

Toward constructing tax efficient withdrawal strategies for retirees with traditional 401(k)/IRAs, Roth 401(k)/IRAs, and taxable accounts

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Abstract

We construct an algorithm for U.S. retirees that computes individualized tax efficient annual withdrawals from tax-deferred, tax-exempt, and taxable accounts. Our algorithm applies a new approach using information from all years that generates an individualized strategy, in contrast to most previous approaches that chronologically generate a suboptimal strategy. Results equal or improve the chronological “naïve” withdrawal strategies advocated by many financial institutions, as well as the chronological “informed” strategies advanced by academics. Our approach allows us to determine the optimal switching times between tax-exempt and taxable account consumption, as well as between tax-deferred and taxable account consumption. It also allows us to understand the significant impact of an heir’s tax and withdrawal rate on a retiree’s optimal withdrawal strategy. Our model, which can work to optimize either portfolio longevity or the bequest to an heir, accommodates many salient tax code features, including dividends, different taxable lots, and required minimum distributions. © 2020 Academy of Financial Services. All rights reserved.

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1. Introduction

U.S. retirees generally have their equity investments¹ in three types of accounts: (1) tax-deferred accounts (TDAs) like traditional IRAs or traditional 401(k)s, (2) tax-exempt accounts like Roth IRAs or Roth 401(k)s, and (3) taxable accounts. These three accounts are

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governed by significantly different tax rules. For example, TDAs are taxed as income via progressive tax brackets and are subject to required minimum distributions (RMDs); Roth accounts are not subject to tax; and stock in taxable accounts, when sold, is subject to capital gains taxes, although all capital gains amassed by a retiree are forgiven when stock is inherited by an heir.

These differences in tax structure, especially between the taxable account and the two retirement accounts, make answering the important question of how to optimally utilize these three accounts quite complicated. Much attention has been given by non-academic institutions, as well as academic researchers, for how to best build up these accounts in preparation for retirement. Brown et al. (2017), for example, provides a comprehensive list of academic papers in this area. Intrinsic to answering this question, as well as of great interest in its own right, is understanding the somewhat less investigated question of how to best withdraw from these three accounts during retirement. The sequencing of withdrawals that a retiree makes among accounts with varying tax structures can have a significant effect on their portfolio's longevity. Even less investigated is the more general, and more complex question of sequencing withdrawals so as to optimize the usefulness of the retiree's bequest to an heir, as opposed to the portfolio's longevity. In this paper, we provide an algorithm that determines a withdrawal strategy that seeks to minimize the effect of taxes on the retiree and, should the retiree wish to make a bequest, the retiree's heir.

Our algorithm approach is unusual in that it is an "all-time" approach, as opposed to a "forwards-time" approach or a "backwards-time" approach. Forwards-time approaches, which are by far the most common, determine allocation strategies year by year working forwards chronologically in time. Backwards-time approaches generally start from a projected date of death for the investor and then work backwards in chronological order. An all-time approach uses information from all years to determine the allocations in every year. In our case, this includes the annual rate of return for the stock, the stock's dividend rate, the rate of inflation, all tax rates and tax brackets, as well as external annual fixed sources of income. In addition, if the goal is to optimize an heir's bequest, it includes the projected time at which the retiree will die, the effective marginal tax rate of the heir, and the rate at which the heir will consume inherited TDA or Roth money. All-time approaches are able to tailor optimization decisions to each individual retiree's financial situation in a way that forwards-time and backwards-time approaches, by their nature, cannot. The disadvantage of our all-time approach is that it is a complicated algorithm. The full details of this algorithm are found in DiLellio and Ostrov (2018) and are posted on the Academy of Financial Services (AFS) 2018 conference proceedings,² where it was awarded the Best Paper sponsored by the CFP Board of Standards.³ Because the details are posted there, we emphasize the novel features and results of the algorithm in this paper, providing only a brief overview of our algorithm's approach, instead of details, in Section 5.

1.1. Literature review

We provide a quick summary of some forwards-time approaches, which are the most common techniques explored by both non-academics and academics. Non-academic advice for retirees' withdrawal choices, which comes from investment firms, financial advisors, and books on retirement, all agree that retirees should conform with the law and take out

Required Minimum Distributions (RMDs) from their TDAs to avoid the significant penalty for not doing so. Otherwise, their advice often contradicts each other. For example, one very common category of strategies, termed “naïve” strategies by Horan (2006a and 2006b), recommends that retirees completely exhaust one account before moving to the next. The books by Solin (2010), Rodgers (2009), and Lange (2009), suggest sequencing withdrawals so that retirees liquidate taxable accounts first, then TDAs, and finally Roth accounts. This strategy is also endorsed by large retail investment firms Fidelity (see Fidelity (2014) and Fidelity (2015)) and Vanguard (Vanguard (2013)).

In contrast, other financial authors, such as Larimore, Lindauer, Ferri, and Dogu (2011), recommend first liquidating taxable accounts, but then recommend liquidating Roth accounts followed finally by TDAs. Another Vanguard paper (Jaconetti & Bruno, 2008) recommends that the decision on whether to liquidate the TDA or Roth account immediately following the taxable account should be based on expected future marginal tax rates. Coopersmith and Sumutka (2011) estimate the suboptimality of these naïve strategies to be approximately 16%. This agrees with DiLellio and Ostrov (2017), who provide illustrations for which the naïve approaches are 10–26% suboptimal.

A second set of strategies, termed “informed” strategies by Horan (2006a and 2006b), use TDA spending up to the top of a given tax bracket in every year. This is advocated in the book by Piper (2013), who suggests filling any remaining consumption needs first with taxable stock, then with Roth money, and lastly the TDA, if the TDA is not already exhausted. We note that there is a different informed strategy for each tax bracket, but each of these informed strategies can be run and then the best of these can then be selected. In this paper, for comparison purposes with our algorithm, we will always use the best of these informed strategies.

Within the context of withdrawal optimization for just TDA and Roth accounts, but no taxable accounts, Horan’s informed strategies were a considerable step towards increasing a retiree’s portfolio longevity using forwards-time approaches. This seminal work was expanded and investigated further in Reichenstein, Horan, and Jennings (2012). Al Zaman (2008) extended this approach to the case where a retiree’s goal is to optimize a bequest, in addition to the easier subcase of optimal portfolio longevity.

Horan’s approach also had a significant impact on forwards-time approaches for increasing the longevity of portfolios that included taxable accounts in addition to TDAs and Roth accounts. For example, Sumutka, Sumutka, and Coopersmith (2012) compare the effectiveness of a wide variety of naïve and informed withdrawal strategies for an array of portfolios. Cook, Meyer, and Reichenstein (2015) cleverly expanded the field of possible strategies by considering the advantages of using conversions from the TDA account in addition to withdrawals for consumption in the retiree’s early years. Geisler and Hulse (2018) investigate the effect of Social Security benefits within this approach.

Backwards-time algorithms include dynamic programming approaches. The paper by Brown et al. (2017), for example, applies dynamic programming to TDA and Roth account withdrawals, but restricts itself to a two-period model due to the computational complexity of this problem, even without a taxable stock account. Because of the well-known “curse of dimensionality” in dynamic programming, backwards time approaches cannot accommodate multiple tax lots.

The all-time approach that we have found in the literature uses linear programming algorithms to optimize withdrawals. This line of research begins with Ragsdale et al. (1994), which worked with a TDA and taxable account. This was extended to include Roth accounts and additional features in Coopersmith and Sumutka (2011) and Coopersmith and Sumutka (2017), who also concentrated on analyzing the effect of different rates of return in accounts, and in Meyer and Reichenstein (2013), Welch (2016), and Welch (2017), who “assign a single rate of return to all accounts to concentrate on the effects of taxes” as we do in this paper. These papers do not determine optimal longevity. Instead they optimize the total amount of a bequest, with no distinction between the types of accounts, at a projected time of death. They assume that taxable stock is to be depleted before consuming from the Roth account. In all cases, the model assumptions of these papers vary considerably from ours, as does their approach, because of the nature of the linear programming technique, which cannot be applied to non-linear phenomena. DiLellio and Ostrov (2017) contains an all-time approach that is not based on linear programming and yields an optimal TDA and Roth account withdrawal strategy for either the goal of optimizing bequests or the subcase of optimizing portfolio longevity. Taxable accounts, however, have very different taxation rules than TDAs and Roth accounts, which necessitated an almost completely new, and far more complex, approach to obtain the all-time approach optimization algorithm in this paper.

1.2. New contributions

Our model includes working with Required Minimum Distributions (RMDs) for TDAs, as is the case with most of the academic research above, but it also considers optimization while accounting for a number of features that are rare, if studied at all, in the above literature. Among the new contributions of our algorithm for optimizing bequests include

- The determination of when the retiree is best off switching from using taxable stock to Roth account money for consumption. In the case of optimized longevity, it is best to switch to the Roth only after the taxable stock account is depleted. However, when we optimize a bequest, the step up in cost basis to the heir creates an important question of when to stop using taxable stock.
- The determination of when the retiree is best off switching from using taxable stock to TDA money for consumption within a tax bracket. If taxable stock is going to be consumed, it is best done early to avoid losses from taxation on dividends. Should a bequest be involved, however, there is the additional complication that it may be better for the retiree to hold onto the taxable stock to take advantage of the step up in cost basis to the heir.
- The effect on the optimal strategy of taking into account the effective tax rate of the heir.
- The effect on the optimal strategy of taking into account the rate at which the heir withdraws money, which changes the worth of inherited taxable stock vs. inherited TDA and Roth accounts, which yield further tax sheltering.
- Quantifying the effect of a retiree’s stock dividends on the bequest size.

Via the functions $L(t)$ and $U(t)$, which we explain later, we are also able to accommodate additional projected fixed sources of money for the retiree that are taxed as income or that

have tax rates that are independent of the retiree's allocation decisions among the TDA, Roth, and taxable accounts, which includes tax-free sources. Using these we can accommodate Social Security in many cases, as will be discussed below.

Although nothing is stochastic in our approach, as is common in the above literature, we note that the annual rates of return, μ_t , at each time t can be chosen independently of each other, allowing results to be generated for any stochastically created set of annual rates of return, as is done in some of the papers above. These rates can be positive or negative as long as there are overall gains in the taxable account, because our approach is designed to minimize taxes in the presence of capital gains, not capital losses. One of the ramifications of our model being deterministic is that it will be optimal to have strictly stock, as opposed to bonds or cash, in the TDAs, Roth accounts, and taxable accounts. That is, because the stock returns are assumed known, the model cannot recommend having bonds or cash given their lower rates of return. Of course, in practice, bonds and cash have an important role to play due to their lower volatility. We note that our model will still be able to accommodate known/projected payouts from bond and cash positions in taxable accounts via $L(t)$ and $U(t)$, as will be discussed below.

The all-time approach in this paper for optimizing withdrawal strategies in portfolios containing TDAs, Roth accounts, and taxable stock accounts makes improvements on previous methods in the following ways:

- Unlike forwards-time approaches, when determining the best allocation in a year, we are able to take into account all future projected annual consumption for the retiree, as well as the projected date of death, and the projected circumstances of the heir. This can have considerable advantages. For example, forwards-time approaches spend TDA money up to the top of one tax bracket, unless RMDs require more to be spent. This can lead to early depletion of TDA money, meaning lower tax brackets in later years cannot be used by the investor to minimize overall taxes. It also prevents optimal strategies where the retiree jumps to the top of new tax bracket at different times so as to maximize the benefit to the heir. Our all-time approach uses the additional information available to it to avoid these potential sub-optimal uses of tax brackets. In general, forwards-time approaches are more suited for optimizing longevity, because bequest information is not used to alter its allocation recommendations. Our algorithm does use this information, so it can effectively optimize bequests.
- Unlike current linear programming approaches, our model can use all-time information to optimize portfolio longevity, whereas current linear programming approaches only look to optimize total bequest size. Our algorithm considers the nature of capital gains and dividends in the taxable account, whereas current linear programming approaches consider taxable accounts only on an after-tax basis. Our algorithm also makes distinctions among the worth of the TDA, Roth, and taxable stock money to an heir, which affects allocation decisions as we will explicitly show. Our algorithm can account for non-linear effects in the tax code, which, by its nature, linear programming cannot.
- Unlike backwards-time approaches, our model is able to be extended to accommodate multiple tax lots, as explained in Section 5 of DiLellio and Ostrov (2018). Further, it can accommodate as many annual time periods as desired, as opposed to only a few time periods due to computational complexity.

