FINANCIAL
SERVICES
REVIEW

Financial Services Review 24 (2015) 15–35

Spread options and risk management: Lognormal versus normal distribution approach

Robert Brooks[a], Brandon N. Cline[b,*]

[a]*Department of Economics, Finance and Legal Studies, The University of Alabama, 200 Alston Hall, Box 870224, Tuscaloosa, AL 35487, USA*
[b]*Department of Finance and Economics, Mississippi State University, 312 McCool Hall, Mississippi State, MS 39762, USA*

Abstract

We provide better tools for managing the downside risk related to the spread between the asset portfolio and corresponding liabilities. These tools are particularly applicable for individual investors. We investigate the spread option valuation model where both underlying instruments follow geometric Brownian motion, and one where both underlying instruments are assumed to follow arithmetic Brownian motion. We show that the risk parameters are often materially different. These results are important in practical applications of risk management for individual investors as well as financial institutions. For most personal financial planning applications, one can safely use the simpler arithmetic Brownian motion model. © 2015 Academy of Financial Services. All rights reserved.

Jel classification: G13

Keywords: Spread options; Basis options; Lognormal; Normal

1. Introduction

Wealth management firms claim to provide tailor-made solutions to a client's unique financial situations. The assertion is that custom-made financial advice is provided within the context of a client's unique preferences. These assertions and claims, when actual practices are examined, often fall woefully short.

As evidence, consider that neither the client, nor the wealth management firm can

* Corresponding author. Tel.: +1-662-325-7477; fax: +1-662-325-1977.
E-mail address: brandon.cline@msstate.edu (B.N. Cline)

articulate the relationship between the behavior of the deployed investment portfolio and the behavior of the client's liabilities. We use the term liabilities here to include both a family's contractual liabilities as well as intended uses for funds (e.g., college tuition, charitable giving, and retirement). We advocate wealth management firms take an asset-liability management approach, an approach successfully used by financial institutions and corporations for decades.

One motivation of this study is to provide better tools for managing the downside risk related to the spread between the asset portfolio and liability portfolio. We seek tools that serve the client better and afford the wealth management firm confidence in their value-added proposition, regardless of financial market behavior. Specifically, we explore managing the spread between assets and liabilities with a particular interest in spread options. Spread options are ideally suited for measuring and managing spread risk exposure. The value of spread options and related risk measures provide useful information regarding the current cost of insuring adverse moves in the spread. Many other authors have sought to connect modern quantitative finance tools and practices to improving individual investor performance. See, for example, Dubil (2004, 2007) and Johnston, Hatem, and Scott (2013). Kyrychenko (2008) sought to incorporate nonfinancial assets into the optimal asset allocation process. We follow a similar strategy, but also include liabilities with a focus on downside risk.

Based on prior academic research, closed form solutions for valuing European-style spread options do not exist when the underlying instruments are lognormally distributed.[1] Consequently, numerical techniques and approximations must be used for these option pricing models. Industry practice is to model spread options assuming the spread is *normally* distributed even when the underlying distributions are known to be non-normal. This assumption is often made because the spread can be and often is negative. For example, when preferred retirement living standards are considered liabilities, then in the context of financial planning, spreads are often negative.

Although assuming the normal distribution is pragmatic, this internal inconsistency creates significant integrity concerns for the risk management systems of many financial institutions. These integrity concerns are especially manifest for risk management of large portfolios and may result in risk measurement errors for individual investors.

For options on underlying instruments other than spreads, current industry practice is to model the underlying instrument options assuming they are *lognormally* distributed. This assumption is often justified by the fact that the financial instrument prices are non-negative because of limited liability. Continuous time models based on geometric Brownian motion (GBM) imply the terminal distribution of the underlying instrument is lognormal. We refer to the spread option model that assumes both underlying instruments follow the lognormal distribution as the base model for comparison purposes. Industry practice, however, is to model spread options assuming arithmetic Brownian motion (ABM) because the spreads are often negative. These continuous time models based on ABM imply the terminal distribution of the underlying instrument is normal. We refer to spread option pricing models using the normal distribution as the alternate models (to contrast them from the base model). The alternate model would be easier to deploy for individual investors and their financial planners, but the base model is common with broader industry practices.

These modeling procedures are internally inconsistent because the difference between two variables that are lognormally distributed does not follow a normal distribution.[2] If it can be shown, however, that the ABM assumption on the spread (alternate models) provides essentially the same risk-measure results as the GBM (base models), the internal inconsistency can safely be disregarded for risk management purposes. Consequently, an alternate model using ABM can be deployed that has a much larger degree of tractability; thus, minimizing the integrity concern.

In this study, we examine whether the associated risk management costs of this practical departure are significant. Specifically, the question is raised: Does implementing a more parsimonious model and erroneously assuming the spread is normally distributed significantly bias risk measures commonly used by financial institutions? Can individual investors and their financial planners use the simpler alternate model?

This article contributes to the spread option literature in three ways: First, we report a new simplified computational method for computing the value of spread options under GBM. Our option pricing model involves estimating three single integral expressions of the standard normal cumulative distribution function (CDF). This procedure involves no iterative search routines or approximations outside of estimating a single integral and the standard normal CDF. Second, we provide a practical methodology for evaluating whether two option pricing models yield different risk parameters. Option pricing models are considered different if the measure of the difference in the risk parameters is larger than the measure of the difference in the risk parameters based on the bid-offer spreads. Finally, we illustrate model comparisons by varying moneyness, time to maturity, correlation, and the strike price. We demonstrate that when the only concern is the risk parameter δ, the ABM assumption does not cause a material error when the time to maturity is short or the correlation is high (spread volatility is low). Thus, the normal assumption can be safely made and may even be preferred because it provides a greater degree of tractability. For longer maturities and lower correlations (or higher volatilities), however, the two models are shown to be materially different and poses significant problems with respect to risk management.

It is important to note that we do not address the empirical question of which spread option pricing model is most suitable for particular spread options during specific periods of history. Instead, we address a risk measurement issue. Assuming both underlying instruments are known to be lognormally distributed, by erroneously assuming the spread is normally distributed, are various risk measures significantly biased? In practice, parsimonious models are strongly favored because of the practical difficulties in estimating input parameters. In this light, the alternate model would be much easier to deploy by individual investors and financial advisory firms.

