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Determining the return-maximizing portfolio leverage and its limitations

Robert A. Ott, CFA[a,*], Timothy E. Zimmer[a]

[a]*Assistant Professor, School of Business, 1400 East Hanna Avenue, Indianapolis, IN 46227, USA*

Abstract

Leverage in the risk allocation of an investment portfolio can be an effective strategy in achieving overall portfolio goals. While the literature on portfolio leverage is robust, quantifying the amount and discussion of its limitations are often minimized. This article focuses on the limitations by explicitly including the volatility drag from leveraging the expected portfolio returns. Maximizing the expected portfolio returns with respect to leverage results in a return-maximizing condition that balances the gains from leverage with the losses in the volatility drag. The return-maximizing condition is graphically illustrated over a range of investment returns to produce a return-maximizing leverage curve. © 2016 Academy of Financial Services. All rights reserved.

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1. Introduction

Behavioral finance recognizes the limitations of individual decision making when the process is complicated by an overwhelming amount of information. Rather than facilitating better decisions, the profusion of information can potentially lead to suboptimal outcomes by intimidating individual investors and acting as an impediment to the decision-making process. In this manner, information availability can be its own moral hazard that restricts individual access. To compensate and simplify the decision-making process, individuals

* Corresponding author. Tel.: +1-317-788-3317; fax: +1-317-788-3586.
E-mail address: ottr@uindy.edu (R. A. Ott)

frequently implement rules-of-thumb decision devices that approximate value-maximizing decisions.

Individuals increasingly find themselves in the role of investment decision makers, especially regarding retirement funds. As defined contribution retirement vehicles grow in number, there is an ever increasing need for individuals to gain the required knowledge to appropriately manage these accounts. According to the Employment Benefit Research Institute (2011), the use of 401(k) and Individual Retirement Accounts (IRAs) among working individuals 21 to 64 has increased from 1996 through 2009 in the United States. In 2009, about 33% of workers utilized a 401(k) and over 20% used IRAs. Providing investment tools and research in portfolio management are essential if the public are to become better stewards of their retirement accounts.

A key concern for individuals managing retirement accounts is the preservation of retirement funds while also providing for sufficient growth to maintain future purchasing power. While people wish to be conservative with retirement funds, there is a real risk of capital exhaustion during retirement. Excess caution with retirement funds management can be as disastrous to a retirement account as excessive weighting towards risk. An investment portfolio should appropriately balance risk and risk-free allocations to ensure current income and asset growth. Using leveraged investments within the risk allocation of an investment portfolio can help provide the desired asset growth. Current research assesses the validity of adding leverage to the risk allocation of an investment portfolio. This article adds to current research by deriving an algebraic equation for the return-maximizing level of leverage.

2. Review

As individuals are increasingly responsible for management of investment portfolios, particularly retirement portfolios, the volume of research examining the methods of management has increased. Given limited interest and time availability in respect to the volume of research, simplified approaches which approximate optimized strategies are desired and well received by the public. One of the most cited of these approaches is the 4% rule provided by William Bengen (1994). According to this rule, from a well-constructed portfolio, 4% annually can be removed for consumption. It is argued that this approach will provide the portfolio owner with sustainable spending and limit the possibility of portfolio exhaustion. In the years since publication, numerous studies have attempted to enhance the approach (Bengen 1997, 2001; Cooley, Hubbard, and Walz, 2003; Guyton, 2004; Guyton and Klinger, 2006; Stout, 2006, 2008) in response to critiques. The strategy still maintains significant interest in academic literature and public use.

Examinations of utility models are some of the earliest forms of investment portfolio research. Seminal works on utility approaches include Merton (1969) and Samuelson (1969). These approaches advocate constant risk portfolios with a mix of fixed return assets combined with risk assets. The risk assets such as stocks allow for growth, while fixed return assets such as bonds ensure a positive income stream even in the event of investment volatility. However, requiring a fixed ratio between bonds and stocks can be problematic, especially during periods of market volatility. In periods of a severe stock market correction, the income generated from the fixed asset portion of the portfolio may be unsustainable as bonds are sold to purchase stocks to maintain the fixed asset ratio. Attempts to modify this approach and create a sustainable structure during stock market declines are introduced by Perold and Sharpe (1988). They propose selling stocks and buying bonds in market downturns to reduce risk and increase sustainability. The approach is called constant proportion portfolio insurance (CPPI).