As with most of the recent literature above, we will compare the effects of our algorithm to the naïve strategies (also known as the common rule or common strategy throughout the linear programming approach literature) and to the best of the informed strategies. Our all-time approach equals or exceeds these forwards-time approaches in every case that we have examined, meaning it obtains a higher bequest or a longer portfolio longevity. This is because the minor approximations used in our approach are only needed to accommodate more complicated phenomena than these strategies consider. Our approach is quite different, but compatible, with the conversion method in Cook, Meyer, and Reichenstein (2015). There are cases where our method, by itself, will equal or exceed theirs, such as when taxable stock is not present, so it cannot be used for consumption, as is needed for the conversions described in their paper; there are others where their method, by itself, will do better. However, the strengths of both approaches can be combined in a simple way by using their method forwards in time for the first few years until conversions run out and then applying our algorithm to the remaining years. They can also be combined in a more complicated, but also more effective, way that integrates the two methods, such as the one detailed in Section 7 of DiLellio and Ostrov (2018).

While the algorithm works to optimally determine how much a retiree should spend each year from their TDA, Roth, and taxable stock accounts – that is, these are the three decision variables to be determined each year – it also accommodates, as indicated above, two other sources of external, fixed, annual income. The first source, which we will call $L(t)$ in this paper, because it will be geometrically represented in the lower part of our graphs, encompasses any projected, fixed sources of money, other than the TDA account, that are subject to income tax. Therefore, $L(t)$ is part of the allocation of withdrawals. These include earned income, some pensions, annuities bought with pretax money, and the earnings from annuities bought with post-tax money. The second source, which we will call $U(t)$ in this paper, because it will geometrically be represented in the upper part of our graphs, encompasses any projected, fixed sources of money that (a) have tax rates that are independent of the retiree's allocation decisions among the TDA, Roth, and taxable accounts, and (b) unlike $L(t)$, have no effect on the taxation rate of the TDA or taxable account. Examples include tax-free gifts and tax-free accounts like Health Savings Accounts, some pensions, and the principal from annuities bought with post-tax money.

Therefore, for example, if we have fixed amounts of cash or bond money coming to the investor, our model can accommodate them, since bond coupons and interest are part of $L(t)$, while consumption of principal and the par value of bonds at maturity are part of $U(t)$.

Social Security, unsurprisingly, is far more complicated.⁴ Tax on Social Security is determined from the retiree's "base amount," defined as the investor's AGI plus non-taxable interest plus half of the retiree's Social Security benefits. If the retiree has a small base amount, meaning, as of 2018, that their base amount is below \$25,000 if single or \$32,000 if married and filing jointly, then there is no tax. In this case, Social Security can be accommodated in our model by putting all Social Security income into $U(t)$. Our model can also accommodate investors with a large base amount, generally meaning, as of 2018, that 85% of their annual Social Security benefit is less than \$4,500 + 85% of the excess of their base amount over \$34,000 if single or \$6,000 + 85% of the excess of their base amount over \$44,000 if married and filing jointly. In this case, 85% of Social Security is subject to income tax, so 85% of the Social Security payment is incorporated into $L(t)$, while the other 15% is put into $U(t)$.

In cases where the investor is in the area between these amounts, the taxation rate on Social Security depends on the allocation between the TDA, Roth, and taxable stock accounts, which is outside our model. Retirees in this area are subject to the “tax torpedo,” which is discussed in Meyer and Reichenstein (2013), who predict that there are likely less than 10 million out of a total of approximately 140 million tax returns that are subject to this issue.

The organization of this paper is as follows: In Section 2 we introduce some basic definitions for our model. Section 3 contains our model’s assumptions. Section 4 lists five guiding principles that will define how our algorithm prioritizes spending in order to maximize the retiree’s bequest. Section 5 defines the algorithm’s objective, explains the bar graphs used to report our results, and summarizes our four-stage algorithm. Section 6 shows how Section 5’s algorithm for optimizing a retiree’s bequest can be used for the subcase where the retiree instead wishes to optimize portfolio longevity. Section 7 demonstrates additional results obtained from our algorithm, such as a comparison of our algorithm’s strategy to the naïve and the informed strategies, a comparison of portfolio longevity under the new 2018 tax legislation versus the previous 2017 tax law, and a sensitivity analysis of our results to varying a wide variety of the financial parameters used by our algorithm. In Section 8, we discuss our main conclusions.

2. Definitions

Definition of basic variables:

t = time (in years) during retirement

t_{death} = value of t when the investor dies

μ_t = annual rate of return, in real dollars, for stock in all accounts before dividends are distributed. The value of μ_t can be chosen to vary with time, t .

d_t = annual dividend rate, where all dividends are assumed to be qualified, distributed at the end of the year, and may be consumed or reinvested. The value of d_t can be chosen to vary with time, t . Capital gain distributions from mutual funds can also be included here if the tax rates τ_{div} and τ_{gains} , defined just below, are equal.

Definition of tax rates:

τ_{div} = tax rate on qualified dividends

τ_{gains} = tax rate on long-term capital gains

τ_{marg} = the marginal income tax rate associated to a given tax bracket.

τ_{heir} = effective marginal income tax rate for the heir or heirs, which is applicable to distributions from an inherited TDA

Definition of a , the heir’s discount factor for inherited taxable stock: Should the heir immediately liquidate their inheritance, they will fail to take advantage of the additional tax advantages that occur over time for a TDA or a Roth account, though not for a taxable stock account. These advantages for the inherited TDA or Roth accounts are maximized by the heir only taking out RMDs. The value of what we will call the heir’s discount factor a will measure how much additional tax efficiency the heir receives due to the speed at which they liquidate their inherited TDA and Roth account. Comparing the worth of inherited Roth money to inherited TDA money is straightforward, because at $t = t_{death}$ inheriting a dollar of

Roth money is equivalent to inheriting $\frac{1}{1-\tau_{heir}}$ dollars of TDA money. Comparing the worth of inherited Roth money to inherited taxable stock money is where a comes in. We define the discount factor $a \leq 1$, so that inheriting a dollar of Roth or TDA money at $t = t_{death}$ is equivalent to inheriting $\frac{1}{a}$ dollars of taxable money at $t = t_{death}$. If an heir immediately liquidates their inherited TDAs and Roth accounts, then $a = 1$. It is beneficial to the heir to obtain a lower value of a low by stretching out the liquidation of these accounts. In Appendix 2 of DiLellio and Ostrov (2018), we compute explicit formulas for the lower bound on a , corresponding to an heir being wise and only taking RMDs from inherited TDAs and Roth accounts. Under typical circumstances, we find this lower bound to be near 0.75; that is, $0.75 \leq a \leq 1$.

Definition of the index j , “lot j of stock,” and j_{max} : Because the cost basis of a stock’s lot depends on the stock’s purchase date, we attach a new index, j , to each successive stock purchase in the retiree’s taxable account. All stock purchased at the same time, indexed by j , will be referred to as the “lot j of stock.” For example, “lot 4 of stock” would be the fourth oldest lot of stock in the retiree’s taxable account. The index j_{max} corresponds to the total number of different tax lots held by the retiree. When dividends are not immediately used for consumption, they will be used to purchase more stock,⁵ forming a new lot and increasing the value of j_{max} by one. If a lot is completely consumed by the retiree, then j_{max} is reduced by one.

Definition of ω_t^j : For the lot j of stock at time t , we define

ω_t^j = the fraction at time t of the worth of lot j of stock that is equal to its cost basis.

For example, let’s say that \$20 was used to purchase the original stock in the portfolio. If, 15 years into our algorithm’s projection, that lot of stock is projected to be worth \$100, then $\omega_{15}^1 = 0.2$. Note that when a new lot of stock such as reinvested dividends is created, we initially have $\omega_t^{j_{max}} = 1$ for this new lot $j = j_{max}$ of stock, because there are no capital gains yet. Each year, each group of stock, in real dollars, becomes worth $(1 + \mu_t)(1 - d)$ times its previous year’s worth. Given this, for each lot j in year $t + 1$, we have that

$$\omega_{t+1}^j = \frac{\omega_t^j}{(1 + \mu_t)(1 - d_t)}.$$

For an investor that sells stocks with losses, we have that⁶

$$\omega_t^1 \leq \omega_t^2 \leq \dots \leq \omega_t^{j_{max}}.$$

3. Financial and model assumptions

We make the following financial and model assumptions in this paper:

1. Because we work in real dollars, we assume the inflation rate is known or projected. Therefore, for example, if $\mu = 5\%$ and the rate of inflation is 3%, then the annual nominal rate of return for the stock is 8.15%, since $1.05 * 1.03 = 1.0815$.

2. Tax rates and tax brackets:
 - a. We assume that τ_{heir} , τ_{div} , and τ_{gains} are known/projected constants. Note that τ_{heir} may be a projected average or effective tax rate over time and/or over a number of heirs.
 - b. We assume, as is typically the case in tax law, that the nominal tax bracket thresholds adjust with the rate of inflation. This means the projected tax brackets in our model are constant in real dollars, which is why our model uses real, instead of nominal, dollars.
 - c. We assume that τ_{marg} , the tax rates for each tax bracket, are known/projected constants.
3. We assume the investor's total after-tax consumption needs, $C(t)$, are known/projected in each year of retirement. We note that the model can easily be rerun with various projections/scenarios for $C(t)$, as well as any of our other parameters, enabling an investor to experiment with these to better understand their financial implications.
4. We assume $L(t)$ and $U(t)$ are known/projected in each year t . The money from these funds is used strictly for consumption, not, for example, to purchase stock. We emphasize that $L(t)$ is subject to income tax rates and could include non-qualified dividends. The tax rate for all funds in $U(t)$ must be fixed and cannot depend on the manner in which spending is allocated among the TDA, Roth account, and taxable account, nor can the value of $U(t)$ affect the manner in which the TDA or taxable account is taxed.
5. Stock:
 - a. We assume that μ_t , the annual rate of return for stock in real dollars, and d_t , the annual dividend rate, are known or projected at each time t . The values of μ_t may be positive or negative as long as the stock, overall, has capital gains, not capital losses.
 - b. We assume there are no transaction costs for buying or selling stock, and that stock can be sold in any quantity, including fractional shares, as is available with mutual funds.

Appendix 3 of DiLellio and Ostrov (2018) gives the subroutine by which our algorithm annually updates the TDA, Roth account, and each of the lots in the taxable stock account to address consumption, growth, dividends, and taxes.
6. We assume there are no additional contributions to the TDA, Roth, or taxable stock accounts, nor are there any conversions from one of these three accounts to another. However, in Section 7 of DiLellio and Ostrov (2018), we discuss how to incorporate allowing Roth conversions from non-RMD TDA money. Section 7 of DiLellio and Ostrov (2018) also considers years where RMDs are greater than the consumption needs, $C(t)$, in which case the excess RMDs may be used to buy taxable stock, since the IRS prohibits RMDs from being converted into a Roth account (see, e.g., Rosato, 2015).
7. We assume that t_{death} is a known/projected time. If desired, t_{death} can be projected from IRS life expectancy tables or the algorithm can easily be reapplied with various values of t_{death} to obtain strategies for different t_{death} scenarios.

8. We assume the inheritance is not high enough such that the estate tax is relevant. We note that even under the previous law, only one in 500 estates were so large that any estate tax was due (see Huang & DeBot, 2015). Since then, the estate tax exemption has doubled due to the tax law passed by Congress at the end of 2017, so this figure is now less than one in 1500 estates.⁷
9. We assume the rate at which the heir consumes their inheritance is known/projected. This is only needed to compute a , the heir's discount factor, discussed earlier.

