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The perfect withdrawal amount: A methodology for creating retirement account distribution strategies

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Abstract

We present a new way to develop withdrawal strategies from retirement portfolios. It is derived analytically, instead of from empirical testing, and it iterates always in the same manner. It is based on a new measure we develop, the Perfect Withdrawal Amount, for which we discuss how to construct a probability distribution and how to apply it sequentially. We also derive a new measure of sequencing risk. We present new strategies built with this framework. © 2015 Academy of Financial Services. All rights reserved.

Jel classification: J26; D81; D14

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1. Introduction

The question of how much to withdraw from a savings account once an investor has entered retirement is now more relevant than ever, as current demographic trends make it a matter of vital importance for a growing number of people every year. The problem itself arises because retirees generally wish to use their retirement funds to support a standard of living that is as high as possible, but without depleting their account so quickly that the years still ahead become difficult to finance—the so-called failure risk. On the other hand, withdrawing "too little" money might simply translate into excessively high balances in their

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accounts at the end of the life span horizon—the so-called surplus risk—and the "golden years" lifestyle would have been restricted unnecessarily.

The goal for the researcher is then to develop formulas or "rules" that dictate withdrawal amounts in each period based on the retiree's age, the assets held in the savings account, and the retiree's willingness to limit consumption in exchange for future safety. There are, of course, other factors that affect the optimality of these rules (inflation, tax implications, covering for emergency withdrawals, etc.), but the three factors mentioned usually have the largest impact on any recommendation.

The research thus far undertaken has produced a considerable body of knowledge. The problem is well understood, and many of the pitfalls of the original, simpler rules have been identified and addressed. The optimal strategies currently available are sophisticated and address a wide range of different situations and scenarios.

Still, it is our opinion that many of these results are, in a sense, heuristic. These approaches typically start with an idea that "makes sense" intuitively and then test it to see if improvement was indeed attained. In contrast, we have tried to develop an entirely *analytical* treatment of the problem, starting from the functional relationship between the relevant variables and building up from there before producing any rules or strategies. A central concept in this effort is a new measure that we have called the *Perfect Withdrawal Amount* (*PWA*). Instead of developing a particular rule for withdrawing, we introduce a fundamentally different way of addressing the problem.

2. Previous research

The origin of the research program on optimal withdrawals in retirement can be traced back to Bengen's (1994) pioneering article, in which he presents the basis for what would come to be known as "the 4% rule." Specifically, he demonstrates that a 4% withdrawal rate from a retirement fund, adjusted for inflation, is generally sustainable for normal retirement periods. A series of studies by Cooley, Hubbard, and Walz (1998, 1999, 2003, and 2011) then strengthened this conclusion, as they report similar findings using overlapping samples of historical stock and bond returns.

From a methodological standpoint, the distinguishing feature of these "first generation" articles is that they rely on a constant withdrawal amount, established from the outset, which is only adjusted to replenish its purchasing power. This has motivated attempts to develop "adaptive" rules, aimed at improving the results by applying midcourse corrections. Guyton and Klinger (2006) develop performance-based rules that decrease or even cancel the inflationary adjustment when return rates are too low, and that modify the withdrawal amount when the implied withdrawal rate falls outside ranges they prescribe. Frank, Mitchell, and Blanchett (2011) use adjustment rules that depend on how much the rate of return deviates from the historical averages. Zolt (2013) proposes curtailing the inflationary adjustment to the withdrawal amount to increase the portfolio's survival rate, and produces different "rules" by varying the degree to which purchasing power is restored.

Another take on the adaptive theme has been to reassess the situation periodically, taking into account the shortening of the horizon period. Spitzer (2008) considers this effect and resets the withdrawal amount every five years. Blanchett and Frank (2009) annually recalculates the probability of depleting the retirement funds too soon. If it becomes higher (lower) than a set of critical values they posit, the withdrawal amount is decreased (increased) by 3%; otherwise it remains constant.

An additional refinement has been to interpret the planning horizon length as a stochastic variable instead of a parameter. Under this view, the goal for the planner is to ensure that the funds in the retirement account "outlive" the retiree (instead of the other way around), no matter the number of years involved. Stout and Mitchell (2006) use mortality tables to make sure that a retirement period of uncertain length can be covered. Stout (2008) decreases the withdrawal amount whenever the account balance falls below a measure of the present value of the withdrawals yet to be made, and increases it when the balance is above this measure plus an additional value reserve. Mitchell (2011) uses different thresholds to trigger adjustments and also addresses the risk of superannuation.

A still more recent approach has treated the selection of withdrawal amounts as a lifetime-utility maximization problem. Milevsky and Huang (2011) posit as the objective function the total discounted value of the utility derived across the entire retirement period, in a setting where this length is a stochastic variable and the subjective discount rate is a measure of "personal impatience." Williams and Finke (2011) use a similar model with more realistic portfolio allocations and also consider other sources of income. Blanchett, Kowara, and Chen (2012) measure the relative efficiency of different withdrawal strategies by comparing the actual cash flows provided by each strategy to the flows that would have been feasible under perfect foresight.

The methodology that we present here can also be used to derive adaptive rules and revisiting schemes, and to perform longevity risk and utility maximization analyses. It is also well-suited for trying out alternative distributions for the rates of return, such as those described in Blanchett and Blanchett (2008), Pfau (2012), and Blanchett, Finke, and Pfau (2014). All of these, however, would now be enhanced by understanding the trajectories that the resulting strategies trace out in the PWA dimension.

3. The perfect withdrawal amount

We begin the development of our methodology by assuming a scenario where annual withdrawals are made on the first day of the year and annual returns accrue on the last day.[1] There's no inflation (or, alternatively, the rates of return used are in real terms) and no taxes. This simplifies the setup without removing any crucial element in the relationship between the main variables and, as will be shown below, any additional pieces needed to represent a real-world situation are easily added on top of this skeleton framework.

We now posit that for any given series of annual returns there is one and only one *constant* withdrawal amount that will leave the desired final balance on the account after n years (the planning horizon). This can be verified by solving a problem that is formally equivalent to that of finding the fixed-amount payment that will fully pay off a variable-rate loan after n years. In other words, we rederive the traditional PMT() formula found in financial calcu-

Year 1	8.6%	Year 11	-3.4%	Year 21	-0.7%
Year 2	19.9%	Year 12	4.5%	Year 22	36.6%
Year 3	-8.6%	Year 13	30.0%	Year 23	-6.4%
Year 4	-6.6%	Year 14	28.9%	Year 24	5.8%
Year 5	2.0%	Year 15	-2.1%	Year 25	25.8%
Year 6	3.7%	Year 16	0.2%	Year 26	23.9%
Year 7	14.4%	Year 17	3.9%	Year 27	-11.4%
Year 8	44.2%	Year 18	12.4%	Year 28	9.9%
Year 9	-2.5%	Year 19	8.2%	Year 29	-12.0%
Year 10	-4.2%	Year 20	-2.7%	Year 30	9.3%

Fig. 1. Sequence of returns produced by Monte Carlo engine through random drawing from S&P 500 historical data.

lators, but with three amendments: (1) interest rates are not fixed but change in every period, (2) the desired ending value is not necessarily zero, and (3) we are dealing with drawdowns from an asset instead of payments to a liability.[2]

The basic relationship between account balances in consecutive periods is:

$$K_{i+1} = (K_i - w)\,(1 + r_i) \tag{1}$$

where K_i is the balance at the beginning of year i, w is the yearly withdrawal amount, and r_i is the rate of return in year i in annual percentage. Applying Eq. (1) chain-wise over the entire planning horizon (*n* years), we obtain the relation between the starting balance K_S (or K_1) and the ending balance K_E (or K_n):

$$K_E = (\{[(K_s - w)\,(1 + r_1) - w]\,(1 + r_2) - w\}\,(1 + r_3)\ldots - w)\,(1 + r_n) \tag{2}$$

And we solve Eq. 2 for w to get:

$$w = \left[K_s \prod_{i=1}^{n} (1 + r_i) - K_E\right] \Big/ \sum_{i=1}^{n} \prod_{j=i}^{n} (1 + r_j) \tag{3}$$

Eq. (3) provides the constant amount that will draw the account down to the desired final balance if the investment account provides, for example, a 5% return in the first year, 3% in the second year, minus 6% in the third year, and so forth, or any other particular *sequence* of annual returns. This figure we call the PWA. If one were to know in advance the sequence of returns that will come up in the planning horizon, one would compute the PWA, withdraw that amount, and reach the desired final balance exactly and just in time.[3]

To provide a concrete example of the relationship between a sequence of returns and its corresponding PWA, we will use one of the estimation runs involved in the exercise presented in the next section to provide a numerical illustration. Suppose that the retirement period is just starting out and that *we know in advance* that in the next 30 years our investment account will yield the returns presented in Fig. 1.

	Starting Balance	Withdrawal	Balance after Withdrawal	Return	Ending Balance
Year 1	$1,000,000	$72,556	$927,444	8.6%	$1,007,266
Year 2	$1,007,266	$72,556	$934,711	19.9%	$1,120,717
Year 3	$1,120,717	$72,556	$1,048,161	-8.6%	$957,630
Year 4	$957,630	$72,556	$885,074	-6.6%	$826,768
Year 5	$826,768	$72,556	$754,213	2.0%	$769,227
Year 6	$769,227	$72,556	$696,671	3.7%	$722,277
Year 7	$722,277	$72,556	$649,721	14.4%	$743,490
Year 8	$743,490	$72,556	$670,934	44.2%	$967,410
Year 9	$967,410	$72,556	$894,854	-2.5%	$872,124
Year 10	$872,124	$72,556	$799,569	-4.2%	$765,627
Year 11	$765,627	$72,556	$693,072	-3.4%	$669,434
Year 12	$669,434	$72,556	$596,879	4.5%	$623,923
Year 13	$623,923	$72,556	$551,367	30.0%	$716,790
Year 14	$716,790	$72,556	$644,235	28.9%	$830,160
Year 15	$830,160	$72,556	$757,604	-2.1%	$741,929
Year 16	$741,929	$72,556	$669,374	0.2%	$670,966
Year 17	$670,966	$72,556	$598,410	3.9%	$621,979
Year 18	$621,979	$72,556	$549,423	12.4%	$617,567
Year 19	$617,567	$72,556	$545,011	8.2%	$589,432
Year 20	$589,432	$72,556	$516,876	-2.7%	$502,907
Year 21	$502,907	$72,556	$430,351	-0.7%	$427,513
Year 22	$427,513	$72,556	$354,958	36.6%	$484,756
Year 23	$484,756	$72,556	$412,200	-6.4%	$385,789
Year 24	$385,789	$72,556	$313,234	5.8%	$331,537
Year 25	$331,537	$72,556	$258,981	25.8%	$325,890
Year 26	$325,890	$72,556	$253,335	23.9%	$313,775
Year 27	$313,775	$72,556	$241,219	-11.4%	$213,608
Year 28	$213,608	$72,556	$141,053	9.9%	$154,968
Year 29	$154,968	$72,556	$82,412	-12.0%	$72,556
Year 30	$72,556	$72,556	$0	9.3%	$0

Fig. 2. Arithmetical confirmation that the PWA for the sequence in Fig. 1, when operating on $1 million starting balance and aiming for $0 ending balance, is $72,556.

