel A. Without match/OI: AGIs range from \$50,000–\$260,000 with incremental changes of \$30,000

AGI	MDI_{BT}	MRI	Match	T_C	T_W	ODI_{BT}	Gain	ODI%	ODI_{AT}	ORI
\$50	\$143	\$117	\$0	0.1800	0.0000	\$48	\$8.59	33.33%	\$48	\$78
\$80	\$240	\$188	\$0	0.2174	0.0000	\$40	\$8.70	16.67%	\$40	\$157
\$110	\$369	\$258	\$0	0.3000	0.1875	\$184	\$20.76	50.00%	\$150	\$129
\$140	\$470	\$329	\$0	0.3000	0.2273	\$274	\$19.92	58.33%	\$212	\$137
\$170	\$601	\$399	\$0	0.3360	0.2607	\$401	\$30.16	66.67%	\$296	\$133
\$200	\$707	\$470	\$0	0.3360	0.2643	\$412	\$29.58	58.33%	\$303	\$196
\$230	\$870	\$540	\$0	0.3792	0.2701	\$435	\$47.40	50.00%	\$317	\$270
\$260	\$1,011	\$610	\$0	0.3960	0.3221	\$758	\$56.02	75.00%	\$514	\$153
\$155	\$551	\$364	\$0	0.3056	0.1915	\$319	\$27.64	51.04%	\$235	\$157
\$73	\$303	\$173	\$0	0.0747	0.1242	\$237	\$17.05	18.60%	\$158	\$56
n.a.	1.00	1.00	n.a.	0.96	0.91	0.95	0.96	0.77	0.96	0.66

Panel B. Without match/OI: Nominal rates of return from 4%–18% with incremental changes of 2%

Returns	MDI_{BT}	MRI	Match	T_C	T_W	ODI_{BT}	Gain	ODI%	ODI_{AT}	ORI
4%	\$142	\$100	\$0	0.3000	0.1647	\$142	\$19.24	100.0%	\$119	\$0
6%	\$244	\$171	\$0	0.3000	0.2184	\$244	\$19.90	100.0%	\$191	\$0
8%	\$423	\$296	\$0	0.3000	0.2294	\$282	\$19.92	66.67%	\$217	\$99
10%	\$744	\$519	\$0	0.3023	0.2197	\$248	\$20.46	33.33%	\$193	\$346
12%	\$1,320	\$915	\$0	0.3067	0.2321	\$293	\$21.89	22.22%	\$225	\$712
14%	\$2,351	\$1,620	\$0	0.3108	0.2274	\$274	\$22.89	11.67%	\$212	\$1,431
16%	\$4,176	\$2,869	\$0	0.3130	0.2285	\$278	\$23.53	6.67%	\$215	\$2,678
18%	\$7,396	\$5,073	\$0	0.3142	0.2365	\$308	\$23.95	4.17%	\$235	\$4,861
11%	\$2,100	\$1,445	\$0	0.3059	0.2196	\$259	\$21.47	43.09%	\$201	\$1,266
4.9%	\$2,534	\$1,737	\$0	0.0061	0.0229	\$52	\$1.83	40.34%	\$36	\$1,718
n.a.	0.88	0.88	n.a.	0.96	0.71	0.77	0.98	−0.95	0.76	0.88

Panel C. Without match/OI: Withdrawal years from 5 to 40 years with incremental changes of 5 years