Dybvig (1995, 1999) uses an expected utility maximization model to determine the optimal investment strategy that includes the preference for sustainable income. However, adoption of this type of strategy is often rigorous, complex, and time-consuming. Scott et al. (2009) successfully extend this line of research by simplifying the application. The authors propose a rule-of-thumb for managing retirement funds while maintaining a given level of spending which is dubbed the floor-leverage rule. Under this rule, 85% of the retirement assets is allocated to risk-free assets such as bonds to maintain current spending levels. The remaining 15% is allocated to a risk portfolio such as stocks that is designed to build wealth to increase future consumption. The authors demonstrate that this method is at least as efficient as other compensation strategies in maximizing utility.

To increase the growth potential of the risk allocation Scott and Watson (2013) recommend a stock portfolio which is leveraged. The authors suggest investing in stocks leveraged three times through exchanged-traded funds (ETFs), despite the limitations of volatility drag as noted by Jarrow (2010) and Sullivan (2009). The choice of leverage in the Scott and Watson model is exogenously provided by the authors. This article builds on this work by endogenously deriving an equation to determine the return-maximizing level of leverage unique to each investment scenario. The equation integrates the presence of volatility drag, which had been previously noted but not incorporated. The proposed method is generalized and available to multiple applications.

3. Methodology

The approach to managing retirement assets suggested by this paper follows that of Scott and Watson (2013) by first setting aside a risk-free portfolio to guarantee a minimum standard of living before using leverage to optimize the residual portfolio. The advantage of this approach is that the level of risk-taking in the residual portfolio will have no impact on the risk-free portfolio. The 85% risk-free allocation they suggest is a useful rule-of-thumb but may not be appropriate for all portfolio holders. The specific allocation should account for the size of the portfolio and the consumption patterns of the portfolio owner. The larger the portfolio and/or the smaller the minimal consumption needs, the smaller the optimal risk-free portfolio requirement. An assumption of the split-account portfolio model is that sufficient income can be generated from the risk-free assets to sufficiently cover consumption needs. In instances where this assumption does not hold, either alternate strategies or reassessment of consumption needs may be required.

In establishing an appropriate allocation mix to achieve both the current income needs and growth, a model should minimally consider the consumption needs of the portfolio owner and the size of the portfolio. If the goal of the risk-free portfolio is to maintain a minimal level of real purchasing power, a conservative approach would simply capitalize the minimum annual consumption needs (S) by the real risk-free rate (R_f). Dividing the resulting value by the size of the portfolio (P) provides a risk-free allocation (Z) of the portfolio as shown in the following equation.

$$Z = [S/R_f]/P \tag{1}$$

This simple formulation illustrates that the lower the minimum consumption needs or the larger the portfolio, the smaller the allocation to a risk-free portfolio.

The risk-free allocation of the portfolio would be invested in real risk-free assets, such as TIPS, to maintain its real purchasing power. The remaining percentage of the portfolio (1 – Z) could be invested in riskier assets that focus on maximizing expected returns. As an example, an individual with an investment portfolio of $3 million, minimal annual spending needs of $50,000, and a real risk-free rate of 2% will have a risk-free allocation of approximately 83%.[1] The remaining 17% of the portfolio would be available to be invested in riskier assets that can be leveraged to maximize expected return.

Eq. (1) provides a simple formula for deriving a risk-free allocation of a portfolio. However, the assumption of an infinite time horizon, thus guaranteeing the preservation of the portfolio's principle in perpetuity, is overly restrictive. A less stringent assumption would allow for a limited time horizon and permitting the principle to be spent over time. Eq. (1) can be modified as a fixed annuity adjusted for inflation. The adjustment allows for principle exhaustion at the end of the time horizon (T), as shown in the following equation.

$$Z = [S \times (1 - (1 + R_f)^{-T})/R_f]/P.^2 \tag{2}$$

For portfolio planning over a finite number of years, Eq. (2) provides a reasonable allocation mix. As an example, by incorporating a 40-year time horizon into Eq. (2), the portfolio allocation of risk-free assets needed is reduced from 83% to 46%.[3] As 46% will now generate sufficient income to minimally cover the consumption needs, the larger proportion of the remaining assets can be invested in riskier assets to produce maximum future purchasing power. Clearly, the decision to allow for possible principle exhaustion has a significant impact on the allocation decision between risk and risk-free assets. Given the long-term application of this method, the allocation mix can be examined annually using Eq. (2) and the portfolio modified when appropriate. Annual adjustment ensures long term sustainability of the portfolio in line with spending rules adjustments as proposed by Waring and Siegel (2015).[4]