Then, assuming that we want to exhaust our account in full (no inheritance) in 30 years, Eq. (3) indicates that each year we should withdraw exactly 7.2556 cents per dollar of initial balance. Fig. 2 confirms.

Therefore, with $1 million as starting balance and a final balance goal of zero, $72,556 is the Perfect Withdrawal Amount for this sequence of returns. We withdraw the same amount every year and reach our desired final balance with no ups and downs in the income stream, no portfolio failure, and no surplus.

In this manner we are characterizing every sequence of returns using one particular figure: the corresponding PWA. Therefore, from here we argue that the retirement withdrawal question is, at its core, a matter of "guessing" what the PWA will turn out to be (eventually) for each retiree's portfolio and objectives. The methodology presented here can then be understood as a way to go about this guessing in a statistically sound way. Specifically, our problem now becomes how to estimate the probability distribution for PWAs from the probability distribution for the returns on the assets held in the retirement account. Before moving on to explore how to derive the PWAs distribution, we should mention several results provided by our approach even at this early stage.

First, Eq. (3) can be restated in a particularly useful way. The term $\Pi_{i=1}^{n}(1 + r_i)$ in the numerator is simply the cumulative return over the entire retirement period, so we'll now call it R_n. We keep the subscript n to underscore the fact that this is the total return, not the average, and thus it depends on the length of the planning horizon.

The denominator, in turn, can be interpreted as a measure of sequencing risk:

$$\sum_{i=1}^{n} \prod_{j=i}^{n} (1 + r_j) = (1 + r_1)(1 + r_2)(1 + r_3)\ldots(1 + r_n) + (1 + r_2)(1 + r_3)\ldots (1 + r_n) + (1 + r_3)(1 + r_4)\ldots(1 + r_n) + \ldots + (1 + r_{n-1})(1 + r_n) + (1 + r_n) \tag{4}$$

We note that, for any given (unordered) set of rates, this expression decreases if "big" rates show up at the beginning of the retirement period and "small" rates show up at the end, because the last rates appear more times than the first rates in the summation.

To make the interpretation more natural, we'll say that it is the reciprocal $[1/\Sigma_{i=1}^{n}\Pi_{j=i}^{n}(1 + r_j)]$ that captures the sequencing effect, because this item goes up when the sequence is favorable.[4] This we call S_n, and we note that two r-vectors can have the same R_n and yet different S_n's—a given cumulative return can come up in different ways, order-wise.

Now Eq. (3) simplifies to:

$$w = (R_n K_S - K_E) S_n \tag{5}$$

This provides us with an expression that captures the impact of sequencing explicitly, through a factor (S_n) that "hits" the withdrawal amount in a perfectly sensible way.

This reformulation is important because the sequencing issue is precisely what makes the optimal withdrawal problem unique. In most financial analysis discussions, an understanding of the total accrued return will suffice, but here it is of the essence to know not just *how much* but *when this much*. In retirement, it makes a huge difference if "good" financial results come first and "bad" ones later, instead of the other way around, because in this stage of the life cycle the rates of return apply sequentially to an ever-dwindling capital base—the retiree is constantly drawing down her savings to support herself.

This is also why previous studies have sought for a way to account for this "sequencing risk" (Frank and Blanchett, 2010; Frank, Mitchell, and Blanchett, 2011). Indeed, even very recent discussions such as Pfau (2014) continue to address this problem, developing proxy variables to measure the correction required because of the sequencing issue. Although it is reasonable to expect that some of these constructs will capture the general features of the adjustment factor, our Eq. (5) comes directly from the simplest, most natural interpretation of the problem—so S_n is not a proxy. Rather, we think S_n is an expression that should be investigated further because it is a measure of *orientation* (return rates going up, going down, up a little then down a lot, etc.), and this is the crucial element that the adjustment factor should capture.

Second, we modify Eq. (5) to express the withdrawal *rate*, instead of the withdrawal amount:

$$w/K_S = R_n S_n - S_n\,(K_E/K_S) \tag{6}$$

Additionally, we note how it is dependent on the ratio K_E/K_S. This means that to derive an optimal withdrawal rate we need to know what *fraction* of the initial account balance is to be bequeathed.

Although this might seem like a trivial adjustment to the existing strategies, which may focus on the zero-bequest case only for the sake of clarity, the specific functional form by which inheritance goals affect the optimal recommendation has not been established before[5] (Bengen, 2006; Bernard, 2011; Spitzer, 2008). It is the *proportion* of the desired final balance to the starting balance that matters and not, as might be assumed, the *amount* by which the initial balance exceeds the target.

For example, a natural way to deal with the complication of a positive bequest goal would be to "put aside" the inheritance money—effectively removing it from the analysis—and work with the excess balance as if it were a zero-bequest case (cf. "Two-Bucket" strategy in Bernard, 2011). However, the separated funds will accumulate returns too, and there is no reason for these returns not to be available for consumption (at least the real part). Should the planner separate the funds to be inherited but then plow back the real returns into the "consumable" balance? Our approach renders these efforts unnecessary by including a bequest term in the equations, making the no-bequest scenario just a specific instance of the general case.

4. Construction of a PWA probability distribution

We now proceed to construct an actual probability distribution for PWAs, assuming a 100% equity portfolio. The decision to use an all-equity scenario is not crucial because we are only illustrating how the methodology works. This allocation was chosen because it maximizes the variance of the returns on the account's assets, and thus produces the PWA distribution with the largest dispersion possible. By examining this "worst case scenario" we can get a clearer idea of what PWA distributions look like in general, keeping in mind that for "normal" portfolio allocations they would be more clustered than what is shown here.

To apply our model we calculated the monthly returns on the S&P500 for the period January 1957 through April 2013 and then used a Monte Carlo engine to draw 360 values at random from this set (one at a time, so it's "with replacement"). These were then interpreted as the monthly returns in a 30-year planning horizon, in the same order as they were drawn, so the first 12 values were compounded to get the annual return for year 1, the next 12 values became year 2, and so on.[6]

The engine repeated this process 20,000 times and computed the cumulative return (R_n) and sequencing factor (S_n) for each series of returns obtained. This provided us with 20,000 (R_n,S_n) pairs, each pair standing in for a realistic vector of return rates with 30 entries.[7] Fig.

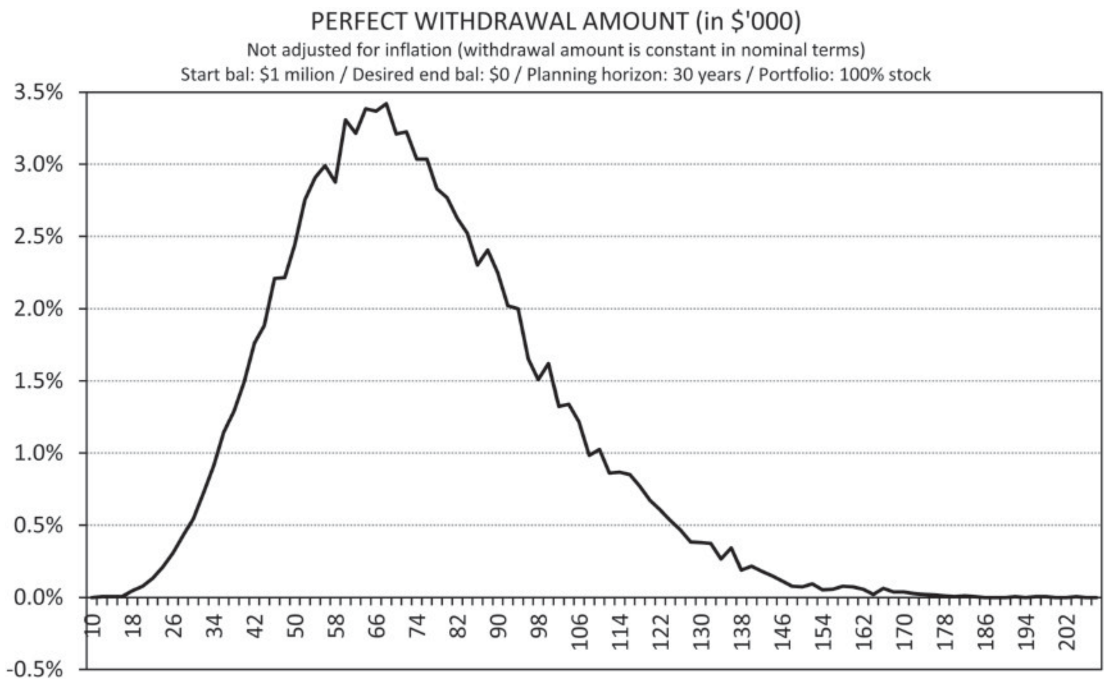


Fig. 3. Frequency distribution of annual PWAs in 20,000 Monte Carlo runs, with the specific parameters shown in the header. This distribution is then interpreted as a probability distribution for the optimal withdrawal amount.

3 presents the frequency distribution of the PWA formula (Eq. (5)) evaluated at each of these 20,000 (R_n,S_n) pairs, using $1 million as starting balance and with $0 as desired ending balance. Fig. 4 is the same but with cutoff points for deciles indicated.

Some of the advantages provided by our model can be seen in these charts. For example, the tabulations used to produce the charts can be used to locate any withdrawal amount, not just milestone values and not only amounts in the lower end of the range. So we can readily advise on the consequences of withdrawing, say, $62,000 every year; from the underlying tabulation we read that 64% of the Monte Carlo runs produced PWAs higher than $62,000, so failure risk for that withdrawal amount is 36%.

This distribution was calculated using $0 as desired ending balance, so the failure risk figure estimates the probability of total ruin. However, if the calculations had included a bequest target, the cumulative areas under the PWA curve would represent probabilities of not reaching this target instead of total depletion. The retiree might be interested in separating these two numbers, and the framework provides a way to do this: just compare the failure risk levels, with and without bequest, at that withdrawal amount. Fig. 5 shows this graphically.

Surplus risk, in turn, can be estimated by inverting the roles of the withdrawal amount and the ending balance as dependent/independent variables in the analysis. For the example above, one would use the same set of (R_n,S_n) pairs to evaluate $K_E = R_nK_S - 62{,}000/S_n$, and then tabulate the resulting figures.[8]

We used this feature to estimate surplus risk in this same scenario using $43,000 as withdrawal amount (which is approximately the cutoff point for the first decile of the PWA distribution). As expected, close to 90% of the runs left a positive balance behind—90.9% “didn’t fail”—but other points along the distribution of ending balances deserve mentioning.