The examination of actual market data in an effort to identify the best model is not within the scope of this article for a variety of reasons. First, it is very difficult to acquire clean market data as exchange-traded spread options are thinly traded, and over-the-counter spread option data are not accessible.[3] Second, our focus here is on risk management and not spread option valuation. We assume the market option price is known and seek to better understand the behavior of various risk measures. Third, it is well known that even the best valuation model for a particular product can change over time as market participants' perspectives change. Hence, even if we provided detailed empirical results it would not necessarily be applicable for other spread options or other periods of time. Finally, for individual investors, their liability portfolios will be unique to their family. Thus, by definition, the measured spread will be unique, and hence the resultant option values and risk measures will be custom-made. The absence of tradeable options is not a major concern here as the goal is improved risk measures and better understanding of the economic costs of spread risk.

The article will proceed as follows: Section 2 presents the two models under consideration: the lognormal spread option pricing model (LNSOPM) and the normal spread option pricing model (NSOPM). Section 3 provides the methodology for comparing two option models for risk management purposes (numerical examples used to illustrate the differences in the models from a risk management perspective are also included). Section 4 contains the conclusion.

2. Models

In this section, we provide two spread option valuation models, one assuming both underlying instruments follow GBM, the second assuming both underlying instruments follows ABM. The general framework of Black and Scholes (1973), Merton (1973), and Black (1976) is followed. Before delving into the particular spread option models, a brief review of the literature will be helpful.

Margrabe (1978) provides a closed-form equation for exchange options that are zero strike spread options. Poitras (1998) advances a pricing formula for European-style spread options by extending a special case of the Bachelier (1964) option-pricing model. The Bachelier model for pricing options on futures spreads provides a methodology for pricing European-style spread options, assuming changes in the underlying futures prices follow unrestricted arithmetic Brownian motion (UABM). The assumption of UABM proves very useful in that it allows negative sample paths to exist, resulting in call options that are priced higher than under plain arithmetic Brownian motion. Poitras points out that because the differences of lognormal variables are not lognormal, a simplification is not possible. Furthermore, he states that if prices are lognormally distributed, it is only possible to have a closed form solution in the special case of an exchange option. Otherwise, some double integral approximation must be used.

While it is true that the normality assumption should be questioned, Schaefer (2002) demonstrates that option values from the Bachelier model are nearly identical to values found using Monte Carlo simulation assuming GBM. However, this is only accurate provided the spread volatility and the time to maturity are both low.

Wilcox (1990) assumes ABM to derive a closed form solution for pricing spread options. It is shown by Poitras (1998), however, that the formula is not consistent with the no-arbitrage argument; thus, it is not a valid option pricing formula. Despite the limitations of the Wilcox formula, many have used it to develop analytic approximations for spread option pricing, including Shimko (1994).

Shimko (1994) provides an analytic approximation based on the Goldman/Wilcox model and a model developed by Rubinstein (1991). The Rubinstein model values options on futures spreads using a double-integral solution where both underlying futures contracts follow GBM. Like Rubinstein, Shimko assumes underlying prices follow GBM. By applying the Jarrow and Rudd (1982) approximations to the Wilcox model, Shimko approximates a single-integral solution for pricing options on futures spreads. At the same time, he assumes a stochastic convenience yield, thus overcoming the limitations of the Wilcox model and the complexity of the Rubinstein model. As a result, Shimko (1994) provides the formula to approximate the "true" lognormal solution.

Schaefer (2002), motivated by the fact that there were no closed form solutions for pricing options on futures spreads under the assumption that the assets follow correlated geometric Brownian motion, provides an analytic approximation for such options. Schaefer compares the Bachelier model to Monte Carlo simulation and the binomial methods that assume the spreads follow correlated geometric Brownian motion. For options on futures spreads with two or three underlying assets, his results indicate that both the Bachelier model and the analytic approximation provide solutions consistent with Monte Carlo simulation and binomial methods. He notes that as volatility and time to maturity increase the disparities between the models become much more pronounced, a result consistent with those reported here. Other analytic approximations have been published, including Alexander and Scourse (2004), Alexander and Venkatramanan (2007, 2012), Benth and Saltyte-Benth (2006), Li, Deng, and Zhou (2008), Carmona and Durrleman (2003, 2006), and Dempster and Hong (2002). Brooks (1995) provides a quadrinomial lattice approach to valuing spread options. Heenk, Kemna, and Vorst (1990) explore Asian options on oil spreads.

Pearson (1995) presents an efficient approximation to pricing spread options assuming the two underlying instruments are lognormally distributed. Pearson shows that the double integral can be reduced to a single integral (not counting the traditional N(d) integral). He proceeds to offer an efficient approximation to the resulting expression. The advantage of the lognormal model presented below is that it is not an approximation. Unfortunately, it does require integration of functions of the standard normal distribution. Because very accurate approximations exist for the N(d) integral, however, standard single dimensional integration can be applied.

Borovkova, Permana, and Weide (2007) offer a spread option model based on a shifted lognormal distribution and then derive approximation formulas based on moments matching. Carmona and Durrleman (2003) provide a detailed overview of spread option pricing models and their uses as well as an approximation formula. The two option pricing models for spread options are reviewed before we turn to appraising model differences from a risk management perspective.

2.1. Lognormal spread option pricing model (LNSOPM)

We briefly review the assumptions and our notation for spread options. The general payoff at expiration, T, of a spread option can be expressed as:

$$CSO_T = \max[0, \alpha_1 I_{1,T} + \alpha_2 I_{2,T} - X], \quad (1)$$

$$PSO_T = \max[0, X - \alpha_1 I_{1,T} - \alpha_2 I_{2,T}], \quad (2)$$

where

CSO_T denotes the call option value at time T,
PSO_T denotes put option value at time T,
$\alpha_1 > 0$ denotes positive constant (index 1 coefficient),
$\alpha_2 < 0$ denotes negative constant (index 2 coefficient),
$-\infty < X < \infty$ denotes strike price,
$I_{1,T}$ denotes the value of index 1 at time T (stochastic), and
$I_{2,T}$ denotes the value of index 2 at time T (stochastic).

If we assume indexes follow GBM with geometric drift, then

$$dI_j = (\hat{\mu}_j - \delta_j)I_j dt + \hat{\sigma}_j I_j dz_j; j = 1,2, \tag{3}$$

where

$-\infty < \hat{\mu}_j < \infty$ denotes the mean growth rate of index j,
$-\infty < \delta_j <$ denotes the carry costs related to index j,
$\sigma_j < \infty$ denotes the *SD* of index j, and
$-\infty < dz_j < \infty$ denotes the standard Wiener process associated with index j.

