Efficient Frontiers in Estate Planning

Ronald R. Crabb

This article explores the nature of the efficient frontier in probabilistic estate planning for 16 different estate plans by considering as random variables ages at death, rates of return on assets, and borrowing rates on debts. The simulation considers two couples, one middle aged, the other elderly. Two 16×16 matrices, one for each couple, are used to record and compare the results of every simulation. That comparative data, in conjunction with the coefficient of variation based efficient frontier, contain useful information for couples who, consistent with their levels of risk, desire to maximize the net present value of assets passing to their heirs. The efficient frontier is shown to be a function of three factors: assumptions, ages of the estate owners, and the discount rates of the heirs. Because of the instability shown in the efficient frontier, estate planners and estate owners must carefully examine not only the estate plans which fall on the efficient frontier but also those estate plans which fall just off that frontier.

I. Introduction

In the second issue of the *Financial Services Review* (FSR), Crabb (1992) demonstrates that traditional point estimate estate planning based on life expectancy is neither realistic nor unbiased, and that "probabilistic estate planning permits modern portfolio theory (mean, variance tradeoffs) to be used to select an optimal estate plan." Rather than ignoring risk by assuming that death occurs at life expectancy, probabilistic estate planning treats ages at death as a random variable. Markowitz's (1952) E-V rule is applied, and an optimal estate plan is defined as one which falls on the efficient frontier.

The purpose of this paper is to explore the nature of that frontier. In Crabb's (1992) article, the only random variable considered was ages at death. Only three alternate estate plans were considered, investment returns on assets were fixed over the duration of the analysis, and a single discount rate was used to compute the net present value of assets passing to heirs. Those rather severe and unrealistic limitations were necessary to demonstrate the superiority of making estate planning

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decisions based on a mean/variance tradeoff versus making estate planning decisions based on the remaining life expectancy(ies) of an estate owner(s).

In the real world, IRAs, TSAs, and 401(k)s are used to accumulate wealth on a tax deferred basis. Annual gifts are used to pass wealth to heirs to avoid estate taxes on both the gifts and growth of the gifts after receipt by the heir(s). Whole life insurance trusts and term life insurance trusts are common estate planning tools. Rates of return on assets may vary with, among other things, inflation expectations and the state of the economy. With the popularity of home equity loans and variable rate mortgages, borrowing rates on debts can vary as well. Wills are revised as income tax, estate tax, and family situations change. Although it is impossible to consider the universe of possible estate plans and to predict changes in income taxes, estate taxes, and family situations, the 16 plans chosen for analysis employ estate planning tools which estate planners commonly recommend to estate owners.

Consider ages at death, rates of return on assets, and borrowing rates on debts as random variables. A computer simulation, which systematically uses alternate wills, annual gifts, whole life insurance, term life insurance, leverage, and tax shelters, can generate a set of attainable estate plans (consisting of the net present values of the after-tax estates passing to the heir(s) and the standard deviations associated therewith). The shape of the efficient frontier emerges from among the set of attainable estate plans. Multivariable probabilistic estate planning can probe the nature of the estate plans which fall on the efficient frontier, and the nature of those which do not.

II. Methodology

The random death selection process is thoroughly explained on pages 144 through 147 of the second issue of the FSR, but a quick review is in order. Given a mortality table and the age of an estate owner, divide the number of persons expected to be alive at some future age by the number of persons alive at the age of the estate owner today. The resulting quotients compute for that estate owner the probabilities of survival to any future age. Except for very old people, the probability survival curve is shaped like a ski slope, with the probability of survival decreasing at an increasing rate through about age 80, and then decreasing at a decreasing rate to allow for the few persons who live into their early 100s. The computer chooses a random number, then interpolates the expected age at death from the survival curve.

Random deviations from the expected rates of return on assets and borrowing rates on debts are computed via a methodology similar to that used to compute random death ages. That is, a random number, defined on the unit internal [0 to 1], can be interpolated to yield a normal distribution, just as a random number can be interpolated to yield an expected age at death.