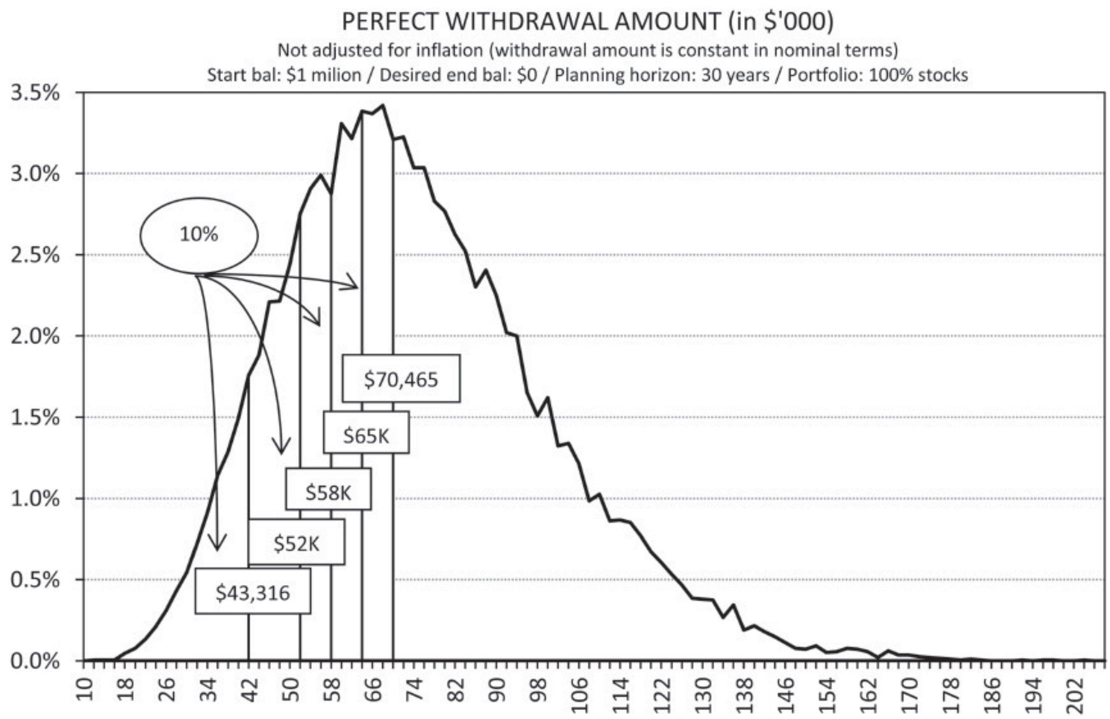


Fig. 4. Same as Fig. 3, but now indicating the figures that delimit 10% areas under the curve. These would be the "milestone" annual withdrawal amounts for the parameter values given in the header.

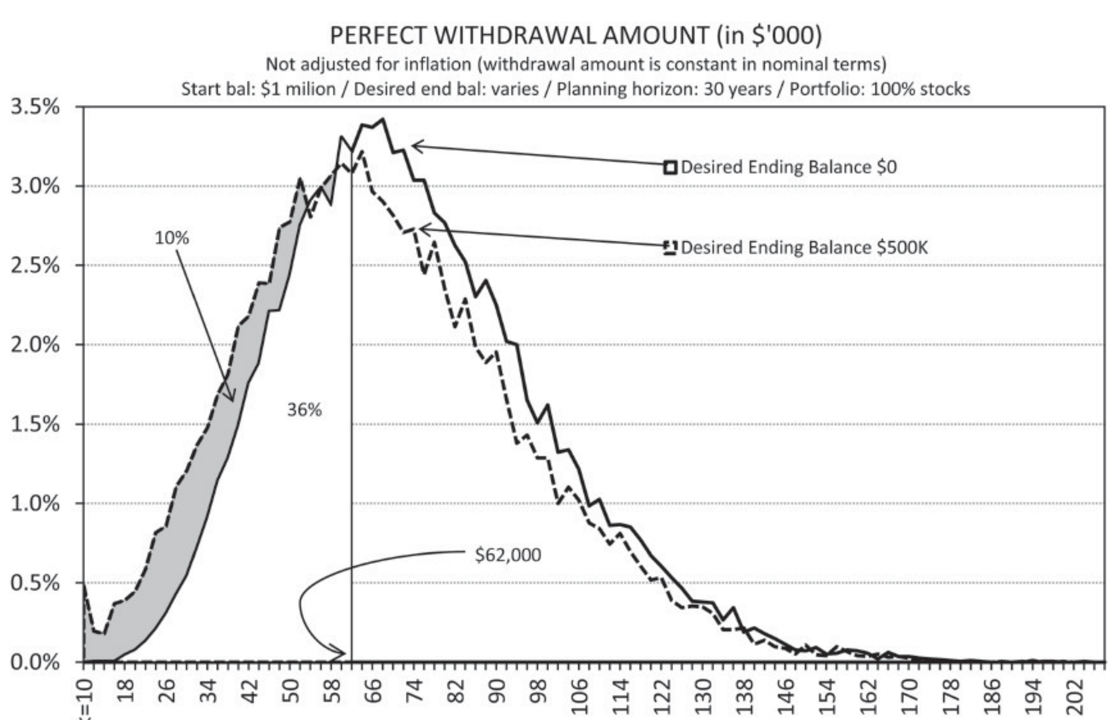


Fig. 5. The continuous line is Fig. 3. The broken line is the PWA distribution when the desired ending balance is $500K. Failure risk at $62K rises from 36% in the no-bequest graph, to 46% in the $500K bequest-goal graph. This means that the probability of a final balance between $0 and $500K is 10%.

For example, 74% of the runs ended up with more money than they began with, that is, the final balance was larger than the starting balance (and the desired ending balance was zero). The final balance is two times or more the starting balance in 58% of the runs. If one were to take this "4.3% rule" at face value, a 100% equity portfolio would have a 12% probability of ending up with 10 times or more what it had at the beginning!

Similar results have been found before and have been fully acknowledged (Cooley, Hubbard, and Walz, 1998), but even so we feel that they call for a reassessment of the "safe withdrawal rate" approach because of the large size of the ending balances associated with it. We think this approach could be used as a conservative guideline to start the series of withdrawals, but understanding that it is very likely that we will be able to increase the withdrawal amount significantly and yet stay inside the desired failure-risk range. Precisely, we now move on to the mechanics of how is the withdrawal amount adjusted each year using this methodology.

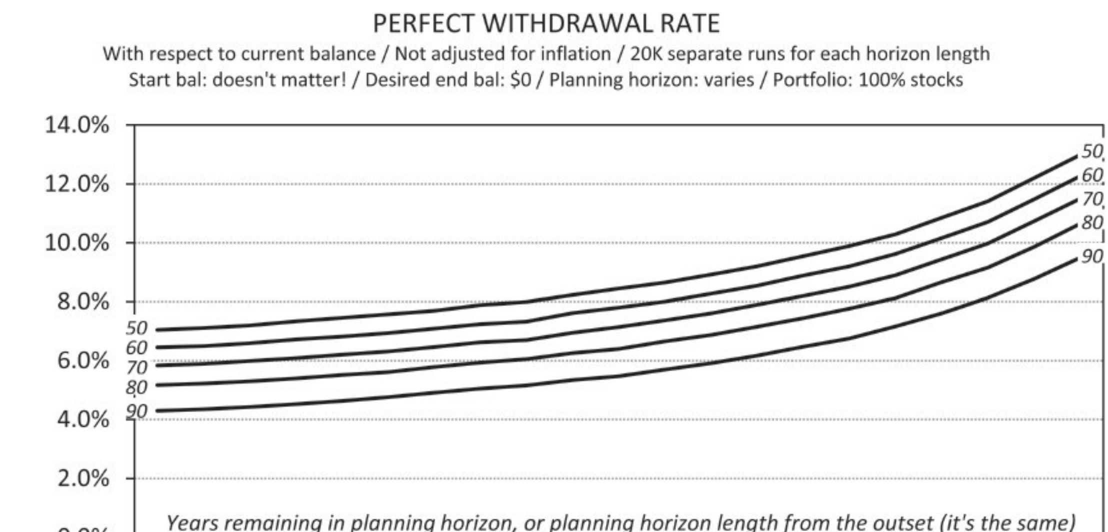
5. Sequential application of the PWA formula

Per our assumptions, we make a withdrawal at the beginning of retirement year 1 and live off this money for the entire period until, at the very end, the yield accrued during the previous 12 months is actually credited to our account. Now our PWA distribution needs to be recalculated using Eq. (5), with the balance actually showing as the new starting balance and shortening the time horizon by one period. The shortening of the horizon is attained by substituting in a new set of 20,000 (R_n,S_n) pairs, obtained with the same procedure described in the previous section, but this time drawing return rate sequences that are shorter than before by 12 data (they represent horizons with one year less to go). From this new distribution we choose the withdrawal amount for year 2, presumably—but not necessarily—using the same risk-tolerance profile as in year 1. Furthermore, this process is simply repeated every year.

We must stress that in PWA the process by which the withdrawal amount is selected is always the same, but this does not mean that the withdrawal amount itself will not change. Actually, it is rare for a withdrawal strategy built with this framework to recommend constant withdrawals beyond a small number of periods. PWA incorporates all new information into the set of data available for the next analyses, and this updating will create adjustment pressures that, in all likelihood, will end up modifying the withdrawal amount at some point.

Also, although a year has gone by, the planning horizon may be adjusted differently. The framework allows the retiree to judge whether *the expected number of years still ahead for her* has indeed decreased by one. If, for example, the retiree has stayed particularly healthy, she may decide to use the same horizon length once more. Or she may even use a longer horizon, if she has just overcome serious illness or made beneficial lifestyle changes. And, sadly, there will be opposite cases where the appropriate length reduction is larger than just one period.

However, other than these discretionary length adjustments, which can be considered part of the information update process, the PWA approach is consistent. It calls for the application of the same procedure to produce the withdrawal menu every year, without triggering different schemes if certain conditions are met.



	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10
50% conf range	7.0%	7.1%	7.2%	7.3%	7.5%	7.6%	7.7%	7.9%	8.0%	8.2%	8.4%	8.7%	8.9%	9.2%	9.5%	9.9%	10.3	10.8	11.4	12.2	13.0
60% conf range	6.4%	6.5%	6.6%	6.7%	6.8%	6.9%	7.1%	7.2%	7.3%	7.6%	7.8%	8.0%	8.3%	8.5%	8.9%	9.2%	9.6%	10.2	10.7	11.5	12.3
70% conf range	5.8%	5.9%	6.0%	6.1%	6.2%	6.3%	6.5%	6.6%	6.7%	6.9%	7.1%	7.4%	7.6%	7.9%	8.2%	8.5%	8.9%	9.4%	10.0	10.7	11.5
80% conf range	5.2%	5.2%	5.3%	5.4%	5.5%	5.6%	5.8%	5.9%	6.1%	6.3%	6.4%	6.7%	6.9%	7.2%	7.4%	7.8%	8.1%	8.7%	9.2%	9.9%	10.7
90% conf range	4.3%	4.3%	4.4%	4.5%	4.6%	4.8%	4.9%	5.1%	5.2%	5.3%	5.5%	5.7%	5.9%	6.2%	6.5%	6.7%	7.2%	7.6%	8.1%	8.8%	9.5%

Fig. 6. Milestone values in the optimal withdrawal rate distribution for the all-equity, no-bequest case. The rates are with respect to the actual balance faced each year, not the balance at the beginning of the retirement period. If, for example, the investor wishes to be 90% safe, each year she should run down the last row and take out the corresponding fraction of her balance at that point, whatever it may be.