The value of spread option today can be expressed generically as,

$$SO_0 = PV[E_0(SO_T)], \tag{4}$$

where the expectation is taken under the equivalent martingale measure (standard finance assumptions are made; see Appendix A). The value of call and put options, using the *lognormal* distribution can be expressed as (base models):

$$CSO_1(I_{1,0}I_{2,0},X,T,\sigma_1,\sigma_2,\rho_{1,2},r) = \exp\{-rT\}\left[\int_0^\infty\int_0^\infty \max[0,\alpha_1 I_{1,T} + \alpha_2 I_{2,T} - X] f_1(I_1,I_2) dI_2 dI_1\right], \tag{5}$$

$$PSO_1(I_{1,0}I_{2,0},X,T,\sigma_1,\sigma_2,\rho_{1,2},r) = \exp\{-rT\}\left[\int_0^\infty\int_0^\infty \max[0,X - \alpha_1 I_{1,T}\right.$$

$$\left. + \alpha_2 I_{2,T}] f_1(I_1,I_2) dI_2 dI_1 \right], \tag{6}$$

where

1	=	(subscript) denotes the LNSOPM,
r	=	risk-free interest rate; annualized with continuous compounding, and
$f_1(I_1,I_2)$	=	bivariate lognormal density function.

Although a closed form solution to the LNSOPM does not exist, there are several single integral representations. Again, we assume the standard finance assumptions that afford using the risk-free rate as the mean for both indexes (see Appendix A). For example, the following single integral version of LNSOPM is used with a standard numerical integration methodology.[4]

$$CSO_{1,0}(I_1,I_2) = I_1 e^{-\delta_1 T} \int_{-\infty}^{\infty} N(d_{1,1}(z)) n(z) dz + \frac{\alpha_2 I_2}{\alpha_1} e^{-\delta_2 T} \int_{-\infty}^{\infty} N(d_{1,2}(z)) n(z) dz$$

$$- \frac{X}{\alpha_1} e^{-rT} \int_{-\infty}^{\infty} N(d_2(z)) n(z) dz, \tag{7}$$

$$PSO_{1,0}(I_1,I_2) = CSO_{1,0}(I_1,I_2) - \alpha_1 I_1 e^{-\delta_1 T} - \alpha_2 I_2 e^{-\delta_2 T} + X e^{-rT}, \tag{8}$$

where

$$n(z) = \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}, \tag{9}$$

$$N(d_i(z)) = \int_{-\infty}^{d_i(z)} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx, \tag{10}$$

$$d_{1,1}(z) = \frac{ln\left[\frac{\alpha_1 I_1 e^{(r-\delta_1)T + \rho^2 \frac{\sigma_1^2 T}{2} + \rho\sigma_1\sqrt{T}z}}{X - \alpha_2 I_2 e^{(r-\delta_2)T - \frac{\sigma_2^2 T}{2} + \rho\sigma_1\sigma_2 T + \sigma_2\sqrt{T}z}}\right] + (1-\rho^2)\frac{\sigma_1^2 T}{2}}{\sigma_1\sqrt{T(1-\rho^2)}} \tag{11}$$

$$d_{1,2}(z) = \frac{ln\left[\frac{\alpha_1 I_1 e^{(r-\delta_1)T - \rho^2\frac{\sigma_1^2 T}{2} + \rho\sigma_1\sigma_2 T + \rho\sigma_1\sqrt{T}z}}{X - \alpha_2 I_2 e^{(r-\delta_2)T - \frac{\sigma_2^2 T}{2} + \sigma_2\sqrt{T}z}}\right] - (1-\rho^2)\frac{\sigma_1^2 T}{2}}{\sigma_1\sqrt{T(1-\rho^2)}}, \quad \text{and} \tag{12}$$

$$d_2(z) = \frac{ln\left[\frac{\alpha_1 I_1 e^{(r-\delta_1)T - \rho^2\frac{\sigma_1^2 T}{2} + \rho\sigma_1\sqrt{T}z}}{X - \alpha_2 I_2 e^{(r-\delta_2)T - \frac{\sigma_2^2 T}{2} + \sigma_2\sqrt{T}z}}\right] - (1-\rho^2)\frac{\sigma_1^2 T}{2}}{\sigma_1\sqrt{T(1-\rho^2)}}. \tag{13}$$

It is important to emphasize that this solution is not an approximation like Pearson (1995), Carmona and Durrleman (2003, 2006), Li, Deng, and Zhou (2008) and others, rather it is an exact result. We do not, however, claim it is closed-form in the usual finance sense. It is still technically a double integral (recall the standard N(d) function is an integral). However, practically it is a single integral because of the existence of very accurate numerical approximations available to compute the standard N(d) function. Even the standard Black, Scholes, Merton option pricing model requires some sort of numerical approximation to N(d).

2.2. *Normal spread option pricing model (NSOPM)*

If we assume that the spread follows ABM with geometric drift, then

$$dS = \mu_S S dt + \sigma_S dz_S, \tag{14}$$

where

$-\infty < \mu_S < \infty$ denotes the mean growth rate of the spread,
$\sigma_S < \infty$ denotes the *SD* of the spread (same units of measure as S), and
$-\infty < dz_S < \infty$ denotes the standard Wiener process associated with the spread.

The value of call and put spread options, based on the *normal* distribution can be expressed as (alternate model):

$$CSO_n(I_{1,0}, I_{2,0}, X, T, \sigma_1, \sigma_2, \rho_{1,2}, r) = \exp\{-rT\}\left[\int_0^\infty\int_0^\infty \max\left[0, \alpha_1 I_{1,T} + \alpha_2 I_{2,T} - X\right] f_n(I_1, I_2)\, dI_2\, dI_1\right], \tag{15}$$

$$PSO_n(I_{1,0},I_{2,0},X,T,\sigma_1,\sigma_2,\rho_{1,2},r) = \exp\{-rT\}\left[\int_0^\infty\int_0^\infty \max\left[0,X - \alpha_1 I_{1,T} + \alpha_2 I_{2,T}\right] f_n(I_1,I_2)\,dI_2\,dI_1\right], \quad (16)$$

where

n = denotes the NSOPM, and
$f_n(I_1,I_2)$ = bivariate normal density function.

The advantage of the normal distribution is that the difference between normally distributed random variables is also normally distributed. Hence, there *does* exist a closed form solution to NSOPM.