Years	MDI_{BT}	MRI	Match	T_C	T_W	ODI_{BT}	Gain	ODI%	ODI_{AT}	ORI
5	\$1,228	\$850	\$0	0.3077	0.2134	\$205	\$19.30	16.67%	\$161	\$709
10	\$711	\$496	\$0	0.3018	0.2222	\$237	\$18.85	33.33%	\$184	\$331
15	\$546	\$382	\$0	0.3000	0.2299	\$273	\$19.16	50.00%	\$210	\$191
20	\$470	\$329	\$0	0.3000	0.2273	\$274	\$19.92	58.33%	\$212	\$137
25	\$427	\$299	\$0	0.3000	0.2272	\$284	\$20.71	66.67%	\$220	\$100
30	\$401	\$280	\$0	0.3000	0.2283	\$300	\$21.56	75.00%	\$232	\$70
35	\$384	\$269	\$0	0.3000	0.2298	\$320	\$22.44	83.33%	\$246	\$45
40	\$373	\$261	\$0	0.3000	0.2316	\$342	\$23.38	91.67%	\$263	\$22
22.5	\$567	\$396	\$0	0.3012	0.2262	\$279	\$20.67	59.38%	\$216	\$200
12.2	\$289	\$199	\$0	0.0027	0.0059	\$44	\$1.66	25.37%	\$33	\$228
n.a.	−0.82	−0.82	n.a.	−0.68	0.80	0.98	0.96	0.99	0.98	−0.85

Panel D. With match and without OI: AGIs range from \$50,000–\$260,000 with incremental changes of \$30,000

AGI	MDI_{BT}	MRI	Match	T_C	T_W	ODI_{BT}	Gain	ODI%	ODI_{AT}	ORI
\$50	\$143	\$117	\$35	0.1800	0.1200	\$36	\$2.15	25.00%	\$31	\$88
\$80	\$232	\$188	\$56	0.1894	0.1647	\$97	\$2.39	41.67%	\$81	\$110
\$110	\$369	\$258	\$77	0.3000	0.2387	\$246	\$15.08	66.67%	\$187	\$86

Table 8 (Continued)

AGI	MDI_{BT}	MRI	Match	T_C	T_W	ODI_{BT}	Gain	ODI%	ODI_{AT}	ORI
$140	$470	$329	$99	0.3000	0.2323	$196	$13.25	41.67%	$150	$192
$170	$601	$399	$120	0.3360	0.2642	$301	$21.59	50.00%	$221	$200
$200	$707	$470	$141	0.3360	0.2679	$295	$20.05	41.67%	$216	$274
$230	$863	$540	$162	0.3747	0.3120	$504	$31.57	58.33%	$347	$225
$260	$1,011	$610	$183	0.3960	0.3325	$674	$42.79	66.67%	$450	$203
$155	$549	$364	$109	0.3015	0.2415	$293	$18.61	48.96%	$210	$172
$73	$303	$173	$52	0.0793	0.0709	$209	$13.83	14.39%	$136	$69
n.a.	1.00	1.00	1.00	0.95	0.96	0.94	0.96	0.62	0.95	0.82

Panel E. With match and without OI: Nominal rates of return from 4%–0.18 with incremental changes of 2%

Returns	MDI_{BT}	MRI	Match	T_C	T_W	ODI_{BT}	Gain	ODI%	ODI_{AT}	ORI
4%	$142	$100	$30	0.3000	0.1848	$142	$16.39	100.0%	$116	$0
6%	$244	$171	$51	0.3000	0.2324	$244	$16.47	100.0%	$187	$0
8%	$423	$296	$89	0.3000	0.2249	$176	$13.24	41.67%	$137	$173
10%	$744	$519	$156	0.3023	0.2599	$248	$10.50	33.33%	$183	$346
12%	$1,318	$915	$275	0.3057	0.2785	$221	$5.98	16.67%	$159	$763
14%	$2,360	$1,620	$486	0.3136	0.2958	$94	$1.67	3.98%	$66	$1,555
16%	$4,190	$2,869	$861	0.3153	0.3297	$0	$0.00	0.00%	$0	$2,869
18%	$7,412	$5,073	$1,522	0.3156	0.3903	$0	$0.00	0.00%	$0	$5,073
11%	$2,104	$1,445	$434	0.3066	0.2745	$141	$8.03	36.96%	$106	$1,347
4.9%	$2,541	$1,737	$521	0.0071	0.0648	$101	$7.05	41.77%	$76	$1,796
n.a.	0.88	0.88	1.00	0.94	0.97	−0.72	−0.98	−0.93	−0.76	0.88