This consistency allows us to compute the optimal withdrawal rates in the no-bequest case for the entire length of the planning horizon, and for all confidence ranges, at the same time. If no bequest is being sought, Eq. (6) becomes:

$$w/K_S = R_n S_n \tag{7}$$

Fig. 6 presents the results of evaluating this equation with the sets of (R_n,S_n) pairs corresponding to different horizon lengths (obtained as described above) for the 100% equity case considered here.

For example, let us consider a retiree that starts out with $1 million, has no desire to leave behind any money, and thinks that 30 years is a reasonable length for her planning horizon. She has assets indexed to the S&P 500 and wants to be 90% certain that she will not have to lower her withdrawal amount later on. Our subject only needs to read from the chart to obtain the corresponding rate. She does this and, following its advice, pulls out $43,000 (4.3%). Her first year turns out badly; the stock market goes down 10%, so her remaining balance of $957,000 (after withdrawing the $43K) shrinks to $861,000. Should she withdraw $43,000 again in the second year? Well, $43K represents 5.0% of her now-current balance and the chart tells us that, *with 29 years remaining*, that would take her out of the 90% confidence range and into the 80–90% region. Those are the factors that need to be considered in the decision—nothing else matters.

We make this point to highlight the simplicity of our approach, which cuts through the Gordian knot of some of the adaptive rules found in the previous literature (Bernard, 2011; Blanchett and Frank, 2009; Guyton, 2004; Mitchell, 2011; Pye, 2000; Robinson, 2007; Stout and Mitchell, 2006). For example, in Guyton and Klinger (2006) we find that "withdrawals are to increase from year to year to make up for inflation, except that there is no increase after a year where the portfolio's total return is negative and when that year's withdrawal rate would be greater than the initial withdrawal rate" (p. 5), or "when a current year's withdrawal rate has risen more than 20% above the initial withdrawal rate, the current year's withdrawal is reduced by 10%; this rule expires 15 years before the maximum age to which the retiree wishes to plan" (p. 7). One can find more rules like these in other studies.

The PWA approach would call this type of adjustments into question, claiming instead that the statistically appropriate thing to do is to calculate the probability distribution of PWAs for the current situation and choose from there—every year. All the relevant information currently available is embedded in that distribution and all new information is captured by the way the distribution changes as time elapses. An adaptive rule is simply a movement into different confidence ranges of the corresponding distribution and this, unless properly understood and consciously chosen, is an arbitrary call.

It must be stressed that these simplified results apply only if no bequest is being sought. When the retiree wishes to leave a certain amount behind, the withdrawal rate is given again by Eq. (6), which is the general-case expression:

$$w/K_S = R_n S_n - S_n\,(K_E/K_S) \tag{6}$$

This can be read as "the withdrawal rate for the no-bequest case, reduced by the sequencing factor *times the fraction of the current balance* that is to be bequeathed." The performance of the investment portfolio in previous years then comes back into play, but it is fully captured by the balance actually showing in each period, which is relevant only insofar as it affects the bequest-to-balance ratio.

6. What confidence ranges really tell us

At this point it becomes important to clarify what is meant by "confidence range" in this framework. Because the framework is adaptive, now an expression such as "90% safe" no longer means that there is a 90% chance that the portfolio will not run out of funds. What it means is that there is a 90% chance that *you won't have to lower your withdrawal amount in the future*, to achieve your bequest target.

We think this interpretation is more informative for the investor because normally in a real life situation a strategy will not be allowed to fail—the withdrawal amount will be reduced before that happens. So the truly pressing question is then *how much* would the withdrawal amount need to be lowered if we go into the red zone. The model can produce these quantitative assessments by computing and analyzing alternate PWA distributions, which are relatively easy to obtain because they can be derived from the original distribution.

For example, the distribution of the PWA after withdrawing an arbitrary amount w* this year can be obtained using a modified version of Eq. (3):[9]

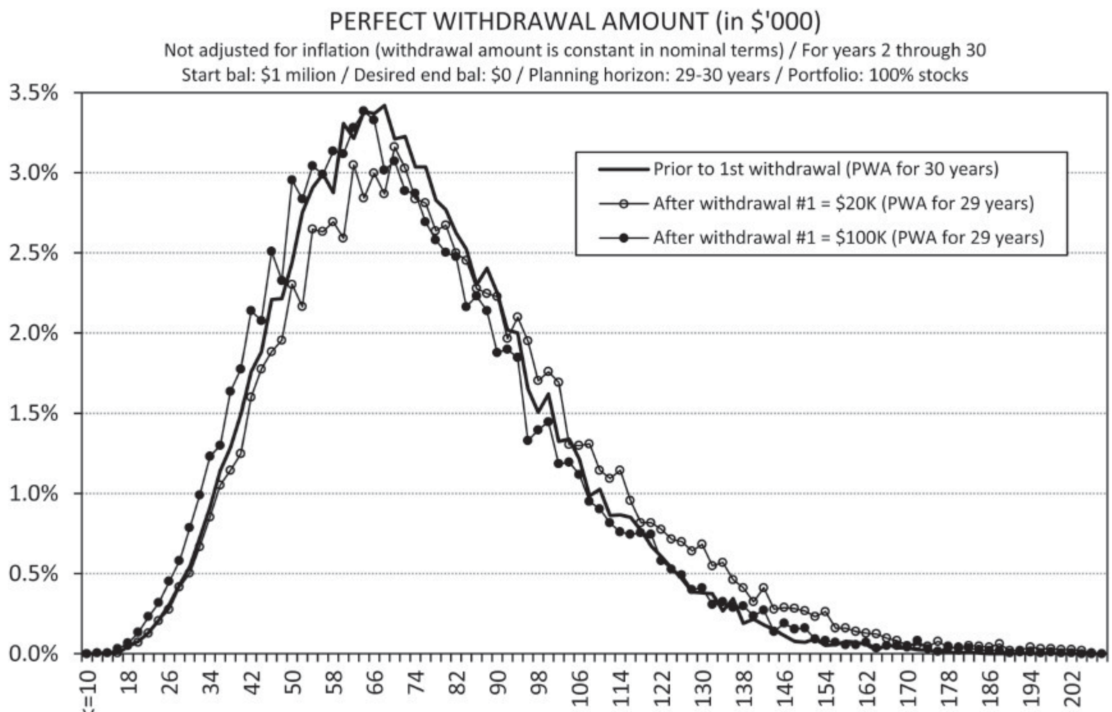


Fig. 7. PWA distributions immediately after making the first withdrawal, but before the first yield accrues. The thicker line is Fig. 3 (the original distribution). These charts answer the question "what happens if I withdraw x dollars this year," and are the PWA alternative to failure-risk figures.

$$w' = [R_n (K_S - w^*) - K_E] / \sum_{i=2}^{n} \prod_{j=i}^{n} (1 + r_j) \tag{8}$$

where w' is the new (modified) PWA for the subsequent years, w* is the withdrawal amount being considered for this year, and we leave the expression in the denominator in its explicit form to note that the summation now starts at period 2. We see that the denominator is now "missing" the product of all the returns from year 1 to year n (the summation starts with the product of returns from year 2 to year n), which is R_n, so we can express it as $\frac{1}{S_n} - R_n$. And we now rewrite Eq. (8) as:

$$w' = [R_n (K_S - w^*) - K_E] S_n/(1 - R_n S_n) \tag{9}$$

Finally, we can identify some of these terms as our original PWA, $w = (R_n K_S - K_E) S_n$, to see that the modified PWA is a linear transformation of the original:

$$w' = (1 - R_n S_n)^{-1} w - R_n S_n (1 - R_n S_n)^{-1} w^* \tag{10}$$

We can use this expression to compute different PWA distributions using the set of (R_n,S_n) pairs that we already have. Fig. 7 presents two of these alternate distributions, using w*=$20,000 and w*=$100,000.

These alternate distributions are then the answer that the framework provides to our pressing question above. Where most other approaches simply provide the change in the failure-risk figure, here we can compute the withdrawal amounts corresponding to different

Probability that the PWA going forward (years 2-30) will be less than...	Withdrawal in Year 1						
	$10,000	$20,000	$ 30,000	$ 40,000	$ 50,000	$ 60,000	$ 70,000
$ 20,000	0.2%	0.1%	0.2%	0.2%	0.2%	0.2%	0.2%
$ 30,000	1.4%	1.5%	1.8%	1.8%	1.9%	2.0%	2.1%
$ 40,000	5.7%	6.5%	6.9%	7.2%	7.8%	8.0%	7.8%
$ 50,000	14.7%	16.0%	16.8%	17.5%	18.4%	18.7%	19.1%
$ 60,000	27.8%	29.5%	29.8%	31.6%	32.1%	33.0%	33.4%
$ 70,000	42.2%	44.0%	44.8%	46.7%	47.1%	48.5%	49.3%
$ 80,000	56.4%	58.1%	59.7%	60.4%	61.3%	62.8%	62.9%
$ 90,000	68.5%	70.1%	71.2%	72.1%	73.2%	74.2%	74.5%
$100,000	78.1%	79.0%	80.3%	80.8%	82.0%	82.8%	83.5%
$110,000	85.5%	86.2%	86.8%	87.4%	88.2%	88.9%	89.7%
$120,000	90.5%	91.1%	91.8%	92.1%	92.9%	93.1%	93.7%

Probability that the PWA going forward (years 2-30) will be less than...	Withdrawal in Year 1						
	$80,000	$90,000	$100,000	$110,000	$120,000	$130,000	$140,000
$ 20,000	0.3%	0.2%	0.3%	0.3%	0.3%	0.3%	0.4%
$ 30,000	2.3%	2.3%	2.7%	2.7%	2.9%	2.7%	3.0%
$ 40,000	8.4%	9.0%	9.3%	9.9%	10.6%	10.7%	10.9%
$ 50,000	19.8%	20.8%	21.2%	22.1%	23.2%	24.0%	24.6%
$ 60,000	34.5%	36.0%	36.1%	37.6%	39.2%	40.2%	41.4%
$ 70,000	50.2%	52.0%	51.8%	53.5%	55.1%	56.0%	57.6%
$ 80,000	64.2%	65.8%	65.9%	67.6%	69.0%	69.7%	71.1%
$ 90,000	75.4%	76.7%	77.2%	78.3%	79.7%	80.3%	81.2%
$100,000	83.7%	85.1%	85.5%	86.4%	87.3%	87.9%	88.4%
$110,000	89.8%	90.9%	91.2%	91.8%	92.4%	93.0%	93.1%
$120,000	93.5%	94.7%	94.9%	95.1%	95.6%	96.0%	96.0%

Fig. 8. Probability that the PWA will drop below a certain amount after a withdrawal of a given size is made in year 1. Assumptions as in Fig. 7 header. For example, the highlighted cell shows that if a withdrawal amount of $40,000 (4%) is acceptable for the retiree, then she can take out $110,000 in year 1 and still be "90% safe" with respect to staying clear of this self-defined "red zone."

safety levels, with one year less to go, after making a withdrawal of a given amount in the current period. From the point of view of the retiree, this is the probability that she will have to settle for a future withdrawal of x dollars or less, after making a withdrawal of y dollars in the current period. Fig. 8, which is also made with Eq. (10) but using a table form instead of a chart, shows these results.[10]

These are examples of a more general feature made available by the PWA framework: the possibility of optimizing withdrawal paths meeting a specific profile request, and not just constant streams.[11] This can be achieved by introducing the "shape" of the request into the optimization process.