The second step in this approach to managing retirement assets is to maximize the expected return by leveraging the risk portion of the portfolio. Using leverage in the risk allocation of the portfolio can better achieve the long term capital preservation aim of the portfolio holder. A leveraged investment provides benefit to the holder if the investment return (R_I) exceeds the cost of borrowing (R_B). The leverage and spread between return and borrowing costs can add expected incremental returns to the portfolio that is expressed in the following equation:

$$R_P = R_I + (R_I - R_B)\,(n - 1), \tag{3}$$

where R_P is the expected return on the portfolio using a leverage multiple of n, represented as the ratio of the portfolio to the equity investment. The expression n-1 in this equation is the debt-to-equity ratio. As an example, an investment yielding 8% that is leveraged three times, with borrowings costing 2%, yields an expected portfolio return of 20% [8% + (8%–2%) (3–1)].

As long as the expected investment return exceeds the cost of borrowing, Eq. (2) suggests that investors could benefit from ever-increasing levels of leverage. In a world of constant returns, increasing levels of leverage generate infinite returns on the leveraged portfolio.[5] However, investment returns are not constant and portfolio returns from ever-increasing leverage are bounded. A limiting factor on investment leverage is volatility drag, as implied in the Scott and Watson (2013) analysis. The application of leverage within a portfolio increases the potential of higher returns. However, the downside of increased portfolio leverage is cost of volatility drag as noted by many including Booth and Fama (1992), Messmore (1995), Sullivan (2009), and Jarrow (2010).

Volatility drag is the difference between the arithmetic and the geometric average returns. As an illustration, Scott and Watson (pp. 55–56) used the example of an investment that over the course of 250 days randomly earns 1% half the days while losing 1% on the other half. While this investment yields an arithmetic return of zero, the geometric return is −1.24%. In this example, the volatility drag is the 1.24%. For normally distributed returns, the relationship between the arithmetic and geometric returns can be expressed more generally as[6]

$$R_n = R_P - ½(\sigma_n{}^2), \tag{4}$$

where R_P represents the arithmetic return, R_n the geometric return, and $\sigma_n{}^2$ the variance of the returns. In the previous example, the arthritic return is zero and the variance is 2.50%.[7] If the returns were normally distributed, the geometric return on the investment would be a −1.25%, instead of the actual −1.24%. The volatility drag is one-half the variance of the returns.

The aim of a leveraged investment is to increase the expected portfolio return. However, a leveraged investment also increases the volatility drag because the increase in volatility of leveraged assets carries more risk. The general expression for the variance of an investment leveraged n times is the product of the variance of the investment and the squared multiple as shown below.

$$\sigma_n{}^2 = \sigma^2 n^2. \tag{5}$$

As more leverage is applied, the return volatility of the asset increases as well as the resulting volatility drag. For example, if the investment example of Scott and Watson illustrated above is leveraged three times, the arithmetic return remains at zero percentage, while the geometric return is estimated to lose 11.25%.[8] The increasing volatility drag functions as a binding constraint to increases in the portfolio return from the deployment of portfolio leverage.

The return on an investment with leverage n can be derived by combining Eqs. (3) through (5) as shown in the following expression.

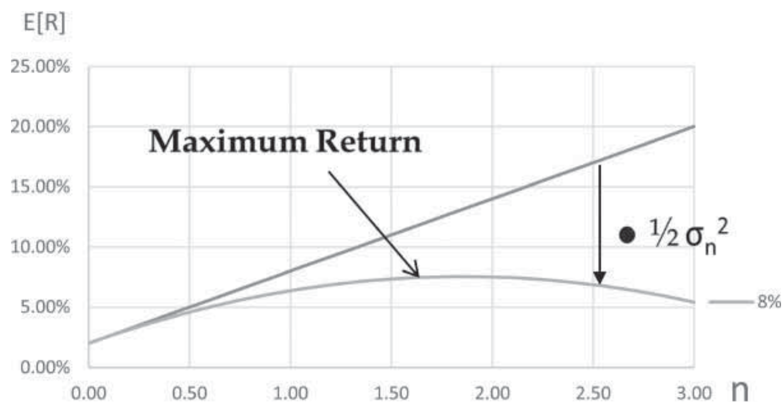

Fig. 1. Leveraged return curve.