Assume, for example, that a particular asset has an expected rate of return of 10% with a standard deviation of 5%. Assume that the computer generates the random number 0.4544. That number falls into the interval between 0.44433 and

0.46414, and that interval is associated with a minus 0.10 deviation from the normal expectation. Hence, assuming an expected rate of return of 10% with a standard deviation of 5%, the 5% standard deviation is multiplied by the –0.10 randomly chosen deviation, and results in a randomized rate of return of 9.5% [10% + (5%)*(–0.10)] for the time period under consideration.

Randomly chosen numbers close to 0.500 result in small deviations from the expectation; randomly chosen numbers close to one or zero result in large deviations (positive or negative) from the expectation. The computer simulation generates a random number for each asset and for each debt for each time period until death, and then interpolates from those random numbers deviations from the expectations, resulting in random and normally distributed expected rates of return for all assets and all debts.

Two types of life insurance are considered: whole life insurance and term life insurance. The whole life policy is a TIAA unisex whole life insurance policy. Although the author would have preferred to be consistent and use TIAA term life insurance, TIAA ends all of their term products at age 70. Hence, a commercial insurance product, with rates guaranteed for 20 years, is used for the term life insurance policy. The term product has a terminal age of 95; that is, at the end of age 94, or the beginning of age 95, the policy is terminated. That could be catastrophic from an individual investor's viewpoint; the premium for the $100,000 policy in the 94th year of life is in excess of $30,000.

Debt (estate leverage) is, for tax purposes, assumed to be secured by a mortgage on the house. That mortgage is assumed to be a variable rate home equity mortgage, and, although in the real world payments would be made on a monthly basis, the simulation mortgage payment is made annually. Hence, on an annual basis, the old balance is increased by the variable borrowing rate and is reduced by the mortgage payment. Since the mortgage payment is a function of the size of the end-of-the-year balance (for computational purposes, the payment is 15% of the end-of-the-year balance), the mortgage will exist for the duration of the simulation. The tax deductibility of the interest is accounted for by reducing the amount of the payment. That is, if the gross payment on the mortgage is $16,500, and the tax savings associated with that mortgage payment is $2,800, then the net payment is $13,700 [the difference between $16,500 less $2,800]. When the mortgage is incurred, an offsetting asset account is created. See Figure 1. Payments on the mortgage are made from that asset account. If the after tax cost of borrowing is equal to the after tax earnings on the asset account, the transaction is, for future net worth purposes, a financial "wash."

The results of this dual transaction (an increase in debt accompanied by an increase in assets) are visually linked together on the computer screen. By comparing the future mortgage balance with the future asset account balance, the individual investor can see the effect of leverage. Obviously, if the after tax rate of return on the asset account is greater than the after tax borrowing rate on the mortgage, then the leverage will serve to increase the net present value of the assets passing to the

heir(s). Conversely, if the reverse is true, the leveraged estate will reduce the heir(s)'s inheritance(s).

Sixteen different estate plans were considered for this simulation. The first estate plan is referred to in Tables 1 through 8 as Plan A, and in the text as either *Simple Will*, as A (*Simple Will*), or, if recently described, as A. Under this estate plan, the husband's assets are passed to his widow when he dies, or the wife's assets are passed to her widower when she dies, and on the death of the survivor the remaining assets are passed to their heir(s).

When estate planners first work with couples, they typically encounter the *Simple Will* estate plan. Often, the first step in estate planning is to show clients the value of the *Exemption Trust Will*. The primary characteristic of this estate planning tool is to take advantage of the $600,000 which can be passed free of Federal Estate Tax on the death of an individual. When the first individual dies, the heir(s) receive $600,000 (in trust), and those assets are not a part of the survivor's gross estate when the survivor dies. As a consequence, the *Exemption Trust Will* increases the amount

AN UNLEVERAGED ESTATE

Assets	Debt	Net Worth		Rate of Return	10%
$600,000	$0	$600,000			
Beginning Assets		$600,000	$643,200	$689,510	$739,155
Interest Earned		$60,000	$64,320	$68,951	$73,916
Tax Effect on Earnings		($16,800)	($18,010)	($19,306)	($20,696)
End of Year Assets		$643,200	$689,510	$739,155	$792,374
Net Worth		$643,200	$689,510	$739,155	$792,374