Consider an investor that is already in retirement but has not yet reached her Social Security full retirement age. She might prefer to avoid reductions in her benefits by delaying the time of claiming and use her investment account as her sole source of income for a number of years. This would then require a two-stage withdrawal plan, taking out relatively high amounts at the beginning and then smaller amounts after the benefits start to come in.

		Bequest Goal					
		$500,000	$400,000	$300,000	$200,000	$100,000	$0
Withdrawal amount that provides the stated confidence level when the bequest goal is as indicated	90%	$ 34,600	$ 36,157	$ 38,451	$ 39,921	$ 41,088	$ 43,316
	80%	$ 44,385	$ 45,527	$ 47,297	$ 48,676	$ 49,990	$ 51,890
	70%	$ 51,779	$ 52,661	$ 54,257	$ 55,665	$ 56,861	$ 58,392
	60%	$ 58,368	$ 59,456	$ 60,750	$ 62,242	$ 62,924	$ 64,650
	50%	$ 64,834	$ 65,925	$ 67,042	$ 68,654	$ 69,234	$ 70,773
	40%	$ 71,705	$ 72,534	$ 73,510	$ 75,093	$ 75,809	$ 77,304
	30%	$ 79,254	$ 80,119	$ 80,917	$ 82,396	$ 83,191	$ 84,401
	20%	$ 88,800	$ 89,135	$ 90,294	$ 91,649	$ 92,468	$ 93,410
	10%	$102,359	$103,054	$104,048	$105,099	$105,991	$106,326

Fig. 9. Withdrawal amounts corresponding to different confidence levels for an all-equity portfolio with $1 M starting balance and 30-year horizon. The amounts depend on the investor's intended bequest. The highlighted cells show that an investor who wants an annual income close to $52K and wishes to bequest $500K will have to accept 70% safety. However, if she does away with her bequest intentions her safety level rises to 80%.

A plan like this can be optimized by rederiving Eq. (3), with the withdrawals after the switch year T equal to the first years' withdrawals minus the benefits B, so that $w_1 = w_2 = \ldots = w_{T-1} = w_T + B = w_{T+1} + B = \ldots = w_n + B$. This scheme produces withdrawal recommendations aimed at providing constant total income throughout; the path of withdrawals from the investment account would have to be "kinked," but this can be handled by the framework without much difficulty.

We close this section by discussing another feature related to confidence levels which enhances the model's flexibility even more. When the retirement plan includes a bequest, the model provides the investor with an additional adjustment lever: the value of that bequest.

By Eq. (6), the safety level of a given withdrawal amount depends in part on the relation between the current balance and the desired ending balance. This means that by modifying the bequest goal we can attain different confidence levels without changing the withdrawal amount. For example, with $1 million starting balance and a 30-year planning horizon, if the retiree intends to bequeath $500,000 the 90% confidence level is at $34,600. But if she decided to lower that goal to $400,000 the 90%-safe withdrawal amount would now be $36,157—a 4.5% increase. Settling for an inheritance of just $200,000 increases this amount an additional 10.4%, to $39,921. Doing away with the bequest goal altogether takes us to Fig. 4, where we see that the 90%-safe withdrawal amount is then $43,316. Fig. 9 shows these bequest-adjustment profiles for several confidence levels.

Of course, modifying the bequest goal is also an option for midcourse corrections, and it can be combined with other adjustment levers. For example, if the account gets high rates of return for a number of years, the retiree may decide to split the benefits of the good run by increasing the withdrawal amount, the confidence level, *and* the bequest goal. The PWA framework will provide the relevant terms-of-trade.

7. Development and testing of a withdrawal rule inside the PWA framework

Once more, the purpose of this article is not to derive a particular withdrawal strategy but to present new tools for constructing and assessing strategies in general. Still, by way of illustration we now present strategies built with this toolkit.

Suppose that a major concern for a strategy under consideration is whether the withdrawal paths obtained from it will be too "jumpy"—that is, we are worried that the recommended withdrawal amount might turn out to be very different between one year and the next. This may arise if, for example, the strategy demands us to stay inside a confidence range that is very narrow, so that the withdrawal amount has to keep moving up and down to fall back into the mandated confidence range.

We may address this concern by proposing a two-tiered strategy that starts by selecting a certain amount of failure risk, but then "tolerates" some degree of variation around this value for the sake of steadiness in the income flow. Suppose we choose 50% as the acceptable initial value of failure risk. Anything "safer" than that is not agreeable to the retiree because she feels it creates too much surplus risk—she has no heirs and wants to consume her savings down to the last penny. We begin by withdrawing the median amount in the 30-year PWA distribution for an all-equity portfolio (because that is what she happens to have).

After making this first withdrawal and getting the return in our account at the end of the year, the withdrawal amount for year 2 is chosen as follows. If the amount withdrawn in the previous year is still inside the 15%-to-85% confidence range (a rather wide 70-point interval) *once the PWA distribution is recalculated starting from the now-current balance and with 1 year less remaining in the planning horizon*, that same amount is withdrawn once more. Otherwise the strategy is "recentered" by taking the median value of the recalculated distribution. This process is then repeated every year.

After setting up our Monte Carlo engine with these parameters, we have run simulations of the withdrawal paths resulting from this strategy using the now customary $1 million starting balance and $0 desired ending balance. Figs. 10, 11, and 12 present three of these simulated paths, each one corresponding to return sequences that give rise to (somewhat) flat, rising, and decreasing withdrawal profiles.[12]

Barring perfect foresight, nothing *strictly* better can be provided for a retiree with these demands and holding these assets. If we want to improve the results of the strategy along either one of the three relevant dimensions (safety, income level, stability), we would have to change either the base safety level, the recentering point or the width of the tolerance range.

This last point is an important realization about the nature of the problem at hand. By the structural relationship between the variables, when the parameters of a strategy are modified to enhance one specific property, it is inevitable for one or more of the other properties to suffer. This in turn means that the design of optimal withdrawal strategies is not a search for the dominant scheme; it is a search for the strategy that best suits the needs of the investor being served.

The next section closes the discussion by looking more closely at this "planner-as-tailor" aspect of the framework.

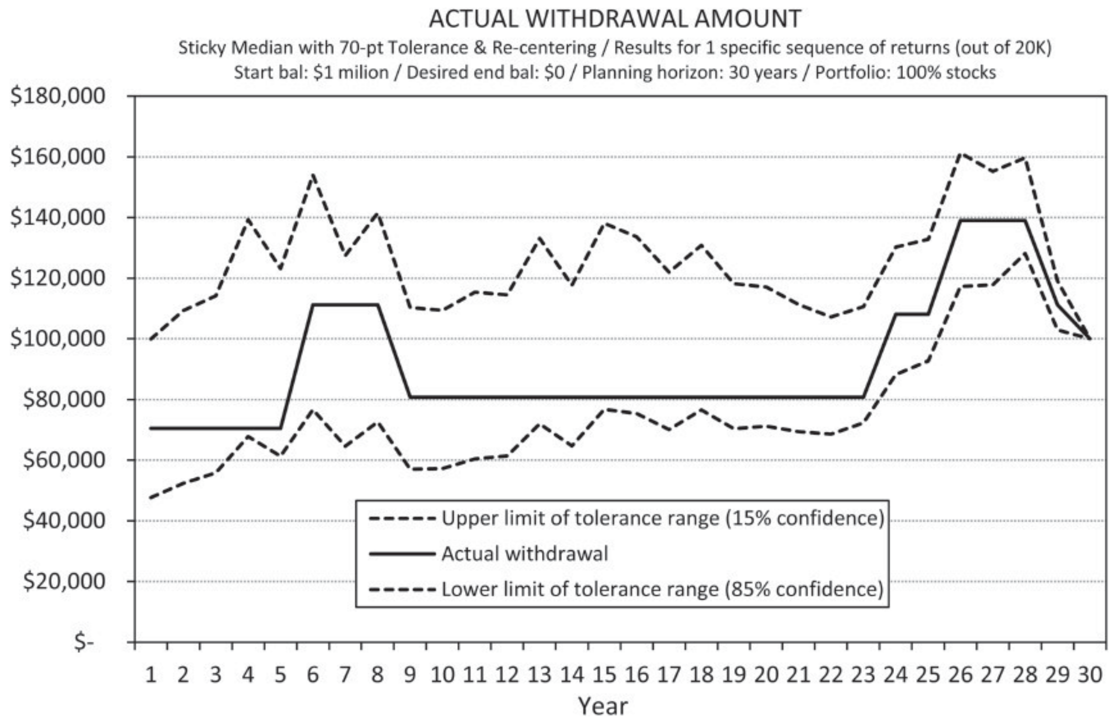


Fig. 10. This strategy kicks off by withdrawing the median value in the 30-year PWA distribution. Then it becomes "sticky," withdrawing the same amount unless it is "too safe" (more than 85% confidence) or "too risky" (less than 15% confidence). When this happens, the strategy again takes the median value of the PWA distribution applicable at that point in time. The PWA for this particular sequence of returns was $84,792 which, by Fig. 4, is somewhat on the high side.

8. A tale of three strategies

We will now use the sequence of returns shown in Fig. 1 to present three different strategies, once again starting from $1 million and ending at $0. As can be seen in Fig. 1, the dispersion in that sequence is significant (standard deviation 14.5 points) and it is quite jumpy, with the values ranging from −12.0% in year 29 to 44.2% in year 8. Therefore, this particular sequence may be considered a worthy challenge for any strategy to handle, even though it's rather "normal" (Fig. 2 informs us that its PWA in this case is $72,556; Figs. 3 and 4 shows this amount is quite close to the median value, and in the peak region of the chart).