4. Guiding principles

At times, we will think about consuming from taxable stock and from dividends separately, even though both originate from the taxable stock account. In Section 5, we will outline how our algorithm looks to optimize withdrawals from four types of money: TDA, Roth, taxable stock, and dividends generated by the taxable stock. This algorithm will be governed by the following guiding principles that stem from U.S. tax law:

Guiding Principle 1: If a given amount of TDA money that is taxed at a constant marginal rate τ and a given amount of Roth money are both going to be spent to address fixed consumption needs, $C(t)$, the allocation/sequencing between the TDA and the Roth to address this consumption does not matter. Similarly, if a given amount of TDA money that is taxed at a marginal rate τ_1 and a given amount of TDA money that is taxed at a marginal rate τ_2 are both going to be spent, the allocation/sequencing between these two groups of TDA money does not matter.

This first statement is proven in Appendix 1 of DiLellio and Ostrov (2018). The second statement can also easily be proven using the method presented in that Appendix. This means that, given a specific amount of TDA money and Roth money to be consumed, we optimize these funds' use by keeping the consumed TDA money in the lowest tax brackets, be they for the retiree or for the heir, as possible.

Guiding Principle 2: It is better to use taxable stock and dividends for earlier, rather than later, consumption by the retiree.

Since the taxable stock and the reinvested dividends have returns that are slowly eroded by the effects of dividends, if we know we are going to use part of our taxable account for consumption, it is better to use that part as early as possible. This means our prioritization of whether to use taxable stock/dividends versus TDA/Roth money to satisfy consumption may be time dependent, with more likelihood of using the taxable stock or dividends at earlier times, since taxation on TDA/Roth spending is not time dependent in the way taxable stock is.

More specifically, if we know we are going to spend some taxable stock money for consumption, it should be prioritized to be consumed before spending any Roth money. The bigger question between the Roth account and the taxable account is whether or not it is worth prioritizing using more Roth money for the retiree's consumption needs so that less taxable stock is used for consumption, enabling more capital gains in the taxable account to be forgiven at death. The question of prioritizing the TDA versus the taxable account can be even

more complex, as the desire to spend the TDA in lower tax brackets may override the desire to spend taxable stock earlier. Our algorithm shows how to answer both of these questions.

Guiding Principle 3: When consuming taxable stock, we consume the lot with the highest cost basis, ω_j^t , that is available at time t . One ramification of this principle is that we always consume dividends before liquidating other lots.

By consuming stock with the highest cost basis, we minimize the amount of tax paid on stock that we need to consume. If we must consume the lower cost basis stock later, we have had the advantage of having a longer time to collect returns accrued from the larger capital gains in the lower cost basis stock. Further, it is more desirable to have stock with a lower cost basis be in the retiree’s account when the retiree dies, because that means that more tax on the retiree’s capital gains will be forgiven, to the greater benefit of the heir. We note that should $\omega_t^1 \leq \omega_t^2 \leq \dots \leq \omega_t^{j_{max}} \leq 1$, guiding principle 3 corresponds to LIFO (last in, first out) being the optimal strategy for an investor. That is, we consume first from lot j_{max} and then, should this lot become exhausted and it is desirable to consume more taxable stock, we consume from lot $j_{max} - 1$, which is relabeled lot j_{max} , and we continue in this manner as long as it remains desirable to consume the lot of taxable stock with the highest remaining j (and ω_t^j) value. Further, since dividends correspond to a taxable lot where $\omega_t^{j_{max}} = 1$, they are always prioritized for consumption before any other lot.

It is worth noting that there may be a material risk associated with a strategy of selling stocks with the highest cost basis, as the taxable account can become very concentrated in stock with strong past performance. This can be partially mitigated by using a broad-based fund, rather than investing in individual stocks.

Guiding Principle 4: We always prioritize using dividends to satisfy the retiree’s consumption needs before using Roth money.

Choosing to prioritize consuming the Roth money, which is not subject to any tax for the retiree or the heir, so that we can retain (after-tax) dividend money used to buy taxable stock⁸ is an inferior choice for three reasons: (1) the erosive effect of taxes on dividends over time with taxable stock, (2) the tax on capital gains should the taxable stock need to be sold before the retiree’s death, and (3) the heir is subject to tax on capital gains accrued after the taxable stock is inherited, even though capital gains are forgiven when the retiree dies.

Guiding Principle 5: We always take out any RMDs.

The 50% fee levied on any RMDs not taken by the retiree from their TDA or the heir from their inherited TDA or Roth cannot be compensated by anything else in the current tax system.

5. Algorithm

In this section we outline how we use our guiding principles to determine the annual allocations from the TDA, Roth account, and taxable stock account that satisfy the retiree’s annual consumption needs and maximize the following objective function for $W_{total}(t_{death})$, the total worth of the bequest to an heir or heirs:

$$W_{total}(t_{death}) = \frac{1}{a}(1 - \tau_{heir})W_{TDA}(t_{death}) + \frac{1}{a}W_{Roth}(t_{death}) + W_{TS}(t_{death}), \quad (1)$$

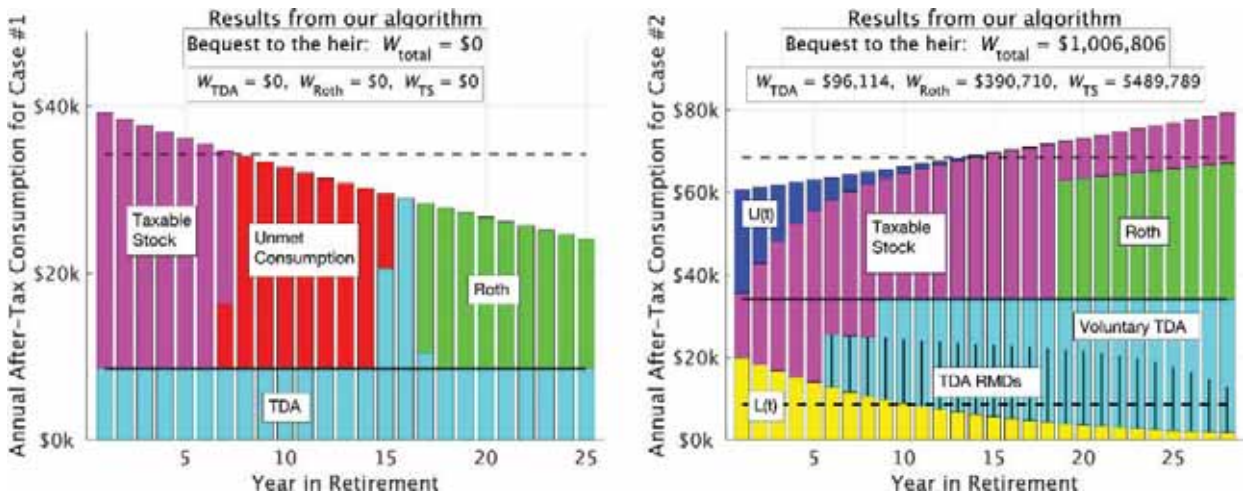


Fig. 1. Annual after-tax consumption for Cases 1 and 2. The parameter values for all of our cases can be found in Table 1. For Case 1 in the left panel, we attempt to address the retiree's consumption needs, $C(t)$, by choosing values of the three decision variables: TDA spending (in light blue), Roth spending (in green), and taxable stock and dividends spending (in magenta). In this case, even without considering RMDs or taxation on dividends, there is too little money in these three sources, so the retiree has unmet consumption needs (in red). More specifically, any red section in the graph indicates that the investor's consumption needs cannot all be fulfilled, no matter how the three decision variables are chosen. For Case 2 in the right panel, we have two additional sources to address consumption, but these are fixed, not decision variables: $L(t)$ (in yellow), which is subject to income taxes and therefore affects the marginal tax rate of TDA spending, and $U(t)$ (in dark blue), which has a fixed tax rate that does not affect, nor is affected by, the tax rates determined by the three decision variables. The TDA is divided between RMDs, which start at age 70 and a half and are represented by the parts of the light blue bars with vertical line segments within them, and voluntary TDA consumption, which is represented by the parts of the light blue bars without vertical line segments. The horizontal lines on the graph represent tax bracket thresholds in real dollars. The solid horizontal line represents H_{heir} , which corresponds to the effective marginal tax bracket for the heir. The tax brackets, starting from the bottom, are: 10%, 12%, 22%, 24%, 32%, 35%, and 37%. The higher brackets are out of the range shown in the figure.

where $W_{TDA}(t)$, $W_{Roth}(t)$, and $W_{TS}(t)$ are the pretax worths of the Roth, TDA, and taxable stock accounts at time t . In the context of this equation, the factor $\frac{1}{a}$ represents the additional benefit to the heir of having the tax advantages of the TDA and the Roth account before they are liquidated by the heir. In Section 6, we will show how to extend this algorithm to optimizing portfolio longevity, instead of optimizing a bequest to an heir or heirs.

Before outlining our method, we explain the features of the bar graphs that we will use for visualization of both our method and many of our results.

5.1. Bar graph visualization and basic set up

We note the example bar graphs in Fig. 1 below. There is a bar for each year $t = 1$ through $t = t_{death}$. The height of the bar in year t is $C(t)$, the known/projected real dollar after-tax consumption needs of the retiree in that year. Just below the title of the bar graph, we present the four quantities in Eq. (1): $W_{total}(t_{death})$, $W_{TDA}(t_{death})$, $W_{Roth}(t_{death})$, and $W_{TS}(t_{death})$.

Because we have assumed the tax bracket thresholds are constant in real dollars over time, as is usually the case, the income bounds for each tax bracket correspond to horizontal lines on the graph. These are represented by dashed lines, with the exception of our using a solid line on the graph at the height, H_{heir} , which we define as the unique height below which $\tau_{marg} \leq \tau_{heir}$ and above which $\tau_{marg} > \tau_{heir}$.

The case number, given in the graph's vertical axis label, corresponds to specific values for parameters, which can be found in Table 1. These parameters are: the initial balances for the TDA, Roth, and taxable stock accounts, the annual consumption needs of the retiree, the values of $L(t)$ and $U(t)$, the values of t_{death} , μ_t , d_t , τ_{div} , τ_{gains} , τ_{heir} , and a . Also, we must specify the age of the retiree at $t = 1$, so we know when the retiree reaches the age of 70 and a half and TDA RMDs begin. We will restrict our computations, although not our algorithm, to the case of the retiree buying only a single lot of taxable stock prior to $t = 1$, so we must specify the initial value of ω for this lot. In all of our cases, we use the IRS tax brackets for a single filer from 2018, which are also given in Table 1.

We note that because our bar graphs are in after-tax dollars, the after-tax values in the final row of our table correspond to the heights of the dashed and solid horizontal lines in the bar graphs.

Since the retiree's consumption needs must be fulfilled with after-tax dollars, the consumption bars must be filled with after-tax money from the retiree's five money sources: $L(t)$ (in yellow), $U(t)$ (in dark blue), TDA money (in light blue), Roth money (in green), and taxable stock money/dividends (in magenta). Consumption needs that are not filled by any source are shown in red.

Because $U(t)$ involves known/projected sources of money for consumption with known fixed tax rates, we can determine the after-tax worth of these sources, which gives us the value of $U(t)$. We then subtract $U(t)$ from $C(t)$ to determine the investor's remaining consumption needs. That is, $U(t)$ essentially lowers the heights of the consumption bars, so we represent this by placing the consumption from $U(t)$ at the top of the bars.