Note that the spread is normally distributed and is denoted as:

$$S_T = \alpha_1 I_{1,T} + \alpha_2 I_{2,T}\,. \quad (17)$$

So that the expected terminal spread is:

$$E[S_T] = \alpha_1 I_{1,0} e^{(\hat{\mu}_1-\delta_1)T} + \alpha_2 I_{2,0} e^{(\hat{\mu}_2-\delta_2)T} = Se^{\mu_s T}\,. \quad (18)$$

The variance of the spread is:

$$V[S_T] = \sigma_S^2 \frac{e^{2\mu_S T} - 1}{2\mu_S}\,, \quad (19)$$

and σS is the *SD* of changes in the spread. We assume the usual finance conditions that afford using the risk-free rate (see Appendix A). Therefore, we have the following version of NSOPM (alternate model):

$$CSO_{n,0}(I_1,I_2) = e^{-rT}[\{E[S_T] - X\}N(d_n) + V[S_T]^{1/2} n(d_n)], \quad (20)$$

$$PSO_{n,0}(I_1,I_2) = CSO_{n,0}(I_1,I_2) - \alpha_1 I_1 e^{-\delta_1 T} - \alpha_2 I_2 e^{-\delta_2 T} + Xe^{-rT}\,, \quad (21)$$

where

$$n(d_n) = \frac{e^{\frac{-d_n^2}{2}}}{\sqrt{2\pi}}\,, \quad (22)$$

$$N(d_n) = \int_{-\infty}^{d} \frac{e^{\frac{-d_n^2}{2}}}{\sqrt{2\pi}} dx, \text{ and} \tag{23}$$

$$d_n = \frac{E[S_T] - X}{V[S_T]^{1/2}}. \tag{24}$$

3. Analysis of model differences

As previously discussed, this article does not seek to address a particular spread option, rather spread options in general. Therefore, rather than start with a set of option prices and calibrate both models, we generate option prices with the LNSOPM and calibrate the NSOPM so it generates the same original option prices. The advantage of this approach is that the NSOPM has only one spread option volatility parameter (under the normal distribution, volatility is in units not percentage). When we compare appropriately calibrated option models, we are not assuming one model is "correct." In practice, option prices are observable in the marketplace, and an option pricing model is "calibrated" to the market option price. For example, one could compute the implied volatility. Our objective, however, is not to conduct an empirical test of a particular option contract. Our objective is solely to explore the implications for risk management by comparing the lognormal and normal models.

Therefore, we begin this analysis by calibrating the alternate model option price using the base model price. Calibration is a common practice for risk management applications. Traders take prices as given and then use models to infer risk parameters. We examine the resulting differences in risk parameters from the base GBM and the alternate ABM for spread options. By varying the parameters of the option models, we examine under what circumstances the risk parameters are significantly different.

Recall the objective here is to examine whether assuming the spread is normally distributed, when the two underlying instruments are lognormally distributed, results in materially different risk parameters. The LNSOPM requires three volatility parameters, the percentage *SD* of each asset and the correlation between assets. The NSOPM only requires one volatility parameter, the per unit *SD* of the spread. Hence, we assume the base model option prices (LNSOPM) are market prices and then calibrate the alternate model (NSOPM). For example, using the parameters specified later in Table 1, the spread volatility is \$20.8. However, the implied volatility from the NSOPM is \$19.4. The \$19.4 volatility generates the call price of \$7.56 as do the LNSOPM parameters.

If the distribution of the underlying instruments is normal, then the implied volatility of the spread option would match perfectly with the analytic spread volatility. The underlying instruments, however, are assumed to follow a lognormal distribution, and hence the difference of lognormal distributions is not lognormal. Thus, the implied volatility would not be expected to equal the analytic spread volatility. The spread option value computed with

Table 1 Varying moneyness
Panel A: Analysis of δ

Moneyness	Call δ		Put δ	
	Error factor	Significance factor	Error factor	Significance factor
−30	0.2167	0.2340	0.0075	0.0098
−20	0.1572	0.1718	0.0234	0.0292
−10	0.1097	0.1212	0.0483	0.0588
0	0.0728	0.0813	0.0785	0.0942
10	0.0455	0.0513	0.1108	0.1318
20	0.0266	0.0302	0.1429	0.1696
30	0.0144	0.0165	0.1741	0.2064

Panel B: Analysis of γ and θ

Moneyness	Call and put γ		Call and put θ	
	Error factor	Significance factor	Error factor	Significance factor
−30	0.3574*	0.1722	0.0474	0.1920
−20	0.2204*	0.0984	0.0474	0.1182
−10	0.1002*	0.0999	0.0474	0.0529
0	0.0060	0.0255	0.0474*	0.0056
10	0.1005*	0.0784	0.0474	0.0585
20	0.1850*	0.1266	0.0474	0.1068
30	0.2611*	0.1709	0.0474	0.1511

Panel C: Analysis of vega and correlation

Moneyness	Call and put vega		Correlation δ	
	Error factor	Significance factor	Error factor	Significance factor
−30	333.2600*	0.1920	0.0785	0.1920
−20	3.1001*	0.1182	0.1020	0.1182
−10	0.7135*	0.0529	0.1136*	0.0529
0	0.0819*	0.0056	0.1169*	0.0056
10	0.4830*	0.0585	0.1142*	0.0585
20	0.7265*	0.1068	0.1070*	0.1068
30	0.8911*	0.1511	0.0963	0.1511

This table reports the error and significance factor for selected risk parameters for the LNSOPM and NSOPM when the option prices are calibrated together. Moneyness of the option is varied from −30 to 30. Time to maturity is set equal to one year, volatility is 30% for both indexes, the risk-free rate is 5%, no dividends are assumed, and the correlation coefficient is 0.8. Index 2 is set to 100.0 and the strike price is set to zero. Index 1 is varied in increments of 10.0, allowing moneyness to vary from −30.0 to 30.0. Trading costs are assumed to be one percentage for both assets; hence, buying the spread would result in a one percentage decline in index 1 and a one percentage increase in index 2. *Denotes the error factor being greater than the significance factor.

the LNSOPM will reflect the non-normal distribution. When the normal distribution is imposed, the NSOPM implied volatility will adjust the assumed normal distribution to reflect the LNSOPM price.

Once we have both the base and alternate models generating the same option prices, we estimate the risk parameters. It is important to emphasize that risk management is not directly an exercise in option valuation. Rather, option valuation models are used to estimate risk parameters, such as δ, γ, θ, and vega. Finally, we assess the magnitude of the risk parameter difference. By varying the initial inputs, we demonstrate that the risk parameter differences between the two models are often not materially different.