The purpose of this exercise is not to establish which strategy is better. The results shown represent not just the strategy followed, but the specific return sequence used as well. To compare strategies one would have to average their respective performance statistics over a large number of randomized runs to determine the frequency with which they produce "good" or "bad" outcomes, where good and bad are defined by the retiree.

One can compare the situation faced by the retiree to that of a shopper in a clothing store right after a big sale. The items in the tables and bins are still misplaced because a horde of customers sorted through them haphazardly the day before. But the most efficient way to find an item of her size and liking is, still, to look in the places where it would normally be; there is a higher probability of finding it there than elsewhere. The

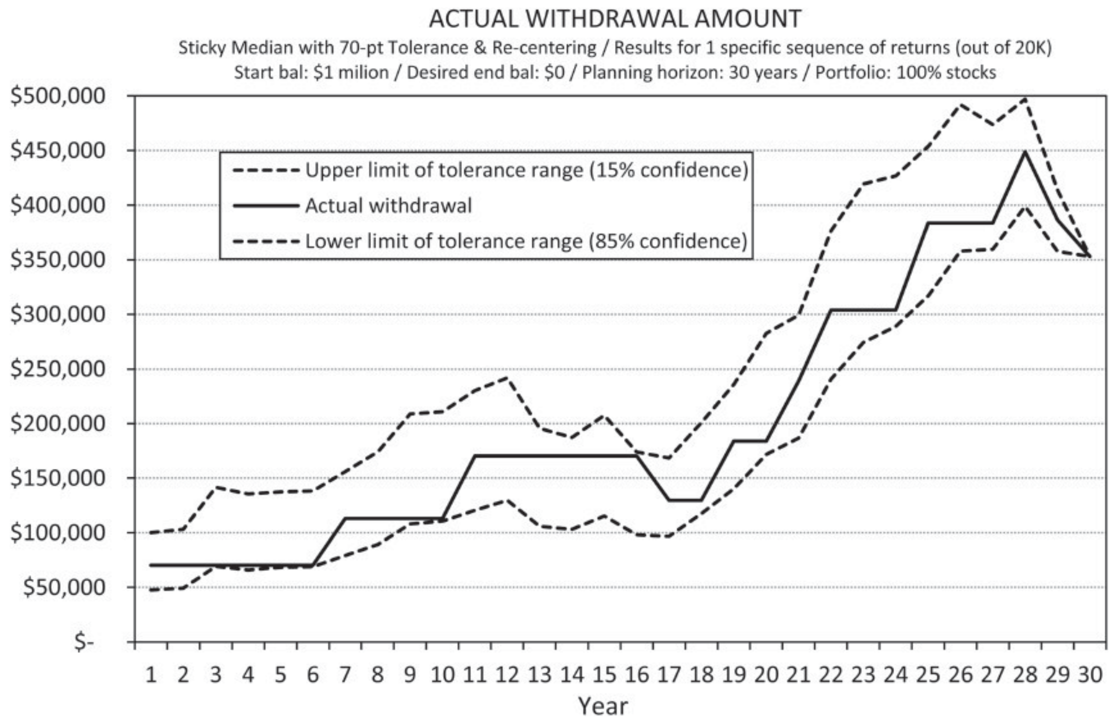


Fig. 11. Same as Fig. 10, but this time the sequence of returns to which the strategy was applied had a PWA of $118,498. Fig. 4 shows this is unusually high, so the strategy is "surprised" and reacts by increasing the withdrawal amount markedly and frequently.

PWA framework would then be the knowledgeable clerk who will ask you what you want and then will tell you where it is more likely that you may find it. Following this metaphor, this section is a sampler of some of the "items carried" in an all-equity portfolio facing a 30-year planning horizon.

The first strategy considered is the 4% rule, which withdraws $40,000 every year and is presented in Fig. 13. We already know that the PWA for this sequence is $72,556, and therefore that this strategy will be too cautious and end up with surplus balance. The balance reaches $3,013,737 in year 30, which shows in the figure as the convergence point of the tolerance range boundaries (only reference points now). These boundaries always come together at the end because in this framework there is only one possible recommendation at that point: take out all you have, because this is the last year. After continuing to withdraw $40,000 (one last time) and accruing the final return of 9.3%, the account bequeaths $3,250,295.

We see that the overaccumulation trend is evident early on, as the confidence level reaches 98% in year 3 after starting out at 93%—perhaps overly high even from the outset. By year 15 the confidence level goes "off the charts," as there is not a single run in our simulation for that horizon length that produces a PWA of $40,000 or lower starting from $1,832,363 (the balance in year 15). These confidence levels "above 100%" persist until the end of the retirement period.

The second strategy is the one we used in the previous section, which we now show in Fig. 14 as "Sticky Median with 70-point Tolerance and Recentering." As discussed above, this strategy uses the central value (the median) in the 30-year PWA distribution as the initial

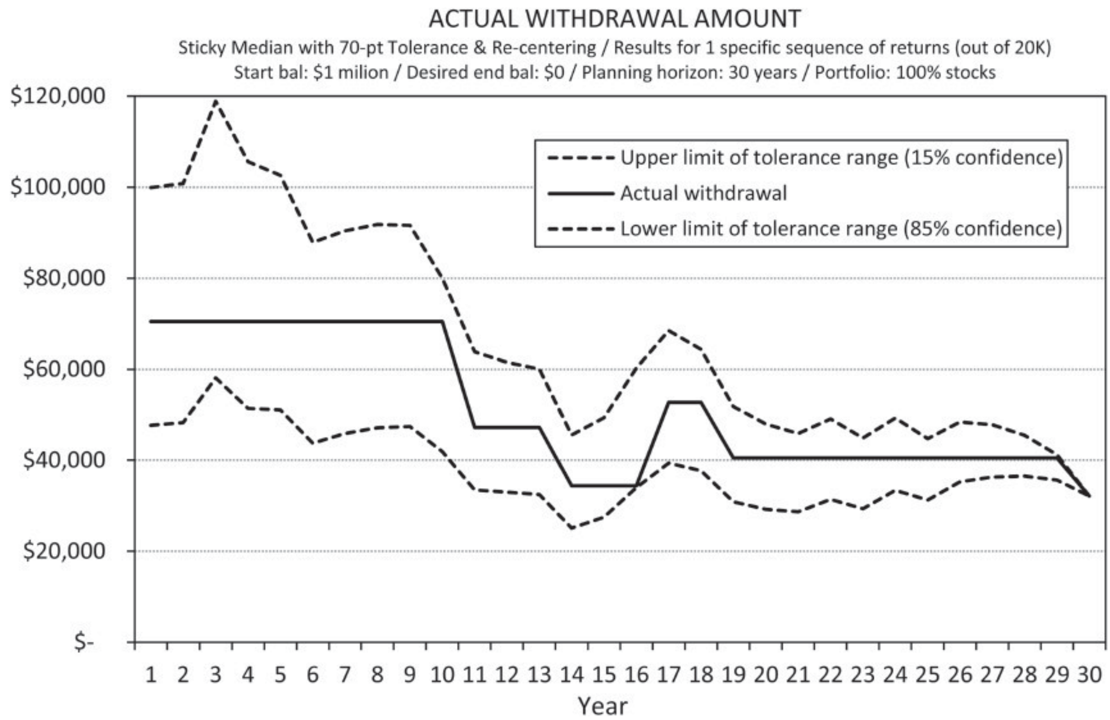


Fig. 12. Same as Figs. 10 and 11, but here the sequence of returns had a PWA of $53,691. Per Fig. 4, this value is in the lower half of the distribution, so the strategy is unable to support the median withdrawal amount and adjusts to a lower level.

withdrawal amount. Then it focuses on stability and withdraws the same amount every year, but only if it is "reasonable" to do so. When taking out the same amount implies falling outside the 70-point confidence interval centered at 50% (between 85% and 15%), we adjust the withdrawal and go back to the center of the current PWA distribution. Therefore, the strategy effectively starts over or "recenters" when it deems that it is no longer prudent to favor steadiness over safety (failure-wise or surplus-wise).

The median of the 30-year PWA chart is $70,465 (see Fig. 4), so the strategy kicks off with that, and then it keeps the amount steady for a number of years. We see how the upper bound gets close to the withdrawal amount in year 7, but the strategy stays the course because the confidence level is 25% (low, but still allowed). The opposite happens in year 20, when the lower bound closes in, but again the reins are held tight because the confidence level only reaches 75%. The tolerance range is not "pierced" until year 23, when holding on to the previous withdrawal amount would put the account at 91% confidence—an unacceptably high risk of overaccumulation. We use the balance at that point ($615,189) to compute the PWA distribution with 8 years to go (year 23 is only starting out), and take out the median value: $94,648. This new withdrawal amount is almost decreased only two years later, when the confidence level drops to 16%, but the tolerance range is only grazed and the strategy holds steady.

The balance at the beginning of year 30 is $71,908, which is taken out in full as final withdrawal and creates the steep drop at the end of the chart (−24%). The drop is of course caused by the terrible return rate in year 29 (−12.0%), but in a sense it can also be attributed to the high volatility of this portfolio allocation. Still, this final amount is higher than what

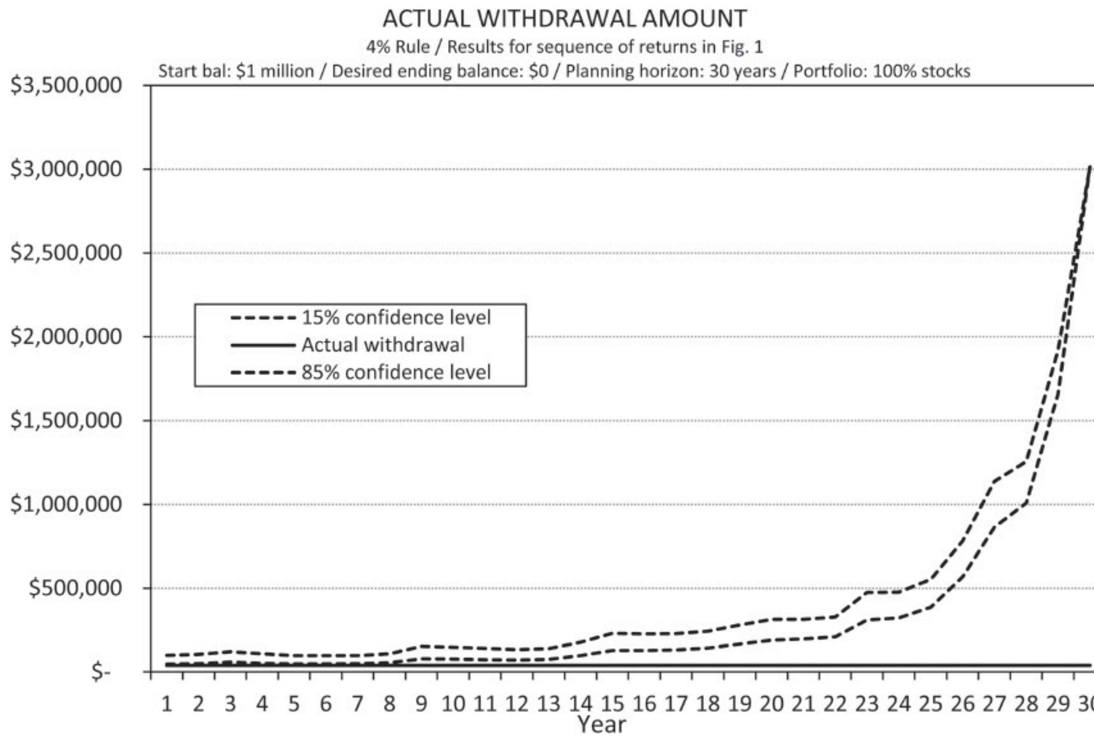


Fig. 13. The 4% rule is used with the sequence of returns in Fig. 1. As the planning horizon shortens and the account balance rises, it becomes less likely that $40,000 will end up being the PWA. This increases the confidence level, even though the withdrawal amount is always the same. After 30 years, the account bequeaths more than $3 million.

the retiree got during the first two decades and is also higher than what would have been provided by the 4% rule.