There are two sources of money subject to income tax, $L(t)$ and the TDA, and we put them – again, in after-tax dollars – at the bottom of the bars, so that their income tax rate is clear. Because $L(t)$ involves known/projected sources of money for consumption, we put it at the very bottom. Because TDA consumption is a decision variable, we ideally choose to spend it in the lower tax brackets, following guiding principle 1. Geometrically, this can be accomplished by thinking of $L(t)$ as a fixed sandy shore at the bottom of the graph and the TDA as calm water on top of it. Vertical line segments placed within the TDA spending indicate RMDs from the TDA. We refer to TDA spending that is not a part of RMDs and, therefore, does not have vertical line segments, as “voluntary TDA spending.”

Consumption from the final two sources of money, Roth and taxable stock/dividends, is represented in the graph above the sand/water geometry of the $L(t)$ /TDA system and below $U(t)$. We place taxable stock/dividend spending above Roth spending when they occur in the same year.

After our initial application of $U(t)$ to the top of the bars and $L(t)$ to the bottom of the bars, our algorithm looks to find the optimal strategy for the three time-dependent decision variables (the TDA, Roth, and taxable stock/dividends) using the four stages outlined in the next subsection:

Table 1. Parameter values for computations the following table gives the parameter values for all the cases presented in this paper

Parameters	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8	Case 9	Case 10	Case 11
t_{death} (years)	25	28	25	28	37	*	30	20	*	20	25
μ_t	5%	6%	5%	6%	5%	5%	5%	5%	7%	5%	5%
d_t	2%	3%	2%	3%	2%	2%	2%	2%	4%	1%	*
τ_{div}	15%	15%	15%	15%	15%	15%	15%	15%	15%	15%	15%
τ_{gains}	15%	15%	15%	15%	15%	15%	15%	15%	15%	15%	15%
τ_{heir}	11%	17%	11%	17%	30%	14%	28%	23%	28%	22%	16%
α	0.9	0.91	0.9	0.91	0.99	0.9	0.91	0.9	0.91	0.85	1
ω at $t = 1$	0.614	0.312	0.614	0.312	0.231	0.614	0.231	0.614	0.713	0.231	0.054
TDA money at $t = 1$	\$160k	\$300k	\$350k	\$325k	\$1,700k	\$230k	\$1,900k	\$1,300k	\$400k	\$1,250k	\$580k
Roth money at $t = 1$	\$60k	\$160k	\$90k	\$160k	\$275k	\$50k	\$300k	\$400k	\$100k	\$1,600k	\$1,200k
Taxable stock money at $t = 1$	\$180k	\$550k	\$150k	\$550k	\$100k	\$150k	\$1,500k	\$300k	\$79k	\$2,250k	\$1,000k
Retiree age at $t = 1$	70	65	70	65	70	70	70	65	30	70	70
Value of A in	\$0	\$20k	\$0k	\$20k	\$10k	\$10k	**	\$30k	\$0k	\$0k	\$0k
$L(t) = Ae^{b(t-1)}$	0	-0.09	9	-0.09	-0.2	-0.2	**	0	0	0	0
Value of b in	\$0	\$25	\$0	\$15k	\$0	\$5k	***	\$0	\$0	\$0	\$0
$L(t) = Ae^{b(t-1)}$	0	-0.3	0	-0.25	0	-0.25	***	0	0	0	0
Value of A in	\$40k	\$60k	\$40k	\$60k	\$81k	\$50k	****	\$140k	\$35k	\$180k	\$100k
$U(t) = Ae^{b(t-1)}$	-0.02	0.01	-0.02	0.015	0.01	0.005	****	0.01	0.006	0.01	0.01
Value of r in											
$C(t) = A(1+r)^t$											

* Values vary and are listed in the paper.

** $L(t) = \$10,000 + \$80(t - 1)(29.7 - t) + \$9,000(1.1 + \cos(0.524t))$.

*** $U(t) = \$25,000(1.1 - \cos(0.230t))$. **** $C(t) = \$200,000 + \$100(t - 1)(32.1 - t)$.

The following table gives the incomes at which each IRS tax bracket begins for a single filer in 2018 (see <https://taxfoundation.org/2018-tax-brackets/>):

Bracket tax rate	10%	12%	22%	24%	32%	35%	37%
Bottom of the bracket in pre-tax dollars	\$0	\$9,525	\$38,700	\$82,500	\$157,500	\$200,000	\$500,000
Bottom of the bracket in after-tax dollars	\$0	\$8,572.50	\$34,246.50	\$68,410.50	\$125,410.50	\$154,310.50	\$349,310.50

5.2. Outline of our algorithm

While the full details of the algorithm can be found at the Academy of Financial Services online 20018 conference proceedings,⁹ including how to accommodate different taxable lots, our algorithm can be summarized in the following four stages:

Stage 1, Optimizing using just the TDA and Roth account: We ignore the existence of the taxable stock account and the TDA RMDs in this stage. For our first step, we follow DiLellio and Ostrov (2017). Specifically, we fill the bar graph system with TDA “liquid” until it is exhausted or rises up to H_{heir} , the level of the solid horizontal line corresponding to τ_{heir} . We then use Roth money to fill the bars as a light gas would fill them, rising to the top of the bars until the Roth money is exhausted or comes down to the H_{heir} line. If either the TDA or Roth account is exhausted but the other fund is not, we continue to use the other fund as before until it is exhausted or all consumption needs are filled. This gives a graph like the left panel in Fig. 2.

For our second step, working under the restriction of maintaining the same $W_{total}(t_{death})$, we move the TDA and Roth money so that the unused consumption is moved to the left, that is, the earlier years, as much as possible, followed by the TDA being moved to the left as much as possible. This gives a graph like the right panel in Fig. 2, which is ready for taxable stock to be applied in Stage 2.

Stage 2, Optimally using taxable stock if there were no dividends: Setting $d_t = 0$ temporarily, we first fill all previously unmet consumption with taxable stock. In the figures – see the left panel of Fig. 3, for example – this means we replace the red section remaining after Stage 1 with taxable stock in magenta. We then compute for each time and tax bracket “desirability factors” for the taxable stock, the TDA, and the Roth. By comparing these factors, we determine the most advantageous time and bracket to replace the TDA or Roth with

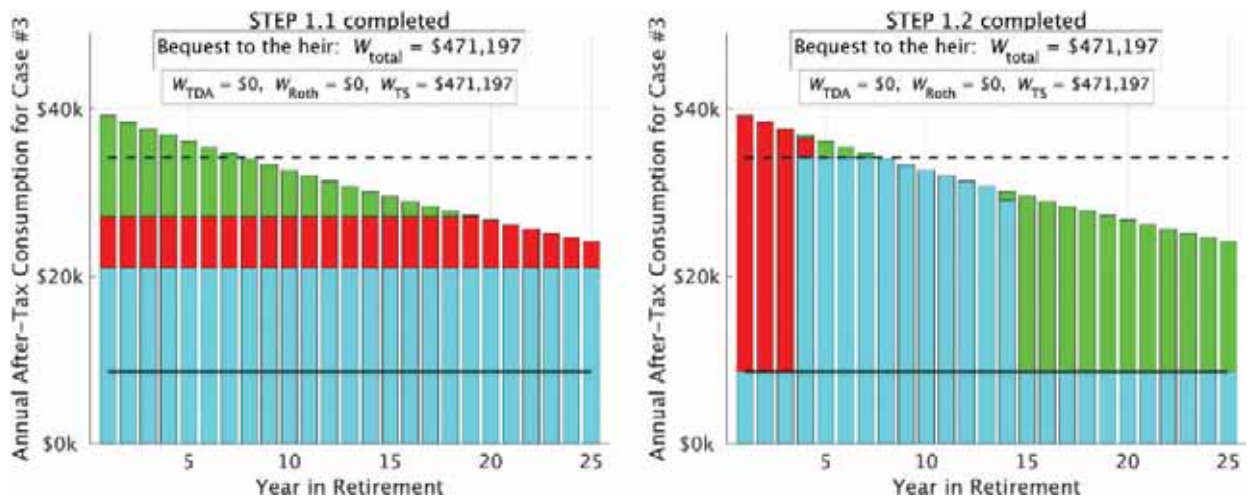


Fig. 2. Stage 1 for Case 3. Left panel: Step 1 produces an optimal strategy for the TDA (in light blue) and the Roth (in green). The unmet consumption needs are in red. Right panel: In preparation for using taxable stock money in Stage 2, in Step 2 we move the unmet consumption to as early as possible in the transition tax bracket and the brackets above it. Then, within the 12% tax bracket, we move the TDA money to earlier than the Roth money, because, being above H_{heir} , it is less desirable for consumption than the Roth. Note that W_{total} is unchanged by Step 2.

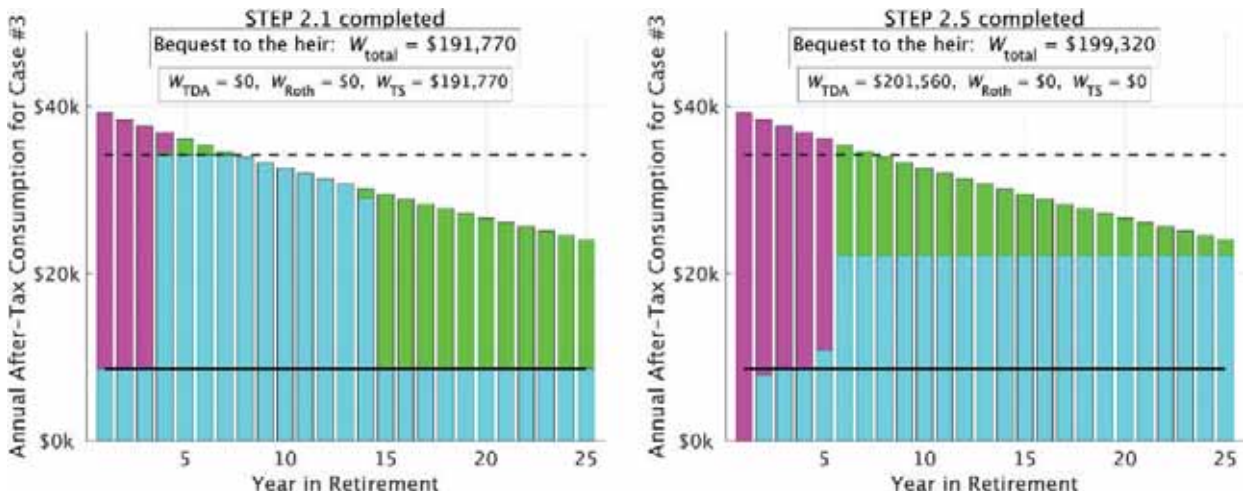


Fig. 3. Stage 2 continued for Case 3. Left panel: Taxable stock fills previously unmet consumption, replacing the red of unmet consumption in the right panel of Fig. 2 with the magenta that represents taxable stock (including dividends). Right panel: Taxable stock, until it is exhausted, has replaced TDA and Roth spending at times and tax brackets where it was most advantageous. After then freezing the taxable stock spending, we reapply Step 1 of Stage 1 giving the horizontal line between the TDA and Roth spending. Note that W_{total} increases because of these changes in Stage 2.

taxable stock, and then make this replacement, continuing in this manner until we run out of taxable stock or all remaining TDA and Roth consumption spending is more desirable than additional taxable stock consumption spending. We then do some cleaning up with some small steps that include freezing the taxable stock spending and rerunning Step 1 of Stage 1 on the TDA and Roth money again. The right panel in Fig. 3 gives an example of the result after this stage.

Stage 3, Incorporating dividends: We reset d_t back to its real value, which reduces $W_{total}(t_{death})$. The generated dividends are first used to replace taxable stock spending. Then they are used to replace Roth spending, unless there is a better case for dividends to replace TDA spending, in which case we do that instead. There are a few additional clean-up steps that may or may not need to occur. By the end of Stage 3, we have changed a figure like the left panel of Fig. 4 to the right panel of Fig. 4. Note that $W_{total}(t_{death})$ increases because of using dividends to replace other spending during this stage.