A LEVERAGED ESTATE

Assets	Debt	Net Worth		Rate of Return	10%
$700,000	$100,000	$600,000		Cost of Debt	10%
Beginning Debt		($100,000)	($93,500)	($87,423)	($81,740)
Interest Expense		($10,000)	($9,350)	($8,742)	($8,174)
End of Year Debt		($110,000)	($102,850)	($96,165)	($89,914)
End of Year Payment		$16,500	$15,428	$14,425	$13,487
Tax Effect		($2,800)	($2,618)	($2,448)	($2,289)
Net Payment (from assets)		$13,700	$12,810	$11,977	$11,198
End of Year Debt Balance		($93,500)	($87,423)	($81,740)	($76,427)
Beginning Assets		$700,000	$736,700	$776,933	$820,895
Interest Earned		$70,000	$73,670	$77,693	$82,090
Tax Effect on Earnings		($19,600)	($20,628)	($21,754)	($22,985)
Net Payment		($13,700)	($12,810)	($11,977)	($11,198)
End of Year Assets		$736,700	$776,933	$820,895	$868,801
Net Worth		$643,200	$689,510	$739,155	$792,374

Note: to keep net worth constant, the payment on the mortgage must be made from the asset account to which the mortgage debt was transferred.

Figure 1.

of assets passed to a couple's heir(s). The *Exemption Trust Will* is referred to as Plan B in Tables 1 through 8, and in the text as either the *Exemption Trust Will*, as B (*Exemption Trust Will*), or, if recently described, as B.

Once clients are familiar with the basics (the *Simple Will* and the *Exemption Trust Will*), estate planners typically expose their clients to additional estate planning opportunities for passing assets to their heir(s): annual gifts and life insurance trusts are two of the most common tools. To increase the size of their future estates, estate planners often show their clients how to avoid current income taxes by using debt (leverage) and tax shelters. Since the particular tool(s) used are often a function of the professional training of the estate planner and the way s/he is compensated for her/his time, estate plans three through 16 do not necessarily represent the order in which an estate planner would introduce her/his clients to alternate estate plans.

Rather, the order of the plans is related to the logic underlying the computer code used to perform the analysis. That is, after the code has examined the *Simple Will* and the *Exemption Trust Will*, gifts are introduced into the estate planning process, followed by whole life insurance, then followed by term life insurance. After eight alternate estate plans (referred to as Plans A through H in Tables 1 through 8) were analyzed, control of the program was returned to the author. Assets were reallocated to consider the use of tax shelters and/or debt. After reallocation, control of the program was returned to the computer, and the *Simple Will* estate plan was modified to include tax sheltering and estate leverage for the age 45 couple, or modified to use only estate leverage for the age 70 couple. As before, gifts, whole life, and term life insurance were sequentially introduced into the analysis to complete the simulation.

Note that the use of debt, tax shelter, gifts, or insurance is not random, since in the real world those would not be random events. The rates of return (or borrowing rates) are random, but the use of the tool(s) is not. Given a set of death ages, a set of input data for the initial estate, and a set of randomly and normally distributed rates of return for all assets and debts over all death ages, the remaining step is to compute the net present value of the wealth which passes to the heir(s) for the 16 different estate plans. A flow chart for the simulation follows. See Figure 2.

A Simulation—A Middle Aged Couple

Initially, eight different estate plans are considered:

A. Simple Will;
B. Exemption Trust Will;
C. Simple Will and Gifts;
D. Exemption Trust Will and Gifts;
E. Simple Will, Gifts, and Whole Life;
F. Exemption Trust Will, Gifts, and Whole Life;
G. Simple Will, Gifts, and Term Life; and
H. Exemption Trust Will, Gifts, and Term Life.