However, the question arises: What is materially different? Specifically, from a risk management perspective, what dictates a significant difference in risk parameters of the models being compared? Are δs of 0.53 and 0.54 materially different? Our measure of materiality is based on the impact of trading costs on the value of risk parameters.

3.1. Significance measure

The bid-ask spread represents an approximation of the marginal cost to the market maker for rebalancing the portfolio. There are clearly other costs, such as fixed costs, market impact costs, and other variable costs. To apply the methodology below with these other costs, one would have to appropriately adjust the estimated bid-ask spread. Our goal here is not to decompose the total spread option price change into the Greek components, rather merely to acknowledge that trading costs impact the ability to actually implement a hedging strategy. The higher the bid-ask spread, the less frequently the portfolio would be rebalanced, and therefore, the risk parameters need not be as precise. When applied by individual investors, the goal is merely to estimate the current market price of mitigating the risk of a decline in the spread between assets and liabilities.

Our significance measure between the risk parameters of the base and alternate models, therefore, is a function of the bid-ask spread. Consider the following measure of significance. *Significance Measure:* Assuming the market maker has the ability to rebalance the portfolio, an insignificant difference in risk parameter (RP) of the base and alternate models being compared exists if the risk parameter error factor between the models is less than the significance factor.

The significance factor of the base model is:

$$\hat{S} = \left| \frac{RP_{Base,Ask}(T) - RP_{Base,Bid}(T)}{\left[\dfrac{RP_{Base,Ask}(T) + RP_{Base,Bid}(T)}{2}\right]} \right|, \tag{25}$$

and the risk parameter error factor between the base and alternate model is:

$$\varepsilon_{RP} = \left| \frac{RP_{Base}(T) - RP_{Alt}(T)}{\left[\dfrac{RP_{Base}(T) + RP_{Alt}(T)}{2}\right]} \right|. \tag{26}$$

Thus, $\varepsilon_{RP} < \hat{S}$ implies $RP_{Base} \approx RP_{Alt}$, where

$RP_{Base,Ask}$ = risk parameter of the option at the "ask" price, base model,
$RP_{Base,Bid}$ = risk parameter of the option at the "bid" price, base model,
RP_{Base} = risk parameter of base model,
RP_{Alt} = risk parameter of alternate model,
T = assumed transaction cost, and
$\approx$ = materially indistinguishable.

The significance factor is a practical measure that seeks to incorporate how sensitive risk parameters are to the value of the underlying assets. If small deviations from the current market price of the underlying asset result in dramatic changes in the risk measures, then it will be difficult to hedge the position when trading costs are high. Therefore, the significance factor measures how sensitive the risk parameter is to changes in the underlying asset price because of the bid-offer spread. The error factor measures the deviation of the risk parameter between the two option models. Thus, an error factor that is lower than the significance factor would be deemed nonmaterial, whereas an error factor in excess of the significance factor would be deemed material.

3.2. Illustrations

We illustrate δ, γ, θ, vega, and correlation δ by varying four parameters: (1) moneyness, (2) time to maturity, (3) correlation, and (4) strike price. Before reviewing these results, a few words regarding these particular derivatives are in order. First, we numerically estimate these derivatives using a highly accurate procedure.[5] The δ, γ, and vegas reported are in terms of the first index value. We also compute the δ, γ, and vegas of the second term, but they are not reported here. Second, correlation δ measures the change in the spread option with respect to a change in correlation. Third, with the NSOPM model, volatility is the *SD* of the spread. Using an iterative search method, we estimate the implied correlation coefficient that yields the implied *SD* of the NSOPM model. Therefore, we can estimate vega and correlation δ for the NSOPM model.

3.2.1. Moneyness

For the first illustration, we vary the moneyness of the option while holding all the other parameters fixed to examine the difference between the risk parameters of the base and alternate spread option pricing models. Moneyness of the option is varied from -30 to 30 units (e.g., dollars), by varying the value of index 1. Time to maturity is fixed at one year, percentage volatility is 30% for each index, correlation is 0.8, strike price is 0.0, and the risk-free rate is 5%. A one percentage trading cost is assumed on both indexes, according to the method discussed in the previous section. By adjusting volatility, the option price of the NSOPM is calibrated to the price of the LNSOPM. The risk parameters are then calculated, and the error and significance factor are reported.

Table 1 reports the results of five risk parameters: δ, γ, θ, vega, and correlation δ. We see that for the parameters selected the δs are not materially different. That is, the error in δ

between the two models is not greater than the difference in δ of the lognormal model, given a one percentage transaction cost. Note that both the error factor and the significance factor are monotonically decreasing in moneyness for calls and monotonically increasing in moneyness for puts. Because call (put) δs increase with moneyness, the percentage difference declines (increases).

The value of γ is maximized around at-the-money spread options. Hence, we observe the error factor and the significance factor are both minimized when the options are at-the-money. Gamma is materially different, however, when the spread option is either in- or out-of-the-money. Very small changes in γ are significant when the value of γ is very small. Both in-the-money and out-of-the-money spread options result in small γs and hence, the error factor is larger than the significance factor.

The error factor for θ is not influenced by moneyness, but the significance factor is influenced. Therefore, θ is materially different at-the-money because the significance factor is minimized. Vega error factors are significant in all cases in Table 1. Again, the significance factor is minimized at-the-money.[6] Vega significance is the same for both calls and puts based on put-call parity for spread options. The risk parameter estimates are dramatically different between these two models. Time decay behavior is different between the lognormal and normal models, in part, because of its influence on volatility.

Volatility risk is difficult to manage, particularly for thinly traded options. We see here that the NSOPM measure of vega is materially different from the LNSOPM. Recall that the NSOPM has only one input for volatility, the volatility of the spread. The LNSOPM, however, has three inputs, the volatilities of each underlying asset and the correlation. Therefore, there are several different measures of volatility risk that can be computed for the LNSOPM. We assume the volatility change was driven by a change in the volatility of the first asset. The goal here is to identify that volatility risk is an important consideration and can be assessed using the significance measure. Correlation δ is also significant when moneyness is around zero.

3.2.2. Time to maturity

Next we vary the time to maturity of the option while holding the other model parameters fixed to examine the difference between the risk parameters of the base and alternate spread option pricing models. Time to maturity of the spread option is varied from 0.25 to 2 years. The remaining parameters are the same as in Table 1. Index 1 and index 2 are each assumed to have values of 100.0 and the strike price is 0, resulting in an at-the-money spread option.