Stage 4, Incorporating RMDs from the TDA: We compute the RMDs for the TDA. If they are already satisfied, then we are done. If they are not, as is the case in the left panel of Fig. 5 where the vertical bars extend past the blue TDA spending, then we apply an iterative procedure to satisfy the RMDs while maintaining TDA spending in the lowest tax brackets possible. An example of the result of this procedure is given in the right panel of Fig. 5.

6. Optimizing portfolio longevity

In this section, we show how determining a retiree's optimal portfolio longevity is a subset of the problem of determining how a retiree can optimize their bequest to an heir. This is

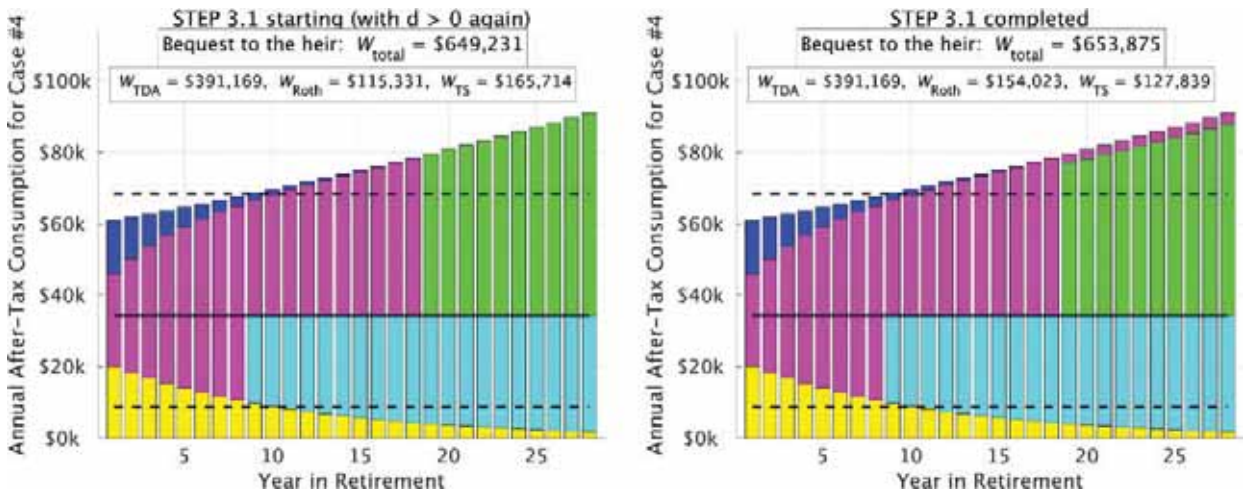


Fig. 4. Step 3 for Case 4. Left panel: The bar graph here is the same as it was after the end of Stage 2. However, the new inclusion of a 3% dividend rate decreases W_{total} at t_{death} from \$667,908 to \$649,231. This decrease, which is strictly through W_{TS} , happens because taxes must immediately be paid on the dividends that, starting in year $t = 19$, are reinvested. Right panel: Following guiding principle 4, we apply the dividends to consumption needs instead of reinvesting them. All the magenta in the bars where $t \geq 19$ represent dividend spending, as opposed to other taxable stock spending. This superior strategy increases W_{total} to \$653,875.

because we can simply run our bequest algorithm from Section 5 repeatedly with progressively larger values of t_{death} if $W_{total}(t_{death}) > 0$ and progressively smaller values of t_{death} if the retiree has consumption needs that cannot be met. A fraction, α , of the final year can be accommodated by multiplying the last year’s annual consumption needs of the retiree by α .

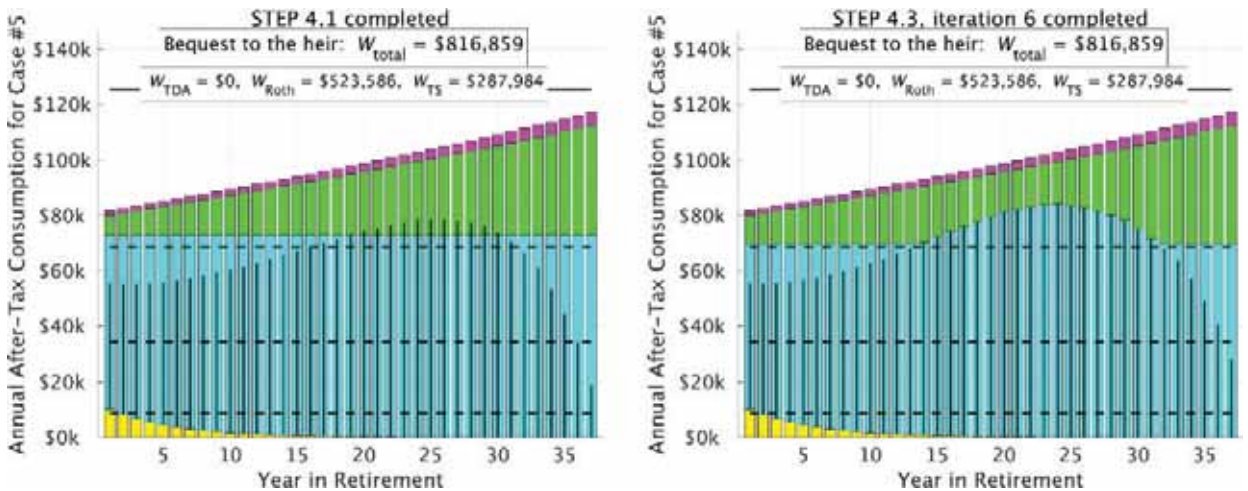


Fig. 5. Stage 4 for Case 5. We note that all the consumption is below H_{heir} , making spending TDA money a priority. Also, no taxable stock other than the dividends are used for consumption. Left panel: The colors represent the results after Stage 3, but the vertical segments representing TDA RMDs for this case are not contained in the light blue TDA spending in years 20 to 30, so RMDs are not yet satisfied. Right panel: TDA expenditures have been moved in such a way that all RMDs are satisfied while continuing to maintain TDA spending in as low tax brackets as possible. In this case, in both panels, the TDA is exhausted and TDA spending covers the entire 22% bracket. The way the remaining TDA spending is spread out in the 24% tax bracket is immaterial to the end result, as stated in guiding principle 1, and confirmed by the identical W_{total} values in the two panels. That is, in this case, the RMDs are able to be satisfied without any negative repercussions to the retiree.

Once we have converged to the value of $t = t_{longevity}$ where $W_{total}(t_{longevity}) = 0$, we have the optimal portfolio longevity for our algorithm. An example of this procedure is given in Fig. 6.

We note that the values of τ_{heir} and a are irrelevant to portfolio longevity since there is no bequest to the heir. Therefore, any values for these parameters can be selected when running the algorithm; they will have no effect on the portfolio longevity, $t_{longevity}$, that is determined.

7. Results

We discuss computed results from our algorithm for a variety of cases, where, as before, each case’s parameter values can be found in Table 1. In all of our cases, we have used the

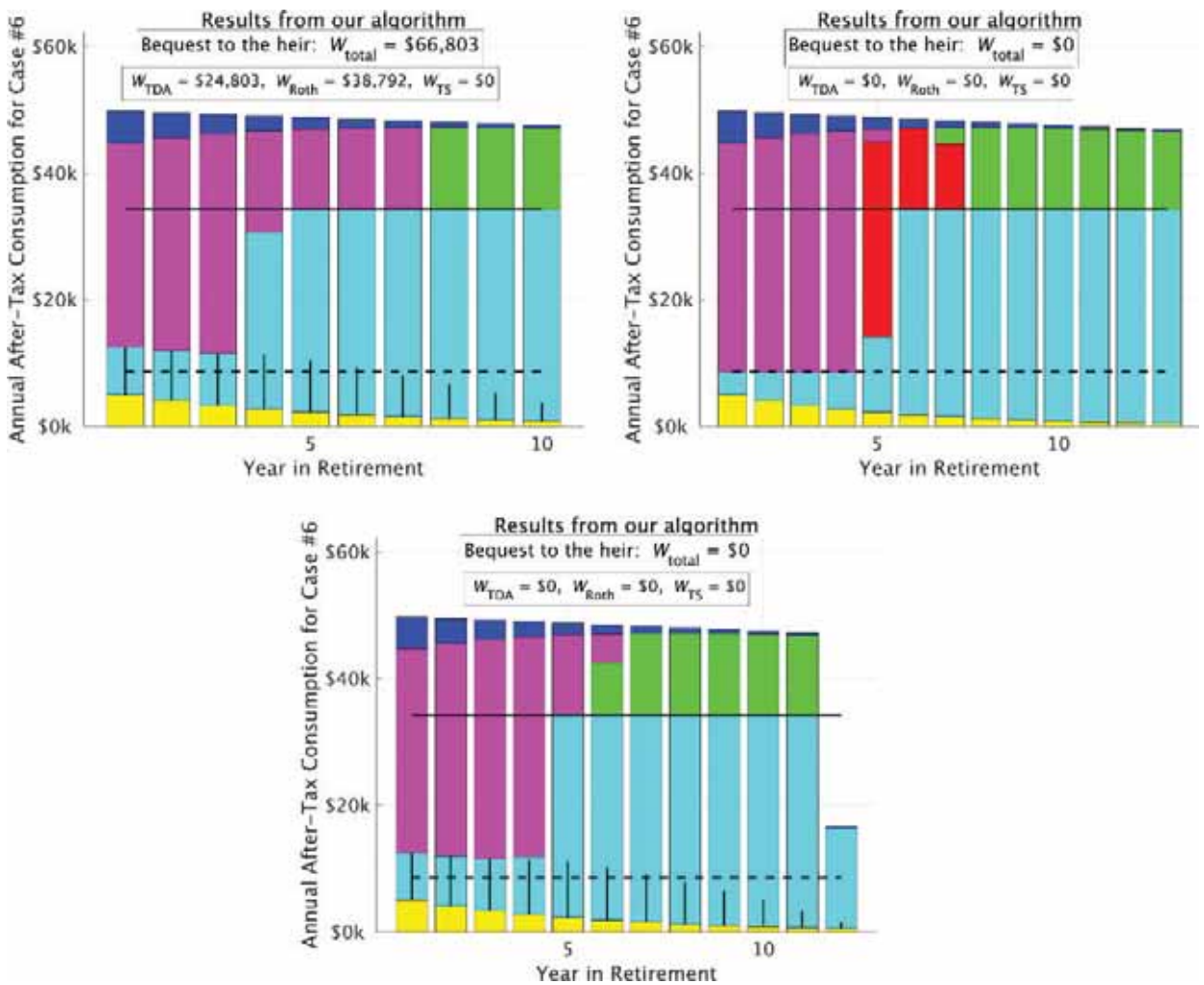


Fig. 6. Obtaining the optimal portfolio longevity for Case 6. Upper left panel: Running our algorithm with $t_{death} = 10$ leads to $W_{total} > 0$, a positive bequest to the heir, so we increase t_{death} . Upper right panel: Running our algorithm with $t_{death} = 13$ leads to unmet consumption (in red), so we must decrease t_{death} . Lower center panel: Continuing in this fashion, we converge on the optimal portfolio longevity of 11.35 years. We note that $\alpha = 0.35$ in this case, meaning that the consumption needs in the final year are reduced to 35% of their normal value.

IRS tax brackets for a single filer from 2018, which are also given in Table 1, although our program can also easily accommodate changes to the thresholds, rates, and number of tax brackets, should Congress change any of these. The MATLAB R2017a computer program for our algorithm typically ran in under 4 seconds on an iMac with a 4 GHz Intel Core i7 processor and 16 GB of 1867 MHz DDR3 memory.

7.1. Examples of our algorithm results

The three computed examples in Fig. 7 demonstrate some of the wide range of behavior that our algorithm captures. The two upper panels of Fig. 7 are the final products of the algorithm for Case 3 and Case 4, whose intermediate steps were presented in Section 5.