Simulation Flow Chart

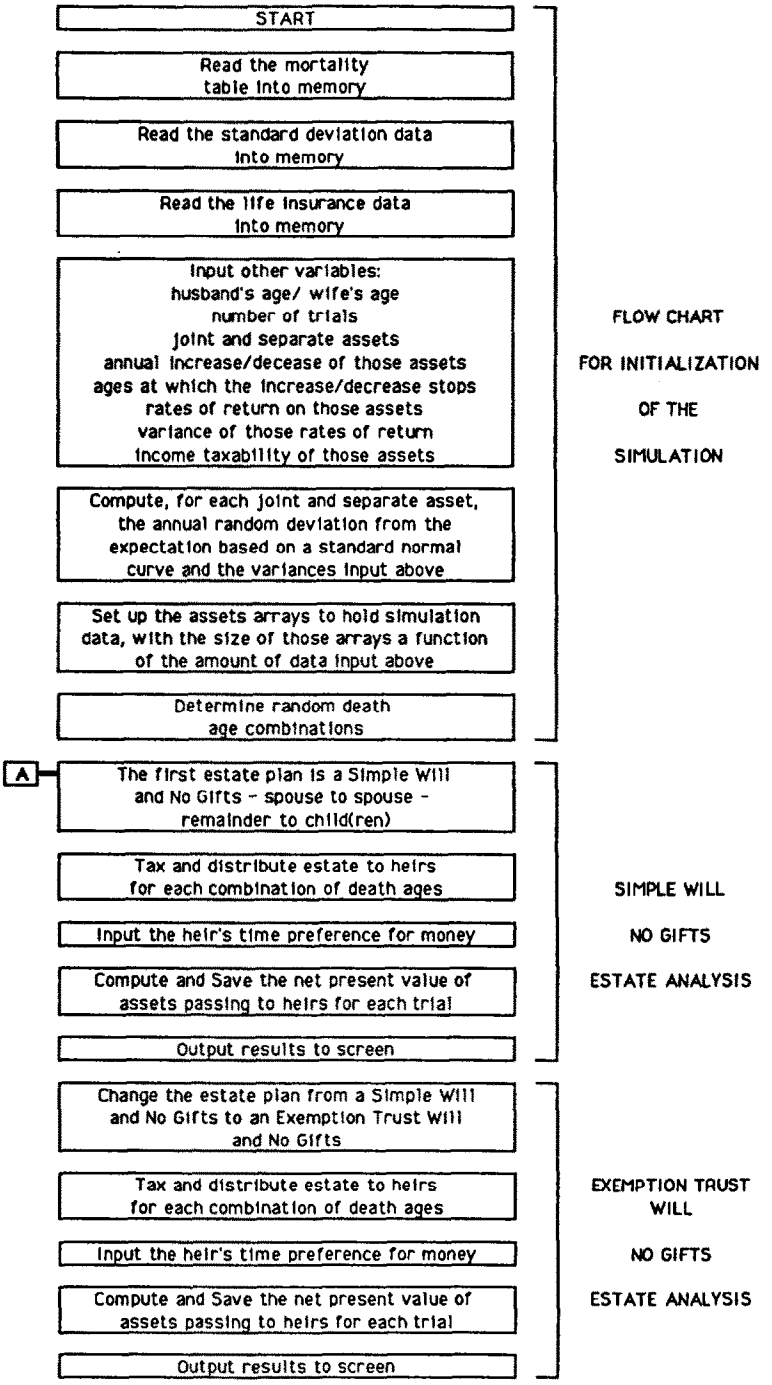

Figure 2. Simulation Flow Chart
(continued)