Table 2 reports the results of this case. We observe for the parameters selected the δs are materially different for time horizons greater than one year. That is, the error in δ between the two models is not greater than the difference in δ of the lognormal model, given a one percentage transaction cost, for maturities less than a year. Note that the error factor is monotonically increasing with time to maturity, whereas the significance factor is monotonically decreasing with time to maturity. Hence, past some inflection point the models are materially different. For longer maturities, the two model δs deviate further, whereas the impact of trading costs on δs declines.

The error factors for γ in this case are very small and the significant factors are much larger. Because γs are relatively large when the options are at-the-money, and trading costs

Table 2 Varying time to maturity

Panel A: Analysis of δ

Time to maturity	Call δ		Put δ	
	Error factor	Significance factor	Error factor	Significance factor
0.25	0.0371	0.1757	0.0386	0.1888
0.50	0.0521	0.1195	0.0550	0.1325
0.75	0.0634	0.0954	0.0677	0.1084
1.00	0.0728	0.0813	0.0785	0.0942
1.25	0.0810*	0.0788	0.0882*	0.0846
1.50	0.0884*	0.0647	0.0970*	0.0076
1.75	0.0951*	0.0593	0.1051*	0.0722
2.00	0.1013*	0.0549	0.1127*	0.0678

Panel B: Analysis of γ and θ

Time to maturity	Call and put γ		Call and put θ	
	Error factor	Significance factor	Error factor	Significance factor
0.25	0.0015	0.0422	0.0118	0.0223
0.50	0.0030	0.0311	0.0236*	0.0112
0.75	0.0045	0.0274	0.0355*	0.0075
1.00	0.0060	0.0255	0.0474*	0.0056
1.25	0.0075	0.0244	0.0594*	0.0045
1.50	0.0090	0.0237	0.0714*	0.0038
1.75	0.0105	0.0231	0.0835*	0.0032
2.00	0.0120	0.0227	0.0956*	0.0028

Panel C: Analysis of vega and correlation

Time to maturity	Call and put vega		Correlation δ	
	Error factor	Significance factor	Error factor	Significance factor
0.25	0.0144	0.0223	0.9990*	0.0223
0.50	0.0374*	0.0112	0.5338*	0.0112
0.75	0.0599*	0.0075	0.1719*	0.0075
1.00	0.0819*	0.0056	0.1169*	0.0056
1.25	0.1035*	0.0045	0.3518*	0.0045
1.50	0.1245*	0.0038	0.5462*	0.0038
1.75	0.1452*	0.0032	0.7091*	0.0032
2.00	0.1654*	0.0028	0.8472*	0.0028

This table reports the error and significance factor for selected risk parameters for the LNSOPM and NSOPM when the option prices are calibrated together. Volatility is 30% for both indexes, the risk-free rate is 5%, no dividends are assumed, and the correlation coefficient is 0.8. Index 1 and index 2 are set to 100.0 and the strike price is set to zero. Time to maturity is varied in increments 0.25, allowing time to maturity to vary from 0.25 to 2.00. Trading costs are assumed to be one percentage for both assets; hence, buying the spread would result in a one percentage decline in index 1 and a one percentage increase in index 2. *Denotes the error factor being greater than the significance factor.

do not considerably influence option value, γs are not materially different. For longer maturities, the γs mildly drift further apart.

Thetas and vegas are materially different for maturities of 0.5 or greater, and correlation δs are all materially different for all maturities. Clearly, the longer the time to maturity for at-the-money options, the greater volatility impacts the difference between these two models.

3.2.3. Correlation

We now vary the correlation of the option while holding the other parameters fixed to examine the difference between the risk parameters of the base and alternate spread option pricing models. Correlation of the spread option is varied from −0.99 to 0.99. The remaining parameters are again the same as in Table 1. We assume the rest of the parameters as previously reported. Varying the correlation is similar to varying the volatility because a spread option becomes more volatile as the correlation between the two indexes declines (holding the other parameters constant).

Table 3 reports the results of this case. We see that for the parameters selected the δs are materially different when correlation declines (or volatility is increased). That is, the error in δ between the two models is greater than the difference in δ of the lognormal model, given a one percentage transaction cost. Note that the error factor is monotonically decreasing with increasing correlation, and the significance factor is monotonically increasing. Hence, past some inflection point the models are no longer materially different. Therefore, as correlation declines (or volatility increases), the two models are materially different with respect to δ.

Gamma is materially different for correlations of 0.25 or below, whereas θ and vega are materially different for correlations of 0.75 or below. Finally, correlation δ is materially different in all cases. Recall that the lower the correlation the greater the spread volatility.

3.2.4. Strike price

We now vary the strike price of the option while holding the other parameters fixed to examine the difference between the risk parameters of the base and alternate spread option pricing models. Again, the market is assumed to provide current market values and these values are used to calibrate the alternate model so that both the base and alternate models yield the same price. Varying the strike price is different than varying the index values as provided in Table 1 because of the influence of positive and negative strike prices. The strike price is varied from −30 to 30 by units of 10. The remaining parameters are the same as in Table 1.

Table 4 reports the results of this case. We see that for the parameters selected the δs are not materially different, similar to Table 1. That is, the error in δ between the two models is less than the difference in δs of the lognormal model (Significance Factor), given a one percentage transaction cost. Note that the error factor is monotonically increasing with the strike price for calls and monotonically decreasing for puts.

The significance factors for the remaining risk factors in this case are similar to those reported in Table 1. For high and low strike prices, the θs are not materially different, whereas γs are materially different. Vegas are always materially different, and correlation δs are materially different for strike prices at 20 or below.