Panel F. With match and without OI: Withdrawal years from 5 to 40 years with incremental changes of 5 years

Years	MDI_{BT}	MRI	Match	T_C	T_W	ODI_{BT}	Gain	ODI%	ODI_{AT}	ORI
5	$1,228	$850	$255	0.3077	0.2834	$205	$4.97	16.67%	$147	$709
10	$714	$496	$149	0.3047	0.2492	$178	$9.91	25.00%	$134	$372
15	$548	$382	$115	0.3018	0.2368	$183	$11.87	33.33%	$139	$255
20	$470	$329	$99	0.3000	0.2323	$196	$13.25	41.67%	$150	$192
25	$427	$299	$90	0.3000	0.2316	$213	$14.59	50.00%	$164	$149
30	$401	$280	$84	0.3000	0.2321	$234	$15.86	58.33%	$179	$117
35	$384	$269	$81	0.3000	0.2333	$256	$17.06	66.67%	$196	$90
40	$373	$261	$78	0.3000	0.2354	$280	$18.05	75.00%	$214	$65
22.5	$568	$396	$119	0.3018	0.2418	$218	$13.20	45.83%	$165	$244
12.2	$290	$199	$60	0.0029	0.0178	$36	$4.27	20.41%	$29	$212
n.a.	−0.82	−0.82	1.00	−0.83	−0.70	0.88	0.96	1.00	0.93	−0.87

Note: The first three panels provide scenario results when the match/OI has no impact. The last three panels give results when OI has no impact. The scenario variables are in the first columns for each panel. The outcome variables (with dollars values expressed in units of 1,000) are in the last 10 columns. The values reported for "Gain" are maximum value and so the values reported for the other outcome variables occur at that maximum point. Panels A and D vary the adjusted gross income (AGI); Panels B and E modify the nominal stock return rate (Returns); and, Panels C and F alter the number of withdrawal years (Years). The following abbreviation are used: MDI_{BT} = maximum annual before-tax DIW; MRI = maximum annual Roth IRA withdrawal; ODI_{BT} = optimal annual before-tax DIW; Gain = $(T_C - T_W)ODI_{BT}$; ODI% = ODI_{BT}/ MDI_{BT} and put in percentage form; ODI_{AT} = optimal annual after-tax DIW = $(1 - T_W)ODI_{BT}$; ORI = optimal annual Roth IRA withdrawal. The last three rows of each panel, respectively, report averages for the prior eight rows, standard deviations for the prior eight rows, and Pearson correlation coefficients with the panel's scenario variable.

variable), the salary is now the same as the AGI, and SSB is adjusted in the systematic fashion described in Section 5.1 each time AGI is changed.

When approximating optimal RVD values for our scenario analysis in Panels A, B and C, we assume withdrawals from zero to the maximum before-tax DIW. The eleven other withdrawals are chosen so that all withdrawals are equidistant from one another. To test if the maximum gain is properly identified, we allow (if necessary) another eleven smaller withdrawals around the initial maximum gain with shorter equidistances between withdrawals. This process is repeated as many times as needed until the maximum gain and optimal RVD can be suitably estimated. For Panels D, E, and F, the first withdrawal is the match and the last withdrawal is the maximum DIW plus the match withdrawal. The 10 other withdrawals are chosen so they are equidistant from one another. As described above, we repeat the process (if necessary) by using smaller equidistance withdrawals until the maximum gain can be correctly estimated. While our scenario analysis procedure uses approximations, we believe they are accurate and serve our purpose, which is to find general relations between scenario and outcome variables. Finally, the last three lines for all panels for Table 8 provide additional statistics. The first two rows include the average and standard deviation for the scenario and outcome variables based on the eight prior rows. The last row in each panel gives Pearson correlation coefficients between the panel's scenario variable and each of the 10 outcome variables.