For Case 3 in the upper left panel of Fig. 7, we see that the solid horizontal line for H_{heir} is at the top of the lowest tax bracket, which is the 10% bracket, because $\tau_{heir} = 11\%$ here, which is less than 12%, the rate for the second bracket. Ideally, the TDA would only fill the area below H_{heir} and nothing above, but RMDs require that more be filled, following guiding principle 5. After that, it is hoped that the Roth and taxable stock accounts can fill the remainder. The optimal strategy requires the taxable stock to be consumed as early as possible, following guiding principle 2. In this case, we fill to the point where the taxable stock is exhausted, causing $W_{TS}(t_{death}) = 0$, then we fill with the Roth. The Roth also becomes exhausted, causing $W_{Roth}(t_{death}) = 0$, so the remainder of the consumption needs must be filled with “voluntary” TDA money. We note that the taxable stock fills all consumption needs that lie in the third tax bracket, which is the 22% bracket, which is optimal since TDA spending in this bracket would be highly taxed. However, by guiding principle 1, the method in which the TDA and the Roth fill the 12% tax bracket does not matter, as long as the Roth is exhausted and all consumption needs are satisfied, as is the case here.

For Case 4 in the upper right panel of Fig. 7, we have known, fixed sources that create $L(t)$ and $U(t)$ at the bottom and the top, respectively, of the consumption bars. In this case, $W_{TDA}(t_{death})$, $W_{Roth}(t_{death})$, and $W_{TS}(t_{death})$ are all non-zero, so the TDA stays below H_{heir} and the Roth stays above H_{heir} . Up through year 18, there is a stronger case to use taxable stock instead of the Roth. That is, during this period, the forgiveness of capital gains for taxable stock when the retiree dies are a weaker effect than the erosive effects of dividend taxes and the inability to shield the heir’s subsequent gains from taxes. Starting in year 19, the forgiveness of capital gains for taxable stock becomes the stronger factor, making using the Roth preferable to using taxable stock, so, starting in year 19, only dividends are consumed from the taxable stock account, as required by guiding principle 4. Similarly, up through year 8, there is a stronger case to use taxable stock instead of the TDA in the 12% tax bracket, but, starting in year 9, this preference reverses. Note that Appendix 5 of DiLellio and Ostrov (2018) contains a table with the specific values for the annual consumption and remaining balances that correspond to the graph for Case 4, as displayed in Fig. 7.

From an economic point of view, $C(t)$, $L(t)$, and $U(t)$ generally exhibit exponential growth or decay. For example, $C(t)$ may need to grow faster than inflation to accommodate increased medical needs; $L(t)$ may represent part-time work that decreases over time after retiring; or $U(t)$ may be a tax-free pension that grows with inflation and is therefore constant

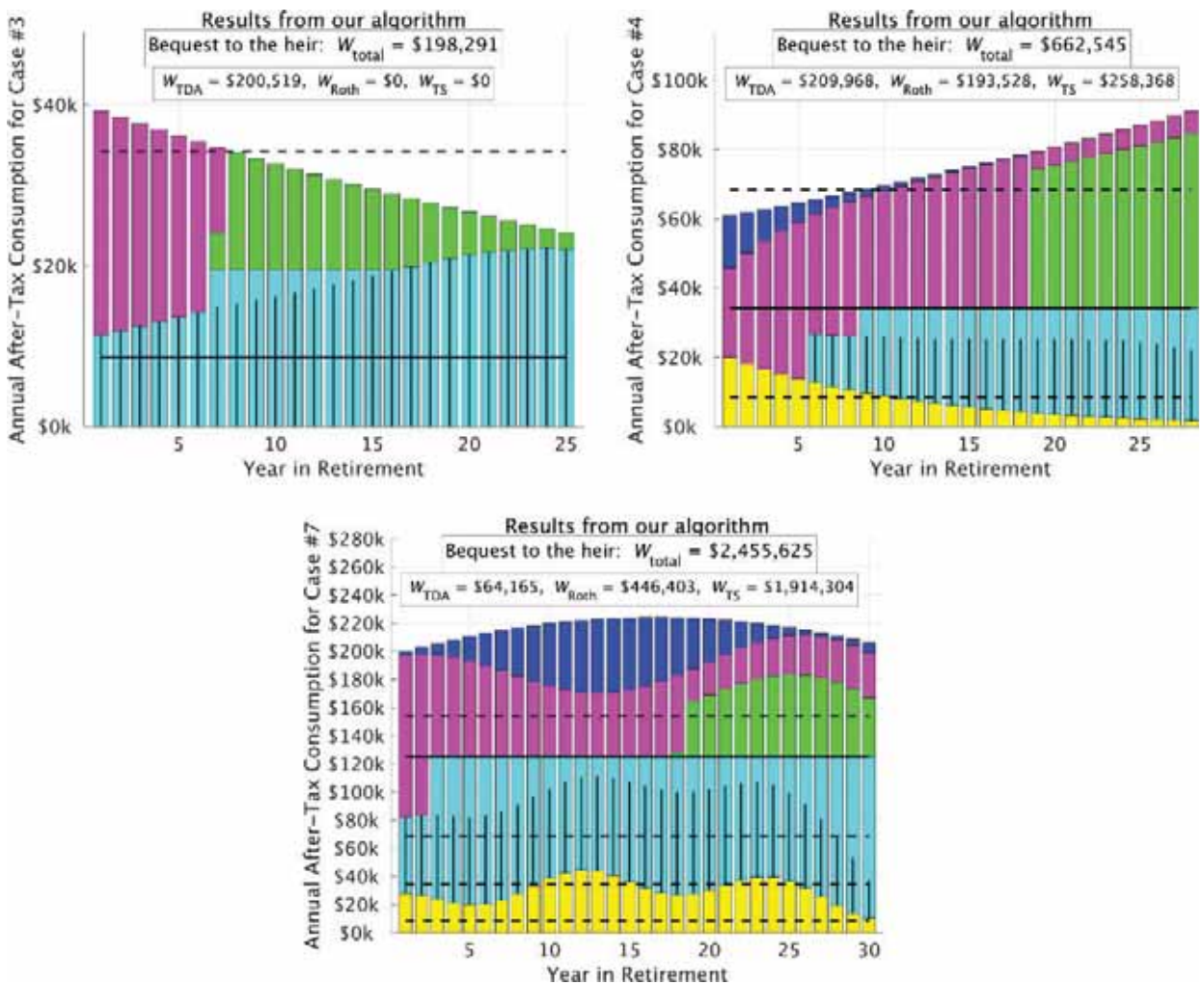


Fig. 7. Algorithm results. Upper left panel: Case 3. Upper right panel: Case 4. Lower panel: Case 7.

in real dollars. However, this restriction to exponential models is only for economic reasons. Our algorithm is capable of handling any functions for $C(t)$, $L(t)$, and $U(t)$ that we wish to model. Case 7 in the lower panel of Fig. 7, for example, uses non-exponential functions for all three.

7.2. Comparison of our results with the naïve and informed strategies

In the introduction, we discussed previous common strategies non-academics and academics have used for drawing down funds in retirement, which Horan called naïve and informed strategies. In all of these strategies, RMDs from the TDA are first satisfied. In naïve strategy 1, the retiree then drains the taxable stock account, followed by the TDA, and finally the Roth. In naïve strategy 2, the retiree then drains the taxable stock account, followed by the Roth, and finally the TDA. In an informed strategy, the retiree drains the TDA up to the top of one of the tax brackets, and then fills any excess consumption needs by first liquidating the taxable stock account, followed by the Roth, and finally the TDA if it has not

already been drained. The best informed strategy selects the tax bracket with the best outcome. Our algorithm generates results that are superior or equal to those generated by either of the naïve strategies or the best informed strategy. This has held in every case that we have run, including not just the cases presented in this paper, but also the numerous other cases we have run but not included here for the sake of space.

Case 8, for example, which is shown in Fig. 8, was produced by considering an investor who had a salary of \$200,000 and followed two standard rules of thumb for retirement: (1) the investor saved 10 times their salary before retiring, and (2) the investor planned on initially spending 70% of their salary during retirement.¹⁰ Comparing the four strategies, we find that the size of the bequest, $W_{total}(t_{death})$, for our algorithm is highest, followed by the best informed strategy, naïve strategy 1, and finally, naïve strategy 2.

In Table 2, we find similar results for Case 6 and Case 9, where we compare optimal portfolio longevity instead of bequest size. Case 6 was previously presented in Fig. 6 Case 9

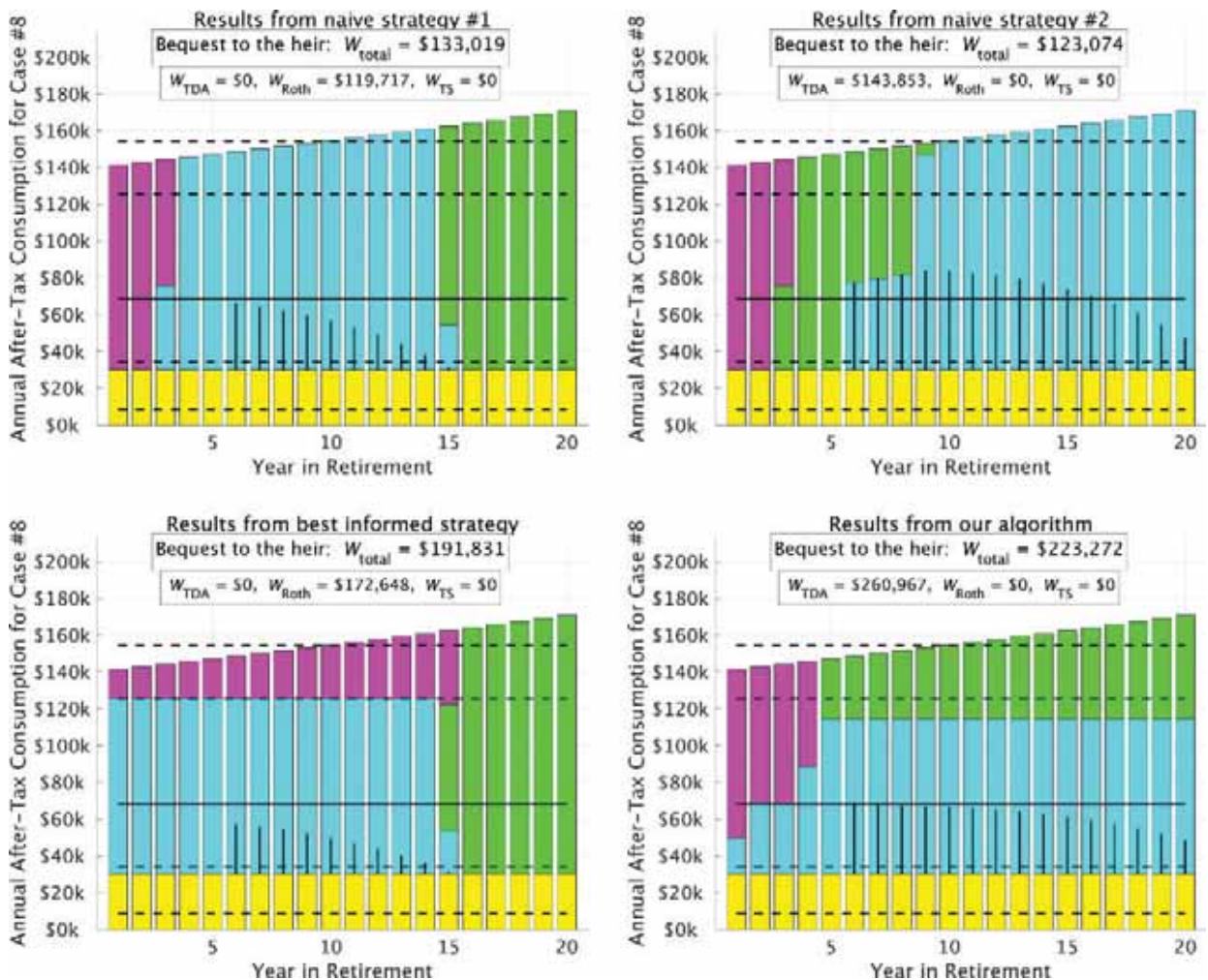


Fig. 8. Comparison of naïve strategy 1 (upper left panel), naïve strategy 2 (upper right panel), the best informed strategy (lower left panel), and the strategy generated by our algorithm (lower right panel). We note that the best informed strategy in this case fills the retiree’s consumption needs with TDA money up to the top of the fourth tax bracket, which is the 24% bracket.