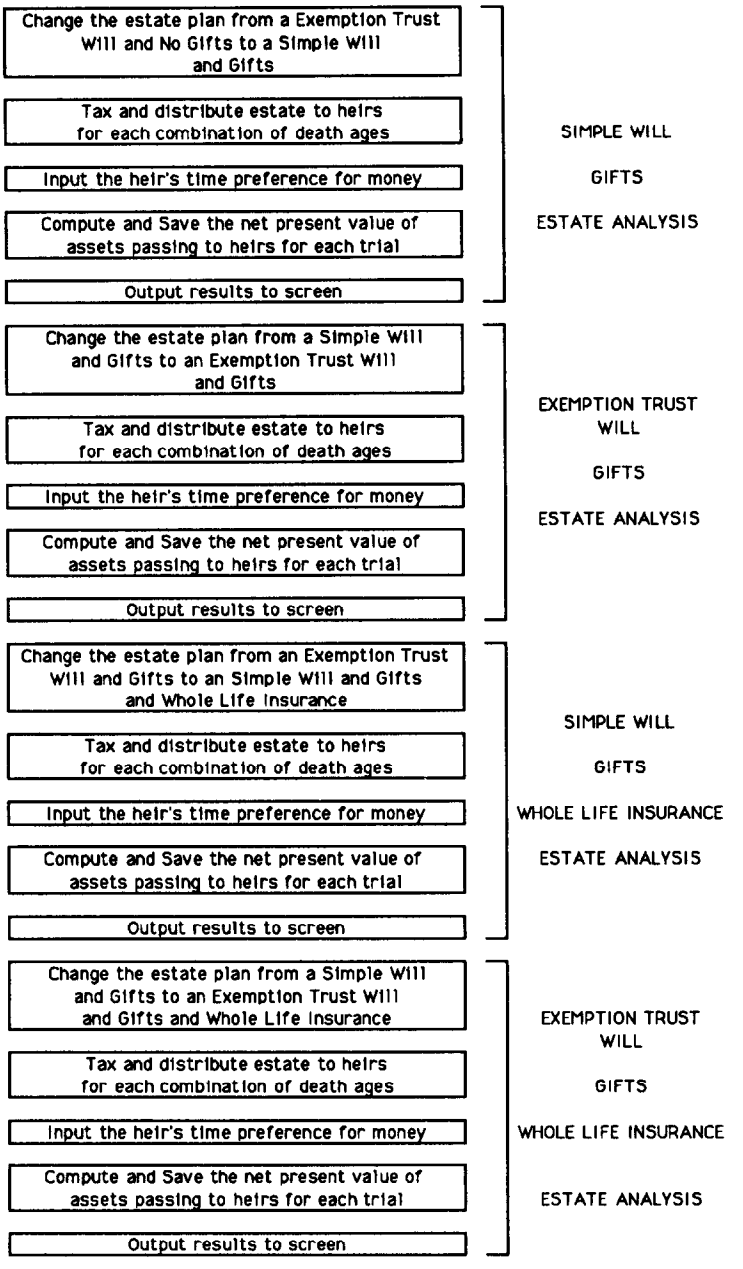

Figure 2. Continued.