$$R_n = R_I + (R_I - R_B)\ (n\text{-}1) - ½(\sigma^2 n^2),^9 \quad (6)$$

where one-half the variance of the leveraged investment of Eq. (5) represents the volatility drag. When no borrowings are used, n equals one and Eq. (6) simplifies to Eq. (4). Using the assumptions $R_I = 8\%$, $R_B = 2\%$, and $\sigma = 18\%$, the leveraged return of Eq. (6) can be expressed as:[10]

$$R_n = 8\% + 6\%\ (n\text{-}1) - 1.62\%\ n^2, \quad (7)$$

and depicted in Fig. 1 as the leveraged return curve. The straight line represents the expected return from leverage with no volatility drag. This leveraged return curve shows that the expected gain from leveraging eventually is dominated by the volatility drag. In this case the maximum return occurs at a leverage multiple of 1.85 times.[11]

When estimating the volatility drag in Eq. (6), the time horizon of the investor becomes an important factor. Blume (1974) showed that the geometric average return R(T) for a particular time horizon T can be approximated as a weighted average of the arithmetic average return and a geometric average return estimated over N years as follows:

$$R(T)_n = R_P\ (N\text{-}T)/(N\text{-}1) + R_n\ (T\text{-}1)/(N\text{-}1). \quad (8)$$

If T equals one, the average return is the arithmetic return and the leveraged return curve is the straight line in Fig. 1. If T equals N, the average return is the estimated geometric return and the leveraged return curve is the curved line in Fig. 1. Depending on the time horizon of the investor, the average return will lie between the arithmetic and geometric returns, with the leverage return curve lying between the two curves in Fig. 1. More important, the volatility drag will lessen as the investor's time horizon shortens, influencing the leverage that achieves the maximum return.

The return on an investment with leverage n shown in Eq. (8) can be modified for the investor's time horizon T by substituting Eqs. (3) and (6) into Eq. (8).

$$R(T)_n = R_I + (R_I - R_B)\ (n\text{-}1) - ½(\sigma^2 n^2)(T\text{-}1)/(N\text{-}1). \quad (9)$$

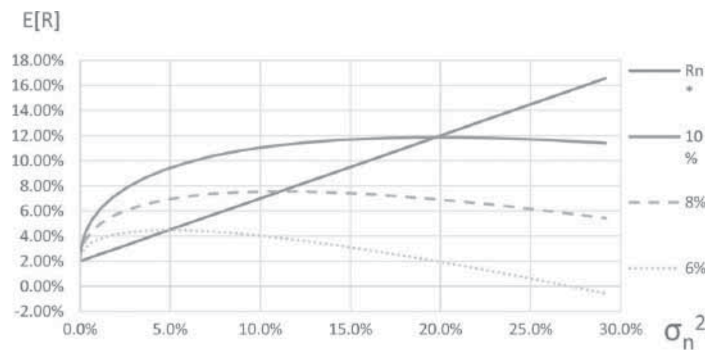

Fig. 2. The leveraged curve (risk measured with variance).

The investor's time horizon must lie between 1 and N. If T is equal to one, the volatility drag is zero and the portfolio return is the arithmetic return. As T approaches N, the volatility drag becomes one-half the variance of the leveraged investment expressed in Eq. (6).

The volatility drag of leverage is a constraint on the portfolio's performance, which is now realized in the equation for the return on a leveraged investment. The maximum expected return with respect to the level of leverage is determined by taking the first order partial derivative of Eq. (9) with respect to the leverage multiple, n, and setting the equation equal to zero. Solving for n provides the following expression for the leverage multiple that maximizes the expected return.[12]

$$n = (R_I - R_B)/(\sigma^2(T\text{-}1)/(N\text{-}1)). \tag{10}$$

An interpretation of Eq. (10) can be better understood by rearranging the equation as the following expression.

$$(R_I - R_B)\ n = \sigma^2 n^2\ (T\text{-}1)/(N\text{-}1). \tag{11}$$

The left hand side of the equation represents the return on the leveraged portfolio (R_P) less the borrowing cost while the right hand side is the variance of the leveraged return for a time horizon of T. An investor will continue to benefit from leveraging as long as the gain from leveraging is greater than the variance of the leveraged return.