8.1. Scenario analysis without the match/OI considered

From the information in the first three panels in Table 8, we offer the following conclusions. *First*, Panel A reveals outcome variables increase because AGI increases as all 10 correlation coefficient in the last row are positive. The two variables that have lower positive correlations are the optimal percentage of the maximum DIW (ODI%) and the optimal contribution to a Roth IRA (ORI). ODI% takes a nosedive from 33.33% to 16.67% when AGI goes from \$50,000 to \$80,000. ODI% then climbs until it reaches an AGI of \$170,000 and then it falls for two consecutive AGIs of \$200,000 and \$230,000 before rising to its highest percentage. We can attempt to explain this roller coaster pattern. Some investors will manage to barely get their tax savings in a higher tax rate bracket and barely attain their highest marginal T_C for dollars they contribute to a deductible IRA even though their AGIs are low compared with other investors who also get the same T_C. Because they have lower AGIs, they can contribute less and possibly withdraw less. With a lower T_W values because of less withdrawals, they achieve larger tax rate differentials (ΔTs) thereby obtaining larger gains even though they have lower ODI% values. This is seen in the fact the correlation between ΔT and ODI% is -0.95 for this panel. We can see in places that when a higher ODI% results, then a lower ORI occurs. Examples of the latter are the rows where ODI% are 50.00%, 66.67% and 75.00% and the ORI takes a dip even though the ORI trend is increasing as AGI increases.

Second, as seen in Panel B, we find aggressive investors, who invest high proportions in equity (95% in the portfolio mix we use), will want less of their IRA contribution to go to a deductible IRA as ODI% falls from 100% to 4.17%. This means the percentage invested in a Roth IRA goes from 0% to 95.83% as returns increase. Consider the returns from 8%

to 10%. ODI% falls in half from 66.37% (for an 8% return) to 33.33% (for a 10% return). Thus, just a 2% increase in returns from 8% to 10% means one's investment in a Roth IRA would double. Unlike Panel A where the gain increases from about $9,000 to over $56,000 as the AGI increases, we find that the gain in Panel B only increases from about $19,000 to $24,000 when returns increase. We conclude AGI (and the amount contributed to an IRA) is a more important factor than returns in getting larger maximum gains. In addition, unlike Panel A where the ORI only went from $78,000 to $153,000, Panel B shows that ORI went from $0 to about $4,861,000 (with $712,000 if the returns are 12%). Thus, returns have a much greater impact on what is put in a Roth IRA compared with the AGI.

Third, from Panel C, we illustrate how investors with smaller years should be putting more of their IRA contribution in a Roth IRA. Those who retire early and/or expect to live long should be placing relatively more of their IRA contributions in a deductible IRA. The reason is obvious as spreading out withdrawals over a longer period can create lower T_W values and larger ΔT values leading to greater maximum gains. As can be seen from comparing the T_C and T_W values in Panel C, ΔT decreases from 9.43% to 6.84% as years increase from five to forty. Finally, we see high positive correlation coefficients between years with gain (0.96) and ODI% (0.99).

8.2. Scenario analysis with match but without OI

From the information in the last three panels of Table 8, we offer the following general conclusions *First*, Panel D is like Panel A, where we find that when AGI increases then the gain increases. Like Panel A, ODI% is not consistently related to AGI. For example, amid the upward climb for ODI% as AGI increases, ODI% is 41.67% for AGIs of $80,000, $140,000 and $200,000 and 66.67% for AGIs of $110,000 and $240,000. In looking at the third to last row, we find that the average gain for all scenarios is around $18,600 and the average ODI% is near 50%.[11] Finally, an analysis comparing Panels A and D reveals that the gain and ODI% both decline with a match with the ODI% decline small.

Second, like Panel B, we find that greater increases in returns require greater IRA contributions to a Roth IRA. There are some differences in the correlation coefficients when comparing Panels B and E. The positive coefficients for ODI_{BT} and ODI_{AT} of 0.77 and 0.76, respectively, have been reverse as they are now negative at −0.71 and −0.76. Thus, both the before-tax and after-tax optimal DIWs are no longer positively related to the increases in returns but are negatively related to them. This is consistent with the notion that the match absorbs lower T_W values causing an investor to place less in a deductible IRA and more in a Roth IRA. Perhaps, most noteworthy, we find a complete reversal in the correlation coefficients between returns and the gain (0.98 to −0.98) when comparing Panels B and E. This shows the devastating effect of the match on the gain. Finally, an analysis comparing Panels B and E indicates that the gain and ODI% both fall when the match is introduced and the fall is greater than when comparing Panels A and D.