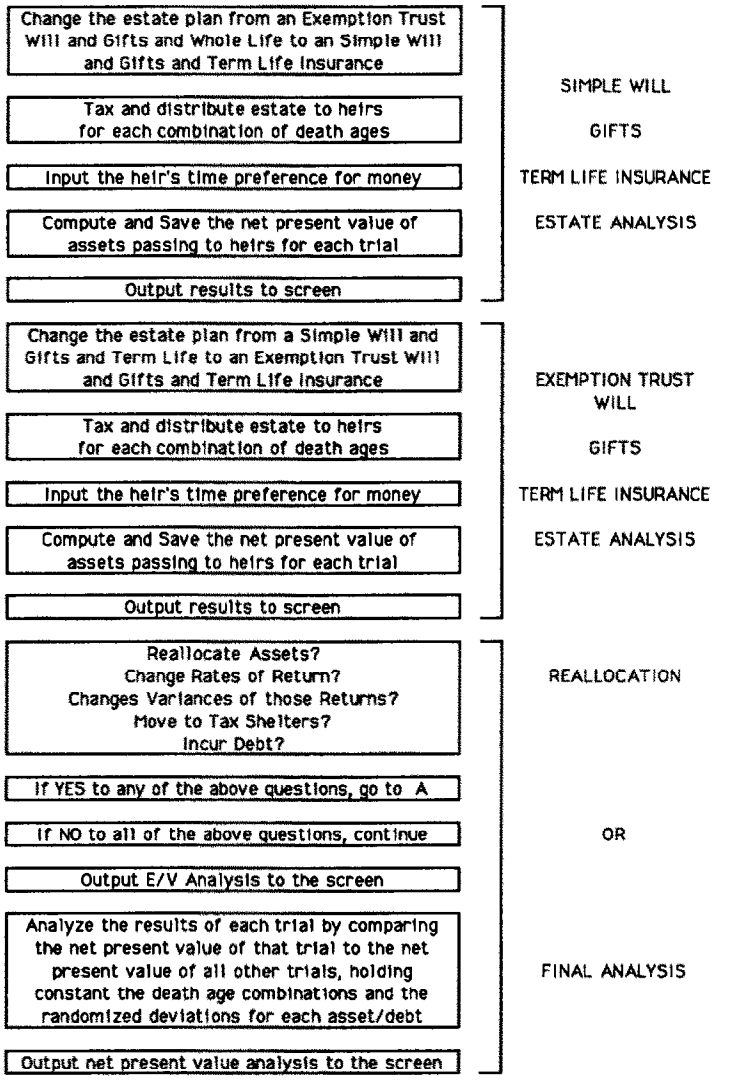

Figure 2. Continued.

Assume the following estate: Husband—Separate Assets—$250,000, with a taxable 10% expected rate of return and a standard deviation of 5%; Wife—Separate Assets—$250,000, with a taxable 7% expected rate of return and a standard deviation of 3%; and Joint Assets (a house) of $250,000, with a non-taxable expected rate of return of 6% and a standard deviation of 2%. Assume that both husband and wife are 45 years of age. Assume a 100 trial simulation.

After these eight plans have been evaluated, the estate assets are reallocated, holding consumption constant. Tax shelter use, to reduce current income taxes and

to compound assets tax deferred, is considered as an asset reallocation. That is, when money is routed into a tax shelter [like a TSA, a 401(k), or an IRA], the source of that money is an existing asset. Consumption is held constant by reducing an existing asset and putting that money into a tax shelter. The reduction amount is adjusted for taxes; that is, if $2000 is invested in an IRA account, then $1440 is removed from an asset account. If $8000 is invested in a TSA account, then $5760 is removed from an asset account. The use of debt, also holding consumption constant, has been previously explained.

The next eight estate plans, referred to as I through P in Tables 1 through 8 and by their names and/or letters in the text of the paper, are:

I. Simple Will, IRA, and Mortgage;
J. Exemption Trust Will, IRA, and Mortgage;
K. Simple Will, IRA, Mortgage, and Gifts;
L. Exemption Trust Will, IRA, Mortgage, and Gifts;
M. Simple Will, IRA, Mortgage, Gifts, and Whole Life;
N. Exemption Trust Will, IRA, Mortgage, Gifts, and Whole Life;
O. Simple Will, IRA, Mortgage, Gifts, and Term Life; and
P. Exemption Trust Will, IRA, Mortgage, Gifts, and Term Life.

For the age 45 couple, the results of the simulation (a total of 1600 separate estate outcomes) are summarized in Tables 1 through 4. Table 1, labelled "Comparative Analysis—Male and Female, Both Age 45—Various Estate Plans," has an alphabetic code at the bottom which refers to the different types of estate plans.

In order to read the data shown in the 16×16 matrix, note first a diagonal of bold zeros (**0**), sloping downward to the right. When reading across the top of the table, from left to right, the numbers lying above that zero diagonal represent the number of times a particular estate plan had a higher Net Present Value to the Heir(s) [NPVH] than the estate plan shown on the left horizontal axis; when reading from top to bottom, the numbers lying below that zero diagonal represent the number of times that the estate plan shown on the left vertical axis had a lower NPVH than the estate plan shown on the top horizontal axis. For example, go across the top line to estate plan C (*Simple Will and Gifts*), drop down to the number 87, and go left on the horizontal axis to estate plan B (*Exemption Trust Will*). Relative to the Exemption Trust Will estate plan, the Simple Will and Gifts estate plan resulted in a higher NPVH in 87 of the 100 trials. Reading in the other direction, go down the left edge to estate plan C, go right to the number 13, and go up to estate plan B. Relative to the Exemption Will Trust estate plan, the Simple Will and Gifts estate plan resulted in a lower NPVH in 13 of the 100 trials.