4. Conclusion

Individual investors and their financial planners should consider modeling both client assets as well as client liabilities. This study offers better tools for measuring and managing the downside risk related to the spread between the asset portfolio and liability portfolio. We

Table 3 Varying correlation

Panel A: Analysis of correlation

Correlation	Call δ		Put δ	
	Error factor	Significance factor	Error factor	Significance factor
−0.99	0.2105*	0.0210	0.2666*	0.0337
−0.75	0.1990*	0.0228	0.2485*	0.0346
−0.50	0.1859*	0.0251	0.2284*	0.0379
−0.25	0.1714*	0.0281	0.2069*	0.0409
0.00	0.1550*	0.0321	0.1834*	0.0434
0.25	0.1359*	0.0382	0.1572*	0.0510
0.50	0.1125*	0.0483	0.1268*	0.0612
0.75	0.0810*	0.0717	0.0882*	0.0846
0.99	0.0168	0.4365	0.0171	0.4495

Panel B: Analysis of γ and θ

Correlation	Call and put γ		Call and put θ	
	Error factor	Significance factor	Error factor	Significance factor
−0.99	0.0596*	0.0205	0.0205*	0.0006
−0.75	0.0524*	0.0206	0.0241*	0.0007
−0.50	0.0449*	0.0207	0.0279*	0.0008
−0.25	0.0375*	0.0208	0.0316*	0.0009
0.00	0.0300*	0.0211	0.0354*	0.0012
0.25	0.0225*	0.0214	0.0319*	0.0015
0.50	0.0150	0.0222	0.0429*	0.0023
0.75	0.0075	0.0224	0.0467*	0.0045
0.99	0.0003	0.1309	0.0503	0.1111

Panel C: Analysis of vega and correlation

Correlation	Call and put vega		Correlation δ	
	Error factor	Significance factor	Error factor	Significance factor
−0.99	0.0402*	0.0006	0.0114*	0.0006
−0.75	0.0463*	0.0007	0.0018*	0.0007
−0.50	0.0524*	0.0008	0.0163*	0.0008
−0.25	0.0584*	0.0009	0.0317*	0.0009
0.00	0.0642*	0.0012	0.0484*	0.0012
0.25	0.0699*	0.0015	0.0669*	0.0015
0.50	0.0754*	0.0023	0.0876*	0.0023
0.75	0.0809*	0.0045	0.1116*	0.0045
0.99	0.0859	0.1111	0.1389*	0.1111

This table reports the error and significance factor for selected risk parameters for the LNSOPM and NSOPM when the option prices are calibrated together. The initial parameters are the same as the previous tables. Correlation is varied in increments of 0.25, allowing correlation to vary from −0.99 to 0.99. *Denotes the error factor being greater than the significance factor.

provide tools that serve the client better and afford the wealth management firm confidence in their value-added proposition, regardless of financial market behavior. Specifically, spread options and related risk measures form a useful framework for articulating client portfolio performance.

Table 4 Varying strike price

Panel A: Analysis of δ

Strike price	Call δ		Put δ	
	Error factor	Significance factor	Error factor	Significance factor
−30	0.0131	0.0154	0.1601	0.1938
−20	0.0259	0.0290	0.1402	0.1640
−10	0.0462	0.0510	0.1113	0.1301
0	0.0728	0.0813	0.0785	0.0942
10	0.1026	0.1156	0.0497	0.0614
20	0.1339	0.1490	0.0292	0.0364
30	0.1665	0.1787	0.0163	0.0201

Panel B: Analysis of γ and θ

Strike price	Call and put γ		Call θ		Put θ	
	Error factor	Significance factor	Error factor	Significance factor	Error factor	Significance factor
−30	0.2377*	0.1589	0.3120	2.1101	0.0365	0.1463
−20	0.1801*	0.1217	0.0641	0.1742	0.0402	0.1072
−10	0.1000	0.0769	0.0513	0.0700	0.0442	0.0602
0	0.0060	0.0255	0.0474*	0.0056	0.0474*	0.0056
10	0.0866*	0.0268	0.0442	0.0495	0.0513	0.0570
20	0.1711*	0.0738	0.0402	0.0969	0.0641	0.1485
30	0.2500*	0.1133	0.0355	0.1365	0.3026	07110

Panel C: Analysis of vega and correlation

Strike price	Call and put vega		Correlation δ	
	Error factor	Significance factor	Error factor	Significance factor
−30	2.2728*	0.2451	0.1197	0.1411
−20	1.4233*	0.0781	0.1229*	0.1030
−10	0.6649*	0.0543	0.1197*	0.0574
0	0.0819*	0.0056	0.1169*	0.0056
10	0.2932*	0.0478	0.1197*	0.0463
20	0.5073*	0.0963	0.1229*	0.0924
30	0.6279*	0.1367	0.1200	0.1309

This table reports the error and significance factor for selected risk parameters for the LNSOPM and NSOPM when the option prices are calibrated together. The strike price is varied from −30 to 30 in increments of 10. Volatility is 30% for both indexes, the risk-free rate is 5%, no dividends are assumed, time to maturity is one year, and the correlation coefficient is 0.8. Index 1 and index 2 are set to 100.0 and the strike price is set to zero. Trading costs are assumed to be one percentage for both assets; hence, buying the spread would result in a one percentage decline in index 1 and a one percentage increase in index 2. *Denotes the error factor being greater than the significance factor.

To meet practical demands many financial institutions model spread options assuming the spread is *normally* distributed even when the underlying distributions are known to be non-normal. We investigate this internally inconsistent practice and test the difference between a spread option pricing model where both underlying instruments are lognormally distributed, and one where both underlying instruments are normally distributed.

Through four sets of option values we demonstrate that the differences between these two spread option pricing models' risk parameters are often not significantly different. As a result, the normal assumption on the underlying can be made in those cases, because it provides a greater degree of tractability. They are, however, significantly different in many other cases. For longer maturities and lower correlations (or higher volatilities), the two models are shown to be materially different. The particular model resulting in the most accurate risk parameters is an empirical issue and likely period and product specific. For financial planners, the goal is to structure asset portfolios that are highly correlated with the known liability portfolios, thus, raising the correlation. Furthermore, downside risk management tends to be focused on the short term. With these two insights, personal financial planners can safely use the simpler ABM model. Therefore, the results presented here provide the foundation for improved client portfolio management by financial planners.

Notes

1 One exception is special cases such as exchange options (see Margrabe (1978)). When referring to "closed form," we use the finance definition of an expression whose numerical complexity is no greater than an "easy to compute" function of cumulative normal distribution functions.

2 Poitras (1998) points out that because the difference of lognormal variables is not lognormal, a simplification of the Bachelier model is not possible.

3 A review of 16 articles related to spread option valuation revealed only one article used actual spread option data. Alexander and Venkatramanan (2007) used less than one year of crack spread option data.

4 Integral solving routines, such as Mathcad, can be used to find reduced form results such as this one. Although complex in appearance, N(d) is easily approximated and standard univariate integration routines can be used. Because bivariate integration is often unstable and therefore unreliable, this single integral solution is very useful.

5 See Eberly (2008). We use a centered difference approach with order of accuracy 4. Order of accuracy 1 assumes ±h (increment) as well as no increment, order of accuracy 2 also includes ±2h, and so forth. Hence, order of accuracy 4 includes ±4h, ±3h, ±2h, ±h, and no increment.

6 Note that the significance factor is the same for θ, vega, and correlation δ because of the options having a zero strike price.