Third, like Panel C, we show in Panel F that investors with short retirement periods should be putting more in a Roth IRA. In comparing Panels C and F, we find the same outcomes found in the previous panel comparisons. For example, with a match present, outcome variables tend to change in the same direction as found previously. A noticeable exception is for T_W as its correlation coefficient changes sign going from 0.80 in Panel C to −0.70 in Panel F. Finally, an analysis comparing Panel C and F indicates once again that the gain and ODI% both fall with a match.

In conclusion, Table 8 illustrate how changes in key variables influence the RVD outcomes. Financial advisors can gather an in depth understanding of just how much a change in a variable can influence a retiree's optimal deductible IRA allocation. Table 8 serves to remind advisors what to expect if any estimation is inaccurate. For example, suppose a future retiree plans to be aggressive by investing in a portfolio heavy in stocks thereby contributing more to a Roth IRA. If something happens and stocks underperform by as little as 2%, then serious consequences can result in terms of one's IRA allocation choice. Similarly, if the AGI unexpectedly changes over time, there are consequences in terms of how the IRA allocation should change as we saw that greater AGIs indicate a general increase in a deductible IRA contribution is warranted. Our scenario analysis also shows that, despite an upward trend in contributions to a deductible IRA as AGI increases, there can be deviations and so an advisor needs to monitor the IRA allocation on a regular basis. In brief, financial advisors should be warned that updating the IRA allocation should be ongoing.

9. Conclusions and disclaimer

This article is motivated by the desire to give financial advisors a concrete tool to help clients fulfill their retirement goal of properly choosing between the two main IRA types: a traditional deductible IRA and a Roth IRA. A survey of the literature suggests this tool is missing and financial advisors state such a tool is needed. In response, we set out to create a Roth IRA versus deductible IRA (RVD) procedure to fill in this missing gap in the personal finance planning area. For this purpose, we developed new formulas and used them within a well-defined computational procedure that includes using an algorithmic method, which is a method that dates back to Euclid in 300 BC. For our RVD procedure, we only require advisors to input 10 values from clients to produce their optimal RVD allocation.

We began our RVD procedure, by introducing the concept of discovery points that are needed to develop a marginal gain formula. Discovery points determine when an additional deductible IRA withdrawal dollar will cause a jump to a higher marginal tax rate. We next provided values for key variables for our couple. We then projected future tax brackets covering the life span of our couple and computed their contribution tax rate using our contribution algorithm. For the next task, we introduced definitions and equations that enabled us to perform standard lump sum and annuity computations based on values from key variables. We then used a variety of different DIWs and placed them within our withdrawal algorithm to illustrate the maximum gain from optimally allocating retirement funds between a deductible IRA and a Roth IRA. We then performed various illustrations showing how the optimal RVD outcomes change when key input variables change.

By using the procedure in this article, financial advisors will have a tool to help facilitate any behavioral change needed in clients who are not allocating their IRA funds optimally among IRA types. By better understanding the retirement decision-making process through knowing the correct RVD outcome, financial advisors should have more confidence in helping clients plan their retirements.

Finally, we offer a disclaimer in terms of our RVD application. Because this article and its RVD procedure is new, anyone trying to duplicate this procedure to make an estimate of a proper RVD allocation should proceed with caution. Subsequent research may cast light on any shortcomings found in our procedure that uses annuities (where RMDs are not a factor) and mean values for the contribution and withdrawal periods. Future research can improve on our procedure as needed providing more accurate estimation of the optimal RVD allocation. Thus, it remains to be seen how accurate the method given in this article will hold up over time.