Table 2 Portfolio longevity for various strategies using 2018 tax brackets and rates

	$t_{longevity}$ (in years), naïve strategy 1	$t_{longevity}$ (in years), naïve strategy 2	$t_{longevity}$ (in years), best informed strategy	$t_{longevity}$ (in years), our algorithm's strategy
Case 6	11.17	11.18	11.26	11.35
Case 9	36.18	35.75	35.99	37.23

Table 3 Portfolio longevity for various strategies using 2017 tax brackets and rates

	$t_{longevity}$ (in years), naïve strategy 1	$t_{longevity}$ (in years), naïve strategy 2	$t_{longevity}$ (in years), best informed strategy	$t_{longevity}$ (in years), our algorithm's strategy
Case 6	10.93	10.94	11.06	11.12
Case 9	33.50	33.11	33.67	35.12

somewhat resembles the pictures for Case 8 in Fig. 8, although it applies to an individual with a longer time horizon, consumption needs that only reach into the 22% tax bracket, no RMDs, and no resources other than the TDA, Roth, and taxable stock. For Case 6, the best informed strategy fills the retiree's consumption needs with TDA money up to the top of the 12% bracket. In Case 9, the best informed strategy fills up to the top of the 10% bracket with TDA money.

As emphasized in the heading for Table 2, the calculations throughout this paper use the 2018 tax brackets and tax rates passed into law by Congress at the end of 2017. These brackets and rates are set to revert to the previous 2017 brackets and rates in 2025 if Congress takes no further action. It is easy to modify our algorithm to use these 2017 brackets and rates, which we have done for Table 3. Comparing Table 2 and Table 3 demonstrates the effect of the 2018 versus 2017 brackets and rates on portfolio longevity. For Table 3, the best informed strategy fills up to the top of the 12% bracket in Case 6 and the 10% bracket in Case 9. As with Table 2, these correspond to the second lowest and the lowest tax brackets.

7.3. Sensitivity analysis

We return to Case 4 to explore how sensitive the optimal decisions are to changes in the underlying parameters. Recall from the withdrawals shown in the upper right-hand panel in Fig. 7 for Case 4, that, within the 12% tax bracket, it is preferable to use taxable stock instead of TDA money in years 1–8 when possible, although TDA money must still be used to satisfy TDA RMDs during these years. Starting in year 9, the situation switches: using TDA money becomes preferable to using taxable money for years 9–28, so we see that year 9 is the first year where voluntary TDA money is applied. Within the 22% and 24% tax brackets just above the 12% bracket, applying taxable stock money for consumption is preferable to Roth money, until we switch at year 19, when it becomes preferable to apply dividends and then Roth money instead of consuming non-dividend taxable stock money.

We begin our sensitivity analysis with the effect of the dividend rate, d_t on these two switching times. From Table 1, we note that d_t is 3% for Case 4. If we reduce d_t from 3% to

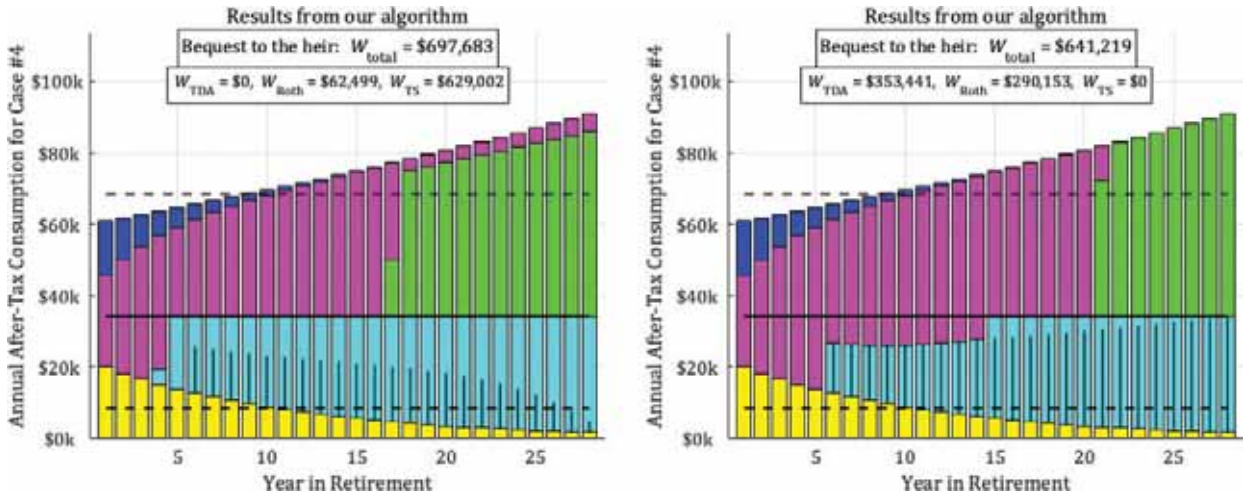


Fig. 9. The effect of dividends on withdrawal choices for Case 4: In the left panel, the dividend rate, $d_t = 1\%$. In the right panel, $d_t = 5\%$.

1%, we get the strategy shown in the left panel of Fig. 9. The corrosive effect of dividends, which we discuss more fully later, is reduced, so there is less push to avoid them by spending down taxable stock early. This means the switching year from taxable stock to TDA in the 12% bracket is reduced from year 9 to year 4, while the switching year from taxable stock to the Roth in the brackets above that is reduced from year 19 to year 17. In fact, ideally, the switching year in the 12% bracket might be earlier than year 4; however, because the TDA is exhausted, there is no additional TDA money to replace the taxable stock within the 12% bracket in earlier years. In the right panel of Fig. 9, we see the opposite behavior: we have increased d_t from 3% to 5%, so we see the switching year in the 12% bracket increase from year 9 to year 15 and then, within the higher tax brackets, the taxable stock exhausts itself at year 21 and consumption switches over to Roth spending. Quantifying where these withdrawal strategy shifts should optimally occur when there is a choice has not been done in other research, to our knowledge.

Because we can quantify these withdrawal switch times, we can measure their sensitivity to changes in other parameters. We note from Table 1 that for Case 4, we have that $d_t = 3\%$, $a = 0.91$, and $\mu_t = 6\%$. We also have initial balances in the TDA, Roth, and taxable stock accounts equal to \$325,000, \$160,000, and \$550,000, respectively. Finally, $\tau_{heir} = 17\%$ and $\tau_{div} = \tau_{gains} = 15\%$. In Table 4, we adjust each of these parameters upwards and downwards from their base case values in Table 1, and see the effect this has on the switching years within the 12% bracket and within the brackets above that. We next explain the qualitative reasons that correspond to the results seen in Table 4.

Recall that the value of a is determined by how long the heir stretches out the removal of funds from their inherited TDA and Roth account. Should they liquidate all of their accounts upon inheritance, then $a = 1$. In this case they are not taking advantage of the additional tax shielding available for the TDA and Roth, making these two accounts less valuable to the heir, so the retiree moves both switching years earlier to spend more of the TDA and Roth. As a gets smaller, both switching years move to later times. We see that this is a large effect for the TDA switching year in the 12% bracket, changing from year 4 when $a = 0.95$ to year

Table 4 Sensitivity analysis for Case 4

Description	Switch year: Taxable to TDA in the 12% bracket	Switch year: Taxable to Roth in higher brackets
Baseline (Case 4)	9	19
$d_t = 1\%$	4	17
$d_t = 5\%$	15	21
$a = 0.85$	12	22
$a = 0.95$	4	17
$\mu_t = 5\%$	10	20
$\mu_t = 8\%$	6	17
Initial TDA balance = \$300,000	9	19
Initial TDA balance = \$350,000	9	19
Initial Roth balance = \$120,000	9	19
Initial Roth balance = \$400,000	9	19
Initial Stock balance = \$450,000	8	17
Initial Stock balance = \$650,000	9	19
$\tau_{heir} = 13\%$	17	19
$\tau_{heir} = 25\%$	4	19
$\tau_{div} = \tau_{gains} = 10\%$	9	24
$\tau_{div} = \tau_{gains} = 20\%$	7	17

9 in the base case where $a=0.91$ to year 12 when $a=0.85$. For the Roth switch in the higher tax brackets, the effect is less pronounced, changing from years 17 to 19 to 22. We also see from Table 4 that as μ_t is increased, the switching years both get a little earlier. This is because the capital gains increase with μ_t , giving a greater incentive not to use taxable stock for consumption and instead take advantage of the step up in basis enjoyed by the heir. There is also an increase in dividends as μ_t increases, which pushes the switching years to be later so as to lower the dividends. This effect, however, is smaller than the effect of the cost basis step up, which we see from the fact that the switching years get earlier, not later.

In Table 4 we see that the initial TDA and Roth balances have no effect on the switching years. This makes sense, since the only effect it should have is when the TDA or Roth accounts are completely drained, which does not happen in the range of values chosen here. Additional initial taxable stock money, however, moves the switching years later because of the need to reduce the additional dividends through earlier expenditures from the taxable stock account.

Increasing τ_{heir} gives incentive to the retiree to spend more TDA money, since it is of less worth to the heir. Therefore, as τ_{heir} increases from 13% to 25%, we see the TDA switching time in the 12% bracket decrease from year 17 to year 4 where the TDA is completely drained. The effects of changing τ_{div} and τ_{gains} work in opposite directions: As τ_{div} increases, the switching years move to later years to decrease the dividends. However, as τ_{gains} increases, the switching years move to earlier years to push more of the taxable stock to the heir, where gains are forgiven. Given that τ_{div} and τ_{gains} have been equal for decades, which effect is stronger? From the final rows of Table 4, we see that the effect from increasing τ_{gains} to move the switching year earlier is stronger than the effect from increasing τ_{div} to move the switching year later, since increasing these rates in unison moves the switching years to earlier times.

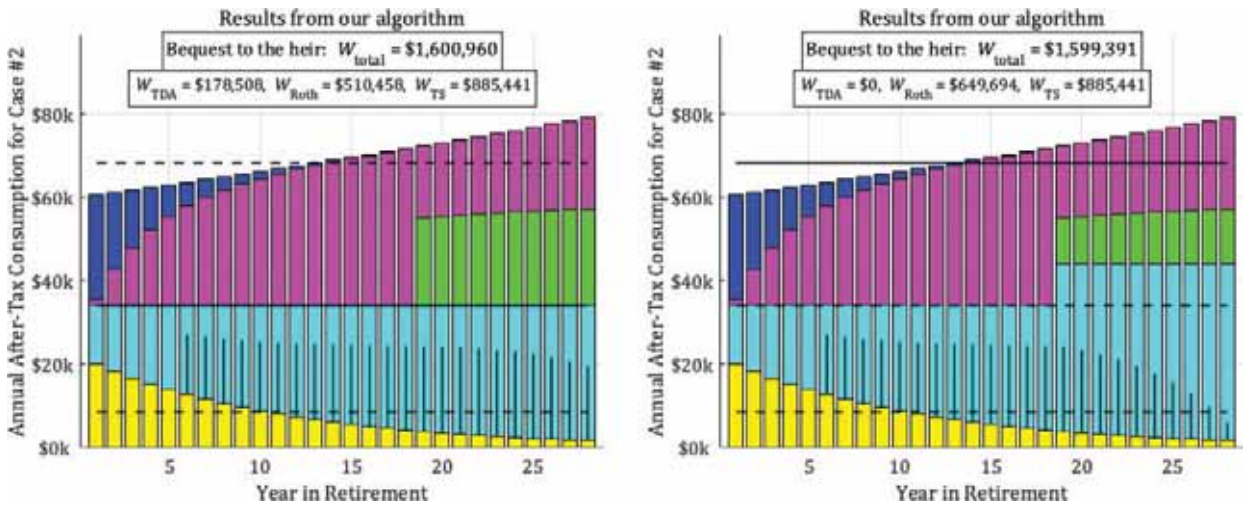


Fig. 10. The effect of the heir’s effective tax rate, τ_{heir} , on the retiree’s withdrawal choices for Case 2, but with an initial TDA balance of \$425,000. In the left panel, $\tau_{heir} = 21.2\%$. In the right panel, τ_{heir} is increased to 23%, which moves it from a value below the tax rate for the 22% bracket to a value above it.