Substituting Eq. (10) into Eq. (9) gives the maximum return with leverage as the following expression.

$$R(T)_n = R_B + ½\ [(\sigma^2)(n^2)((T\text{-}1)/(N\text{-}1))].^{13} \tag{12}$$

The maximum return equals the borrowing cost plus one-half the variance of the leveraged returns. This equation is the straight line depicted in Fig. 2 as the leveraged curve. In examining the application of leverage within ETF's, Cooper (2010) found a consistent relationship between leverage and returns over multiple time periods and indices as that observed in Fig. 2.

4. Leveraging the S&P 500

To test the return maximizing leverage of Eq. (10), we examined the monthly returns to the S&P 500 index, including dividends, over 65 years from November 1950 through

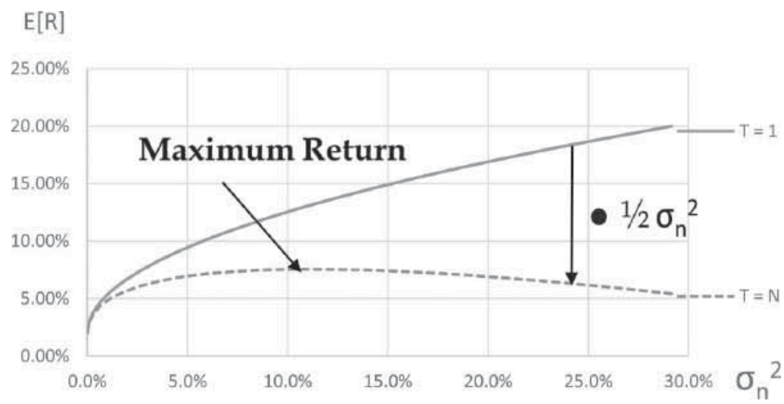

Fig. 3. Risk/return with leverage (risk measured with variance).

October 2015.[14] The average monthly return was 0.688%, representing an annual percentage rate (APR) of 8.25%. The variance of these monthly returns was 0.173% or 2.08% annualized. Using Eq. (4), the estimated geometric return is 0.601%, representing an APR of 7.21%. The actual geometric return over the 65 years was 0.600% or an APR of 7.20%. In this data series, Eq. (4) slightly overestimates the actual geometric return.[15]

To calculate the return maximizing leverage, a borrowing rate must be specified. We chose to simply use the 3-month Treasury bill rate as a proxy for the borrowing rate. Over the 65 years, the 3-month Treasury bill rate averaged 0.366% monthly or 4.40% APR. Using the S&P 500 average return and the variance, the return-maximizing leverage over the 65 years implied by Eq. (10) was 1.85 times. To examine the accuracy of this measure, we calculated the leverage that maximizes the actual return. Calculating the actual monthly return on a leveraged portfolio using Eq. (3), we solved for the leverage multiple that would have generated the greatest return over the 65 years. This optimal leverage multiple was 1.77 times, 4.3% percent less than our estimated value.

5. Discussion

The derived return-maximizing leverage in Eq. (10) allows for easy application within the risk allocation of an investment portfolio. Using the assumptions from Eq. (7)[16] in addition to $N = 85$, and $T = 40$, the leverage multiple that maximizes the portfolio return is 3.99 times. As the investor's time horizon lengthens, the leverage multiple is reduced toward 1.85 times, the leverage multiple if $T = N$. Scott and Watson suggest a leverage multiple of 3 implies a 53-year time horizon. Rather than an exogenous selection of leverage, Eq. (10) provides an investor the means to tailor the leverage to the specific circumstances. The introduction of the volatility drag explicitly in the portfolio return differentiates this study from previous studies. Without volatility drag, ever-increasing levels of leverage increase portfolio returns. Volatility drag limits the portfolio returns from leveraging, eventually outweighing the gains. Continuing the same example, the influence of volatility drag on portfolio returns is displayed in Fig. 3 which illustrates the expected leveraged return across the variances of the leveraged portfolio.[17] As the expected investment return increases, the leverage multiple that maximizes the portfolio return also increases, as expressed in Eq. (12) and depicted in Fig. 2.