The third strategy is a variation of the second one. It is somewhat exotic, as we modify the three main parameters of Sticky Median to illustrate the flexibility in strategy design provided by the PWA framework. This version kicks off by withdrawing the amount at the 75% safety level in the PWA distribution, so it is more conservative than Sticky Median, which starts at 50%. Then we narrow the tolerance range around the base safety level to 30 points, so it now spans the interval from 60% to 90%. This means it stays closer to the level of risk with which the retiree feels most comfortable (compared with Sticky Median), but it demands more frequent adjustments to the withdrawal amount because it allows less "straying." Therefore, the third change is in how are the adjustments applied; instead of going back to the original safety level (75%), this time the strategy "pushes" the withdrawal amount (up or down) by the minimum amount necessary to get back into the tolerance range. This adjustment procedure is more subtle than recentering, but it asks for the retiree to live at the very edge of his risk preference in some years. These changes produce a peculiar but very reasonable strategy, which we have called "Sticky First Quartile with 30 point Tolerance and Nudging" and is shown in Fig. 15.[13]

The most noticeable change in the resulting withdrawal path is its upward trend, which is a result of the higher safety level used. The initial withdrawal amount is $55,025 (75% safe)

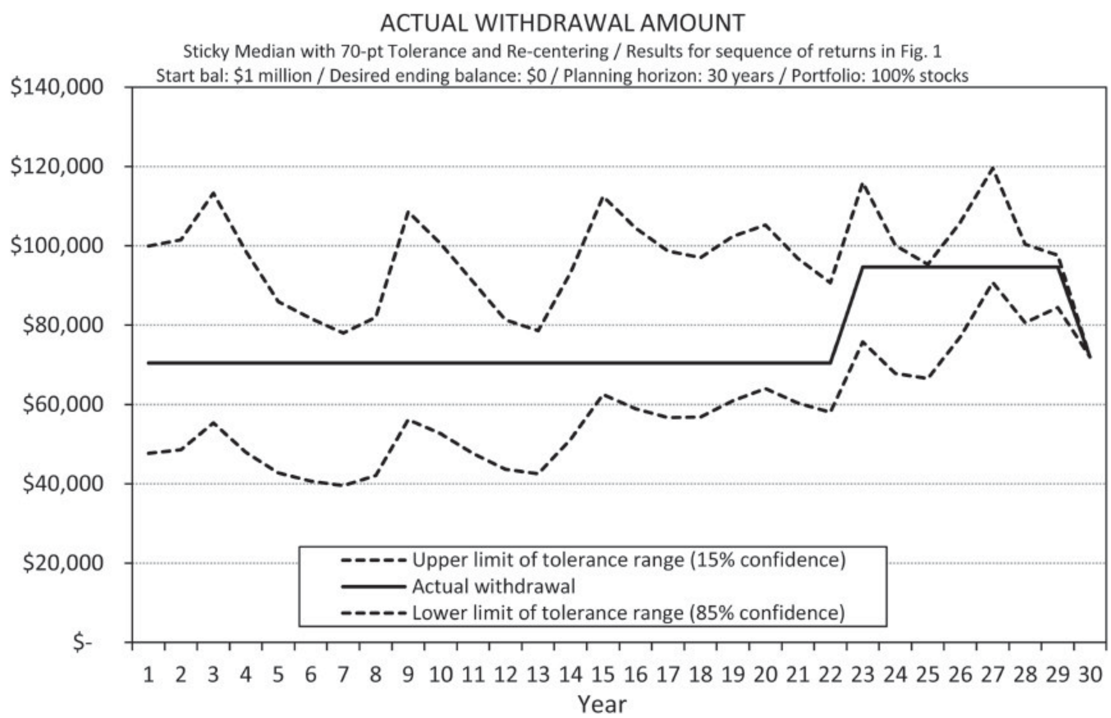


Fig. 14. Same as Fig. 13, but now the strategy followed is "Sticky Median with 70-point Tolerance and Re-Centering." The withdrawal amount is much higher than with the 4% rule and yet retains some degree of stability. Here the entire remaining balance is withdrawn in year 30 (no bequest is desired), but this final withdrawal is still "inside the ballpark" set by the previous amounts.

and from there the lower bound of the tolerance range keeps nudging the amount higher. The only downward adjustment comes at the very end, and is very slight. After withdrawing $199,620 in year 29 and taking the decrease of −12.0%, the balance reaches $199,197; this is the last withdrawal.

These three sets of results shed some light on the consequences of pursuing different goals. An all-out desire to provide withdrawal stability would follow something akin to the 4% rule, even if it means bequeathing an amount much larger than what was intended. An approach that attempts to maximize income without going out on a limb would prefer something like the Sticky Median strategy, even if the withdrawal amount can suffer major jolts in either direction. If the primary concern is safety, and cash flow steadiness is not essential, a strategy similar to Sticky First Quartile may be in order even though it is likely to result in a markedly rising withdrawal path. Fig. 16 superimposes the three withdrawal paths for a more direct comparison.

Again, this exercise must not be taken as evidence in favor or against either strategy. If instead of the sequence in Fig. 1 we had used the one underlying Fig. 12 (the "decreasing" path), the interpretation might have been different. Fig. 17 shows the values in that "bad" sequence, which has a PWA of just $53,691 starting from $1 million and aiming for $0. Fig. 18 is the chart of the resulting withdrawal paths, which may be compared with Fig. 16 for contrast.

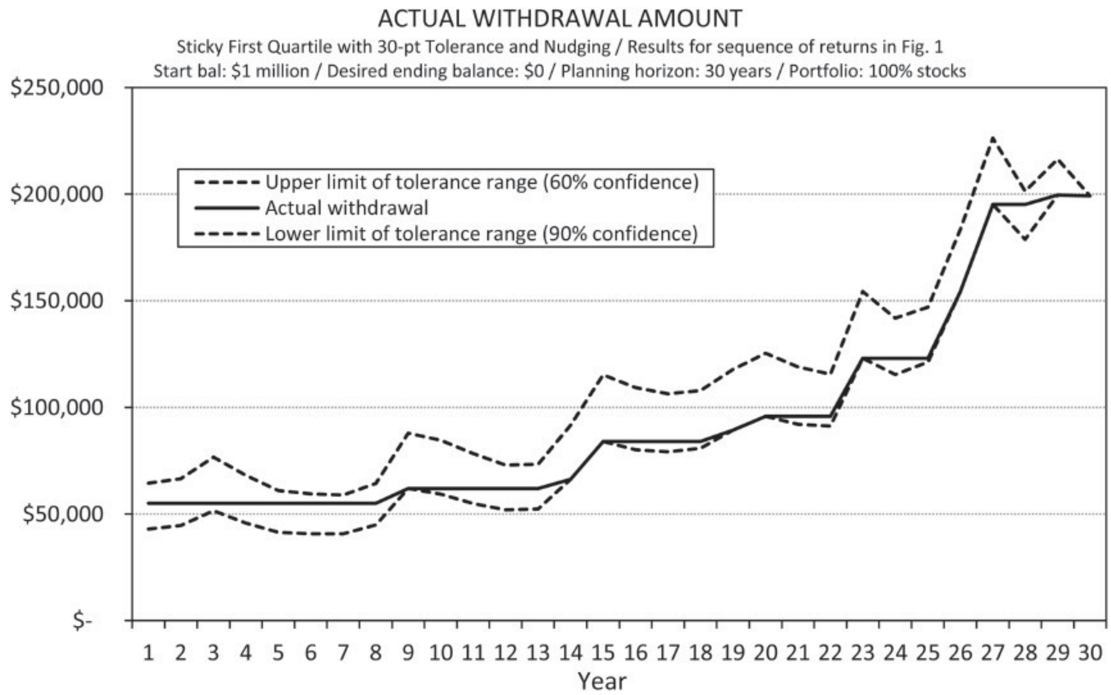


Fig. 15. Same as Figs. 13 and 14, but now the strategy followed is "Sticky First Quartile with 30-point Tolerance and Nudging." Compared with Fig, 14, this strategy is more cautious (it drifts around 75% safety instead of 50%), more strict (tolerance 30 points instead of 70), and more subtle (upon piercing the tolerance the amount is nudged back in instead of recentered).

9. Conclusions and next steps

A sequence of return rates, together with a starting balance and a desired ending balance, determines a "perfect" withdrawal amount. If the retiree's goal is to obtain a constant stream of income from her investment account, this is the amount she should take out in every period. If she withdraws more than the PWA, she will run out of funds (or leave behind less money than she intended). If she withdraws less, she will leave money on the table.

The problem, of course, is that we do not know in advance what the PWA will turn out to be in each case. To make inferences and take decisions one must construct probability distributions, and the methodology presented here is a way to do this soundly. Here we use the historic distribution of return rates as the probability distribution of future returns, assuming it to be independent and identical in all periods. However, any other assumption may be used, as our framework only discusses how to "process" the assumptions (or expectations) held by the researcher.