Notes

1 For educational and nonprofit employees, a 403(b) plan would be used and would be similar to the 401(k) plan in allowing both a deductible IRA contribution and a Roth IRA contribution. Thus, the use of 401(k) can also refer to any similar employee retirement plan.

2 A nondeductible IRA is like an employer's match in two respects. *First*, it creates taxable income during retirement that can raise the withdrawal tax rate. *Second*, it also does not create a tax deduction. However, unlike a match, investors must pay for the nondeductible IRA out of their own pockets.

3 We call it T_C because it would be the same tax rate used in (5) in that marginal context.

4 As seen later in the T_C columns of Table 8, over 60% of the 48 values for T_C cover just one marginal tax rate. The T_W columns reveals just the opposite as rarely does T_W cover one marginal tax rate and then it is either 0% or 12%.

5 We include zero because, as seen later, it is possible to have a DIW that is not taxed.

6 For the most part, we use the "round" function in Excel to get the values to the nearest dollar. Tax bracket values, like $36,656, would be found in the nearest dollar in tax brackets. Thus, for reasons such as this, rounding off errors can occur when reporting and comparing values.

7 The link to the SSB calculator is https://www.ssa.gov/OACT/quickcalc/.

8 We get $1,565,846 by adjusting the actual value of 0.3($5,219,608) = $1,565,882 for a rounding off error of 0.002% caused from earlier computations. As mentioned previously, we use the "round" function in Excel.

9 In formula form using (12), we have: marginal gain = ΔT_1DP1 + ΔT_2(DP2-DP1) + ΔT_3(withdrawal-DP2) = (0.30−0)$50,588 + (0.30−0.12)($87,244–$50,588) + (0.30−0.18)($98,220–$87,244) = $15,176.40 + $6,598.08 + $746.26 = $22,521.

10 For example, the match (like the deductible IRA) for a 403(b) can avoid the payment of state taxes on withdrawals during retirement and so not all of it may be taxed. However, more noteworthy, OI does not necessarily lower T_W because its withdrawal can create income not taxed at the ordinary tax rate but at lower long-term capital gains and dividends rates. If OI consists of nontaxable investments like municipal bonds, then that is another argument that OI is not a factor influencing T_W.

11 Because we use average T_W and gain values, we can extrapolate based on numbers from Figure 2 by roughly estimating that a marginal gain would likely hover around $10,000 with ODI% around 35%.

Appendix 1

Proof that $FV_{Roth} = FV_{ATDed}$ when tax savings from deductible IRA invested in the deductible IRA and $T_C = T_W$. R_C is the expected rate of return on IRA contribution.

Step 1: Get two components for the after-tax future value Roth IRA.

1. Amount of earnings available to invest in a Roth IRA = $X.
2. Amount of after-tax future value Roth IRA upon retirement = $\mathbf{FV_{Roth}} = \$X(1 + R_C)^{Yc}$.

Step 2: Get after-tax future value of deductible IRA.

1. Amount of earnings available to invest in a deductible IRA = $\dfrac{\$X}{(1 - T_C).}$
2. Amount of after-tax future value deductible IRA upon retirement = $\mathbf{FV_{ATDed}} = \dfrac{(1 - T_W)\$X(1 + R_C)^{Yc}}{(1 - T_C)}$.

Step 3: set $T_C = T_W$.

The amount of the future value Roth IRA is not taxed and so $\mathbf{FV_{Roth}}$ remains at $\$X(1 + R_C)^{Yc}$.

If $T_C = T_W$, the amount of the future value deductible IRA is:

$$\mathbf{FV_{ATDed}} = \frac{(1 - T_W)\$X(1 + R_C)^{Yc}}{(1 - T_C)} = \frac{(1 - T_C)\$X(1 + R_C)^{Yc}}{(1 - T_C)} = \$X(1 + R_C])^{Yc}.$$

Thus, when $T_C = T_W$, we get $\mathbf{FV_{Roth} = FV_{ATDed}} = \$X(1 + R_C)^{Yc}$.

Q.E.D.

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