7.4. Additional implications of the heir’s tax rate

Changing the value for τ_{heir} can have more interesting implications on the optimal strategy than just moving the switching year for preferring TDA to taxable stock spending that was seen in the previous example. Consider Case 2, which is shown in the right panel of Fig. 1 and looks similar to Case 4. From Table 1, we note that Case 2 has an initial TDA balance of \$300,000 and $\tau_{heir} = 17\%$. Coincidentally, as with Case 4, within the 12% tax bracket, it becomes preferable to switch from using taxable stock to using TDA money starting in year 9 and within the 22% tax bracket, it becomes preferable to switch from using taxable stock to using Roth money and dividends starting in year 19. As we increase τ_{heir} , as we saw in the previous example, the switch in the 22% bracket stays at year 19, while the switch in the 12% bracket moves from year 9 to earlier years. This consumes more and more of the TDA until it runs out. That is, all the TDA money is spent either in the 10% bracket or in all but the earliest years of the 12% bracket. Therefore, we boost the initial IRA amount from \$300,000 to \$425,000 and continue to increase τ_{heir} . Once τ_{heir} reaches 21.2%, the switch in the 12% bracket has moved to year 1. That is, there is no taxable stock, only TDA, used in the 12% tax bracket. Note that, unlike the Roth which never displaces dividend spending, the TDA displaces all dividend spending in the 12% bracket in this case (see Fig. 10, left panel). Once τ_{heir} increases past 22%, using the TDA becomes preferable to using the Roth in the 22% tax bracket, so the TDA now becomes exhausted replacing the Roth (see Fig. 10, right panel). Increasing the initial TDA amount further causes it to eventually replace all the Roth consumption. Should there be additional TDA money and then τ_{heir} is further increased, the TDA will first move the switching year to progressively early times. After the switching year, the TDA and dividends are used for consumption in the 22% tax bracket. When $\tau_{heir} = 29.02\%$, the switching year is year 4. As soon as we cross $\tau_{heir} = 29.02\%$, the TDA becomes preferable to dividends in years 4 and after within the 22% bracket and, therefore, the TDA immediately replaces all of this dividend spending. Dividends, along with $U(t)$, are

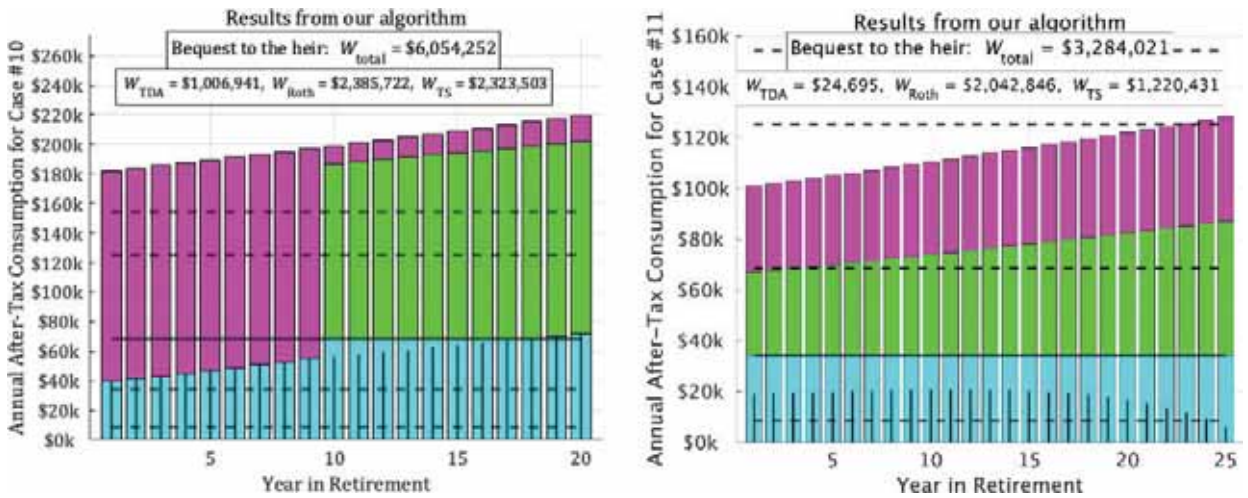


Fig. 11. Left panel: Case 10 for studying how sensitive the optimal strategy is to changes in the heir’s discount factor, a . Right panel: Case 11 for studying the corrosive effect of dividends, d .

still used exclusively in the 24% bracket and, along with taxable stock, for years one through three in the 22% bracket. As τ_{heir} increases further, the switching year decreases, so the TDA resumes pushing out the taxable stock and dividend spending in years one through three in the 22% bracket. If τ_{heir} is increased even further, the TDA begins to push out later dividend spending in the 24% bracket. We note that forward time algorithms are, by their nature, incapable of determining these types of effects of τ_{heir} upon the optimal withdrawal strategy.

Nor are forward time algorithms capable of determining the effect of a , the heir’s discount factor, on the optimal strategy. In the left panel of Fig. 11, we have Case 10, where the value of a is 0.85. Unlike Case 4, the switching time in year 10 is the same for both the TDA and the Roth account. This switching time is remarkably sensitive to the value of a . If a is increased from 0.85 to 0.91, the TDA and Roth are now always preferred, and so they fill all the area, with the exception of the dividends, which are still consumed in all years. On the other hand, if a is decreased to 0.81, taxable stock is always preferred, so it fills all of the area above the TDA RMDs. That is, the decision of the heir to quickly spend down their inherited TDA and Roth versus taking out RMDs from these accounts has a significant – and quantifiable – effect on how the retiree optimally withdraws funds.

Given a choice between two stocks with an expected return of 5%, where one stock returns 4 of that 5% as dividends and the other gives no dividends, the heavy majority of retirees and their advisors will pick the stock that returns dividends for reasons including that dividend paying firms are often considered to be more stable than non-dividend paying firms. From a taxation point of view; however, this is known to be a considerable mistake (e.g., see Demuth 2016). There are many reasons for this. It is well known for long-term stock investing that the loss of deferring capital gains taxes because of an intermediate sale and repurchase of a stock can have a considerable negative impact on the total gains after the final sale of the stock. Assuming $\tau_{gains} = \tau_{div}$, dividends are like a forced intermediate sale, except that dividends are worse for two reasons: (1) The sale of normal stock is at the cost basis determined by ω of the lot being sold, but dividends are like a sale on just the gains. That is, it is a sale on a part of the lot where $\omega = 0$, the worst case. (2) The fact that

dividends then lower ω for the rest of the lot will partially compensate for this if the stock is later sold by the retiree. However, once the retiree stops using taxable stock for consumption, this lot will not be sold by the retiree. It will go to the heir, where all gains are forgiven and the fact that ω was lowered for the retiree becomes irrelevant.

Our algorithm quantifies how problematic these tax issues with dividends are. Consider, for example, Case 11 shown in the right panel of Fig. 11, where the investor starts with a million dollars in taxable stock, $a = 1$, and the dividend rate is $d_t = 4\%$. All the magenta in the panel represents dividends, which are optimally used for consumption by general Principle 4. If we now change the dividend rate to $d_t = 0$, the magenta part will be replaced by the Roth in green, and the value of $W_{total}(t_{death})$ will increase from \$3,284,021 to \$3,608,910. That is, in this case, if the retiree chooses the stock without dividends for their taxable account, their heir will receive \$324,889 more dollars. In other words, we see from our results that the corrosive effect of dividends is considerable, especially for investors with large taxable stock holdings.

8. Conclusions

Previous strategies for how a U.S. retiree should allocate their withdrawals in a tax efficient manner among TDAs, Roth accounts, and taxable stock accounts have not depended on the amounts in these three accounts, nor on the parameters governing the retiree, nor on the parameters governing their heirs. In this paper, we have presented an algorithm that uses these amounts and parameters to develop a strategy that adapts to the retiree's specific circumstances. The development of our algorithm reveals insights into the complex structure governing the trade-offs in using these three accounts to satisfy retiree consumption needs. This is particularly challenging with the taxable account, because its tax structure differs significantly from the tax structure of the TDA and the Roth account.

Our algorithm starts by using current tax law to create five guiding principles that govern prioritizing consumption from the three accounts: (1) If the retiree consumes a given amount of Roth money and consumes given amounts of TDA money at various marginal tax rates, the order/allocation in which these are consumed is irrelevant; (2) In contrast, taxable stock and dividends are better spent earlier rather than spent later; (3) When consuming taxable stock, the lot with the highest cost basis should always be liquidated; (4) Dividends should always be consumed before Roth money; and (5) RMDs should always be taken out of any TDA. These five guiding principles lead to our algorithm for optimizing how to allocate from the three accounts, so as to meet the retiree's projected yearly future total after-tax consumption needs. Our algorithm incorporates a number of standard features studied in the research literature, such as working with RMDs from the TDA account and optimizing portfolio longevity, as well as a number of less standard features, such as incorporating the effect of dividends, optimizing a bequest to an heir, allowing for two types of additional fixed sources of money for the retiree, and accommodating different taxable lots in the taxable stock account.

Because our algorithm uses an all-time approach that takes information from every year to form its withdrawal strategy in each year, it is able to produce a number of results that were previously not possible with other approaches, including forwards-time approaches,

which generate their withdrawal strategy in chronological order. For example, our algorithm can avoid running out of the TDA early, which means that low tax brackets in later years cannot be exploited, or having too much TDA at the end, which means that the TDA can be forced into higher tax brackets, should the other accounts be drained. We are also able to use the financial situation of the heir to guide withdrawals, and have seen that these often have a considerable effect on the optimal withdrawal strategy. Our algorithm shows significant improvements over the previous naïve and informed forwards-time strategies. It is also compatible with the advantages of forwards-time conversion strategies in the literature.

Notes

- 1 In this paper, the term “stock” will be shorthand for a portfolio of stocks that may include mutual funds, exchange-traded funds, as well as a variety of individual stocks.
- 2 <https://academyfinancial.org/resources/Documents/Proceedings/2018/F2%20DiLellio%20and%20Ostrov.pdf>
- 3 <https://academyfinancial.org/page-18138>
- 4 <https://www.fool.com/retirement/2018/09/29/is-social-security-taxable.aspx>
- 5 If the retiree has no earned income, they cannot put dividend money into a TDA or a Roth account. Even if they have earned income, but are older than 70 and a half, they cannot put dividend money into a TDA. See IRS rules: <https://www.irs.gov/retirement-plans/traditional-and-roth-iras>.
- 6 We note that this is optimal in the following sense: for any stock at a loss, it is always optimal for the investor to sell the stock and then buy another stock with similar properties to immediately reap the tax advantage of realized capital losses. The replacement stock cannot be exactly identical because of wash sale rules. So, for example, a total stock market fund would be replaced with another total stock market fund that tracks a similar, but not identical, index. See, for example, Ostrov and Wong (2011).
- 7 Please see Tax Policy Center (2017).
- 8 We assume the retiree is not working, so the money remaining from dividends after taxes cannot be used to purchase stock in the TDA or Roth account.
- 9 <https://academyfinancial.org/resources/Documents/Proceedings/2018/F2%20DiLellio%20and%20Ostrov.pdf>
- 10 See, for example, <https://www.fidelity.com/viewpoints/retirement/how-much-money-do-i-need-to-retire>

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