At this stage, we can only make preliminary recommendations. However, one that seems to take shape intuitively is a major departure from conventional wisdom: the best level of failure risk is 50%. To be more precise, we should use the mode of the PWA distribution—because that is the most likely value that the PWA will end up taking in our particular case. For symmetric or moderately skewed distributions, failure risk at the mode will be close to 50%. If the assumptions used for the distribution of return rates

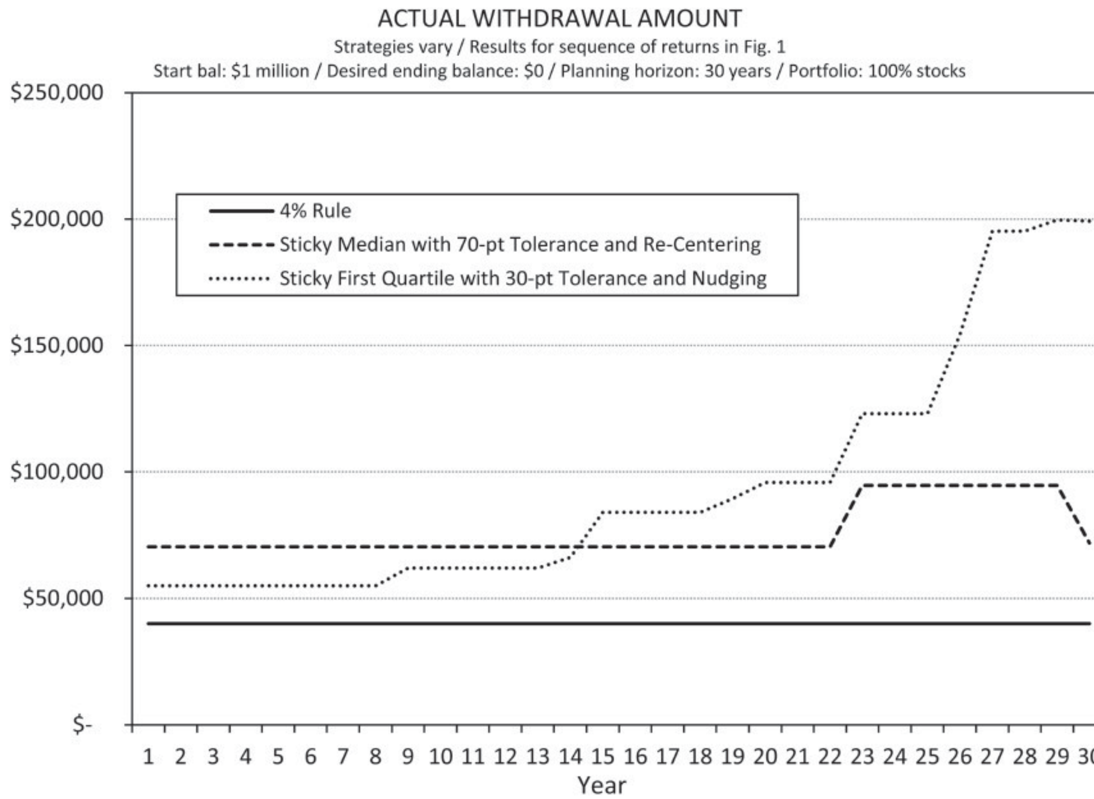


Fig. 16. Strategy comparison. The withdrawal paths in Figs. 13, 14, and 15 are superimposed, with the tolerance range lines removed for clarity. The 4% rule provides a low cash flow and leaves a sizable bequest, so the standard of living in retirement is overly restricted. Sticky Median provides more money, but failure risk reaches very high levels along the way. Sticky First Quartile plays it safer, but at the expense of frequent adjustments to the amount withdrawn.

are sound, then close to half of the retirees who follow this policy will be able to keep withdrawing this amount (or more!) for the entire retirement period. For the other half, the procedure outlined here will steer them clear of funds' depletion or bequest collapse through timely warnings that the amount needs to be decreased—and probably not by

Year 1	7.8%	Year 11	3.5%	Year 21	16.5%
Year 2	26.8%	Year 12	4.2%	Year 22	-1.3%
Year 3	-7.0%	Year 13	-18.0%	Year 23	21.3%
Year 4	3.5%	Year 14	15.7%	Year 24	-2.1%
Year 5	-9.4%	Year 15	29.9%	Year 25	19.3%
Year 6	11.0%	Year 16	18.8%	Year 26	6.3%
Year 7	8.9%	Year 17	0.2%	Year 27	2.8%
Year 8	7.5%	Year 18	-13.1%	Year 28	-0.9%
Year 9	-6.8%	Year 19	-1.1%	Year 29	-5.5%
Year 10	-13.5%	Year 20	4.5%	Year 30	-2.5%

Fig. 17. Another random sequence of returns produced by Monte Carlo engine. This is the sequence that gives rise to Fig. 12.

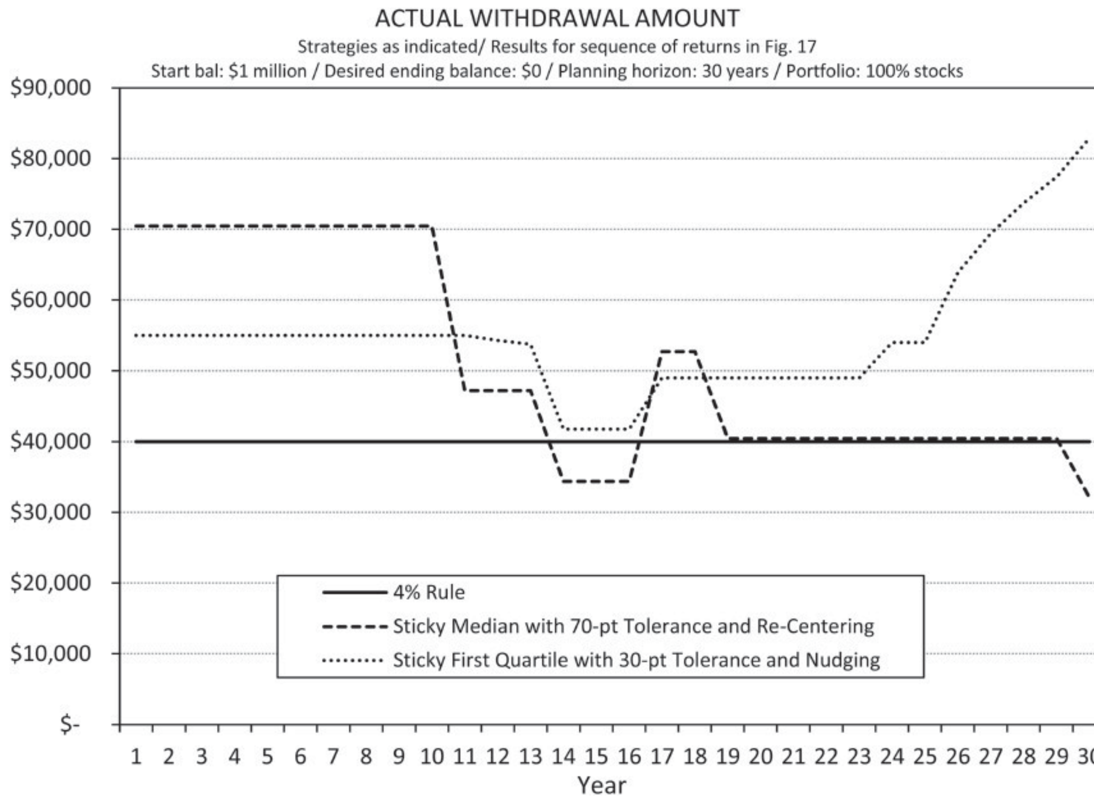


Fig. 18. Comparison of the same three strategies as in Fig. 16, but now the withdrawal paths are the ones produced by the sequence in Fig. 17 instead of Fig. 1. The 4% rule is closer to the other two in this case, but it still results in a significant bequest of $735,059 when none was sought.

much. If ours is indeed the case when the sequence of returns awaiting in our future is a terrible one, then the procedure will keep signaling for downward adjustments, taking the withdrawal path asymptotically to a very low value, but probably not too far below what it would have been under perfect foresight.

The obvious next step for the framework's future development is to use it to derive new "standard" rules—à la Bengen's 4%—for different portfolio compositions and retirement objectives, and to look at the rules currently in favor through the lens of PWA. Also, it would be useful to add longevity risk into the framework by making the planning horizon a stochastic variable instead of a parameter. And it would seem possible to adapt the model to the accumulation phase of the life-cycle, deriving a Perfect Contribution Amount concept along the lines of PWA.

Once we know more about the PWA for different portfolio compositions, we can proceed to the derivation of optimal adaptive strategies that produce specific results along any of the relevant dimensions. Here we would be interested in exploring "gliding" strategies that change the asset-mix over time, as well.

Finally, we see an opportunity to improve the computation of the PWA probability distributions by using more advanced methods to construct the probability profiles for the financial assets' returns. The bootstrapping procedure used here is very practical, but we are aware of its numerous drawbacks.

Notes

1 Although not entirely realistic, the assumption of annual withdrawals has been the norm in withdrawal rate studies since the "founding" of the field by Bengen (1994).

2 The fact that this is a mirror image of the paying-off-a-loan situation means that the variable "r" in the typical formula is now a return rate instead of an interest rate, the ending value is available capital instead of unpaid balance, and the result of the formula is a payment from the account instead of a payment into the loan.

3 Blanchett, Kowara, and Chen (2012) presents a measure similar to PWA called Sustainable Spending Rate (SSR). Our PWA was developed independently, but it turns out to be a generalization of SSR, SSR being the PWA when the starting balance is $1 and the desired ending balance is zero.

4 The sequencing factor formula is the same as the sinking fund factor under variable rates of return, but the interpretation and application is thoroughly different in each case.

5 Actually, considering only formal, mathematically explicit models, the authors were unable to find a single study where the desired bequest is an input in the calculations.

6 Similar "bootstrapping" procedures are used, for example, in Blanchett (2007); Blanchett and Frank (2009); Spitzer (2008); Spitzer, Strieter, and Singh (2007); and Zolt (2013).

7 In the previous section we said that we characterize each sequence of returns by its corresponding PWA, but this is only an intuitive description. The formally correct statement is that we characterize each sequence by its corresponding (R_n,S_n) pair. To obtain an actual PWA for a specific sequence of returns, the current balance and the bequest target must be provided.

8 For the (R_n,S_n) pairs that produce a negative ending balance, the numerical value of this formula would be wrong because the yields cease to apply once the balance moves into the red. But the tabulation can show the size distribution of the positive ending balances, with the rest marked as "failures."

9 Eq. (8) is obtained simply by changing the standpoint of the evaluation: from before the current year's withdrawal is made, to after it has been made but before the current year's return hits the balance. In other words, we perform the evaluation assuming we are in the middle of the year, instead of at the very beginning. This changes the starting balance from K_s to K_s-w*, but leaves the sequence of yield accruals unchanged.

10 We thank an anonymous reviewer for comments that led to the development of the tables in Figs. 8 and 9.

11 Approaching the problem from a different angle, Robinson and Tahani (2010) also produce non-constant withdrawal profiles.

12 The three return sequences involved in the figures were produced randomly by the Monte Carlo engine, but they are not equally likely. We selected them intentionally, out of the set of 20,000 runs, to illustrate different possibilities for the resulting withdrawal path.

13 This strategy developed from suggestions by an anonymous reviewer, which we thankfully acknowledge.

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