Once the return-maximizing level of leverage is determined, Scott and Watson (2013) suggest using leveraged exchange-traded funds (ETFs) to achieve the desired level of leverage. An investor can simply choose the ETF that uses leverage closest to the return-maximizing derived leverage. If the exact leverage required is not available, or more precision is sought, it is possible to invest in multiple ETFs with different leverages, adjusting the investment weight between them to achieve the desired aggregate leverage.

The approach assumes a normal long-term distribution of market returns in an attempt to simplify the model. Acknowledging the non-normal long-term market returns, and incorporating terms to account for distribution skewness and kurtosis within market returns would add to the basic model. This area would be open to further research and certainly expand the application of the approach presented in this article.

6. Conclusion

This article provides endogenously derived equations designed to assist in managing a retirement investment portfolio. The approach is to allocate retirement funds between a risk-free portfolio to support minimum annual consumption spending and a risk portfolio leveraged to provide long-term purchasing power. While leveraging the risk portfolio can enhance expected portfolio returns, this article emphasizes the limitations by explicitly including the volatility drag of leveraging. If the spread of the leveraged portfolio return over the borrowing costs is greater than the variance of the leveraged return, then the investor can enhance expected portfolio returns by increasing leverage. Otherwise, the volatility drag of leveraging diminishes the expected portfolio returns. This limitation occurs at higher leverage multiples if an investor has a shorter time horizon.

While this article focuses on using leverage in a retirement portfolio, the approach is applicable to other issues involving the use of leverage to enhance portfolio returns. In particular, hedge fund managers and managers of financial institutions typically operate with high levels of leverage and may experience the limitations from volatility drag. This topic is also relevant in addressing the optimal capital structure for a firm in corporate finance.

Notes

1 This allocation is close to the 85% suggested by Scott and Watson (2013).
2 As T approaches infinity, the influence of the annuity diminishes back to $1/R_f$ and Eq. (2) reverts back to Eq. (1). In such a case, the principle never depletes and the portfolio will exist in perpetuity.
3 Scott and Watson (2013) used 40 years in their example.
4 Waring and Siegel (2015) propose a spending rule that constantly adjusts the annuity through time so as to never deplete the portfolio.
5 This is Modigliani-Miller's (1958) proposition II.

6 Bodie, Kane, and Marcus (2011) p. 132.
7 $\sigma_n{}^2 = 250 \times 0.01\% = 2.5\%$.
8 $R_I = 0\%$ and $\sigma^2 = 22.5\%$ [or $250 \times 0.01\% \times 9$]. In the Scott and Watson example, the actual loss from 3 times leverage is -10.64%, p. 56.
9 This equation can also be expressed as: $R_n = R_B + (R_I - R_B)\, n - ½(\sigma^2 n^2)$.
10 These are the assumptions used by Scott and Watson (2013, p. 49).
11 When $n = 0$, the portfolio is leveraged -1 times. This is equivalent to selling the investment portfolio and investing it at the borrowing rate.
12 $R_n = R_I + (R_I - R_B)(n\text{-}1) - (½\sigma^2)(n^2)((T\text{-}1)/(N\text{-}1))$,

$$\partial R_n/\partial n = (R_I - R_B) - (2n)(½\sigma^2)((T\text{-}1)/(N\text{-}1)),$$

$$0 = R_I - R_B - (n)(\sigma^2)((T\text{-}1)/(N\text{-}1)),$$

$$n = (R_I - R_B)/(\sigma^2)\ ((T\text{-}1)/(N\text{-}1)).$$

13 Eq. (12) can also be expressed as a function of the spread between the investment yield and the borrowing cost as $R(T)_n = R_B + ½\,(R_I - R_B)^2/\sigma^2\ (T\text{-}1)/(N\text{-}1))$.
14 Data series is from Yahoo Finance.
15 The precise relationship is derived from continuously compounded rates that are lognormally distributed. See Jacquier, Kane, and Marcus (2003).
16 $R_I = 8\%$, $R_B = 2\%$, and $\sigma = 18\%$.
17 Figure 2 is the same curve as Figure 1, but with the variance as the dependent variable instead of leverage.

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