Appendix A: Standard finance assumptions

Consider the standard set up for modeling prices (see, e.g., Harrison and Kreps (1979) and Harrison and Pliska (1981)):

1. $[0,\hat{\tau}]$, for fixed $\hat{\tau} \geq t \geq 0$, finite time horizon, a finite horizon economy, $0 \leq t \leq \hat{\tau}$.
2. $(\Omega, \Im, P)$, uncertainty is characterized by a complete probability space, where the state space Ω is the set of all possible realizations of the stochastic economy between time

0 and time $\hat{\tau}$ and has a typical element ω representing a sample path, $\Im$ is the sigma field of distinguishable events at time $\hat{\tau}$, and P is a probability measure defined on the elements of $\Im$.

3. $F=\{\Im(t):t\in[0,\ \hat{\tau}]\}$ the augmented, right continuous, complete filtration generated by the appropriate stochastic processes in the economy, and assume that $\Im(\hat{\tau})=\Im$. The augmented filtration, $\Im(t)$, is generated by Z. $\Im(0)$ contains only Ω and the null sets of P.
4. F is generated by a K-dimensional Brownian motion, $Z(t)=[Z_1(t),\ldots,Z_K(t)],t\in[0,\hat{\tau}]$ is defined on $\{\Omega,\Im,P\}$, where $\{\Im(t)\},t\in[0,\hat{\tau}]$ is the augmentation of the filtration $\{\Im^Z(t)\}t\in[0,\hat{\tau}]$ generated by $Z(t)$, and satisfies the usual conditions.
5. $E_P(\cdot)$ denotes the expectation with respect to the probability measure P.
6. All stated equalities or inequalities involving random variables hold P-almost surely.
7. P is common for all agents implying uniqueness of the nature of the stochastic processes.
8. Conventional perfect market conditions are also assumed, such as no transaction costs, no taxes, unrestricted short selling, and no regulatory or institutional constraints.

References

Alexander, C., & Scourse, A. (2004). Bivariate normal mixture spread option valuation. *Quantitative Finance, 4,* 637–648.

Alexander, C., & Venkatramanan, A. (2007). *Analytic Approximations for Spread Options*. ICMA Centre Discussion Papers in Finance DP2007-11, 15th August 2007, 1–23.

Alexander, C., & Venkatramanan, A. (2012). Analytic approximations for multi-asset option pricing. *Mathematical Finance, 22,* 667–689.

Bachelier, L. (1964). Theory of speculation (J. Boness, Trans.). In P. Cootner (Ed.), *The Random Character of Stock Market Prices* (pp. 17–78). Cambridge, MA: MIT Press. (Original work published 1900).

Benth, F. E., & Saltyte-Benth, J. (2006). Analytical approximation for the price dynamics of spark spread options. *Studies in Nonlinear Dynamics & Econometrics, 10,* 1334–1355.

Black, F. (1976). The pricing of commodity contracts. *Journal of Financial Economics, 3,* 167–179.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy, 81,* 637–659.

Borovkova, S., Permana F. J., & Weide, H. V. D. (2007). A closed form approach to the valuation and hedging of basket and spread options. *The Journal of Derivatives, 14,* 8–24.

Brooks, R. (1995). A lattice approach to interest rate spread options. *Journal of Financial Engineering, 4,* 281–296.

Carmona, R., & Durrleman, V. (2003). Pricing and hedging spread options. *SIAM Review, 45,* 627–685.

Carmona, R., & Durrleman, V. (2006). Generalizing the Black-Scholes formula to multivariate contingent claims. *Journal of Computational Finance, 9,* 43–67.

Dempster, M. A. H., & Hong, S. S. G. (2002). Spread option valuation and the fast fourier transform. In H. Geman, D. Madan, S. R. Pliska, & T. Vorst (Eds.), *Mathematical Finance–Bachelier Congress 2000* (pp. 203–220). Berlin: Springer.

Dubil, R. (2004). The risk and return of investment averaging: An option-theoretic approach. *Financial Services Review, 13,* 267–283.

Dubil, R. (2007). Economic derivatives markets–new opportunities for individual investors: A research agenda. *Financial Services Review, 16,* 89–104.

Eberly, D. (2008, March 2). *Derivative Approximation by Finite Differences.* Retrieved from http://www.geometrictools.com/ Documentation/FiniteDifferences.pdf.

Harrison, J. M., & Kreps, D. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory, 20,* 381–408.

Harrison, J. M., & Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process and their Applications, 11,* 215–260.

Heenk, B. A., Kemna, A. G. Z., & Vorst, A. C. F. (1990). Asian options on oil spreads. *Review of Futures Markets, 9,* 510–528.

Jarrow, R. A., & Rudd, A. (1982). Approximate option valuation for arbitrary stochastic processes. *Journal of Financial Economics, 10,* 347–369.

Johnston, K., Hatem, J., & Scott, E. (2013). Individual investors: Asset allocation vs. portfolio insurance (puts or calls). *Financial Services Review, 22,* 291–310.

Kyrychenko, V. (2008). Optimal asset allocation in the presence of nonfinancial assets. *Financial Services Review, 17,* 69–86.

Li, M., Deng, S., & Zhou, J. (2008). Closed-form approximations for spread option prices and Greeks. *Journal of Derivatives, 15,* 58–80.

Margrabe, W. (1978). The value of an option to exchange one asset for another. *Journal of Finance, 33,* 177–186.

Merton, R. C. (1973). Theory of rational option pricing. *Bell Journal of Economics, The RAND Corporation, 4,* 141–183.

Pearson, N. D. (1995). An efficient approach for pricing spread options. *The Journal of Derivatives, 3,* 76–91.

Poitras, G. (1998). Spread options, exchange options, and arithmetic Brownian motion. *Journal of Futures Markets, 5,* 487–517.

Rubinstein, M. (1991). Somewhere over the rainbow. *Risk, 4,* 63–66.

Schaefer, M. (2002, April). *Pricing and Hedging European Options On Futures Spreads Using the Bachelier Spread Option Model.* Paper presented at NCR-134 Conference on Applied Commodity Price Analysis, Forecasting, and Market Risk Management, St. Louis, MO. Retrieved from http://purl.umn.edu/19055.

Shimko, D. (1994). Options on futures spreads: Hedging, speculation, and valuation. *Journal of Futures Markets, 14,* 183–213.

Wilcox, D. (1990). Energy futures and options: Spread options in energy markets. New York, NY: Goldman Sachs & Co.