The next three tables, labelled "E-V Analysis - . . . - ? %," where ? is equal to 4, 5, or 6, summarize the results of the 1600 trials discounted at 4, 5, and 6 percent. One hundred death age combinations, with randomized rates of return for each asset and borrowing rates for each debt until death for both husband and wife, evaluated

for estates plans A (*Simple Will*) through P (*Exemption Trust Will*, IRA, Mortgage, Gifts, and Term Life), resulted in the set of NPVHs and Standard Deviations associated therewith. In the top left rectangle, the several estate plans are NPVH ranked, listed from the highest NPVH to the lowest NPVH. In the top right rectangle, the several estate plans are Coefficient of Variation ranked, listed from the highest Coefficient of Variation to the lowest Coefficient of Variation.

In the two lower rectangles, the set of possible outcomes has been reduced to the efficient set, both on a Standard Deviation basis and on a Coefficient of Variation basis. For example, in the 4% discount based table, two plans, M (*Simple Will*, IRA, Mortgage, Gifts, and Whole Life) and K (*Simple Will*, IRA, Mortgage, and Gifts) are eliminated from consideration. Their means are lower than those of plan H (*Exemption Trust Will*, Gifts, and Term Life), and their Standard Deviations are higher. After other plans have been eliminated on the same decision criteria, 10 plans remain, and the individual investor, given his/her risk preference, could choose from among those 10.

In the 4% discount based table, on a Coefficient of Variation basis, the 16 plans can be reduced to six plans, and the individual investor would only need to consider six alternate plans, rank order:

N. Exemption Trust Will, IRA, Mortgage, Gifts, and Whole Life;
P. Exemption Trust Will, IRA, Mortgage, Gifts, and Term Life;
F. Exemption Trust Will, Gifts, and Whole Life;
H. Exemption Trust Will, Gifts, and Term Life;
O. Simple Will, IRA, Mortgage, Gifts, and Term Life; and
G. Simple Will, Gifts, and Term Life.

Unfortunately for the individual investor, the efficient frontier is a function of the individual investor's discount rate, and in the 6% table, the Coefficient of Variation based efficient frontier is:

N. Exemption Trust Will, IRA, Mortgage, Gifts, and Whole Life;
L. Exemption Trust Will, IRA, Mortgage, and Gifts;
D. Exemption Trust Will and Gifts;
M. Simple Will, IRA, Mortgage, Gifts, and Whole Life;
O. Simple Will, IRA, Mortgage, Gifts, and Term Life; and
E. Simple Will, Gifts, and Whole Life.

See Figures 3, 4, and 5 for the Coefficient of Variation based efficient frontiers for discount rates of 4, 5, and 6%. The Standard Deviation based efficient frontier also changed as the individual investor's discount rate changed.

Decision criteria problems for the individual investor continue. When discounted at 4 and 5%, plan A (*Simple Will*) is on the Standard Deviation based efficient frontier. However, in the "Comparative Analysis Table," the *Simple Will*

TABLE 1.
Comparative Analysis—Male and Female, both Age 45—Various Estate Plans

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
A	**0**	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
B	0	**0**	87	100	99	100	95	100	11	100	89	100	100	100	100	100
C	0	13	**0**	100	100	100	39	83	0	21	100	100	100	100	79	95
D	0	0	0	**0**	0	100	4	39	0	1	12	100	40	100	6	79
E	0	1	0	100	**0**	100	9	77	0	7	70	100	100	100	58	94
F	0	0	0	0	0	**0**	0	9	0	0	8	77	12	100	0	58
G	0	5	61	96	91	100	**0**	100	0	10	81	100	96	100	100	100
H	0	0	17	61	23	91	0	**0**	0	0	36	83	50	99	12	100
I	0	89	100	100	100	100	100	100	**0**	100	100	100	100	100	100	100
J	0	0	79	99	93	100	90	100	0	**0**	87	100	97	100	93	100
K	0	11	0	88	90	92	19	64	0	13	**0**	100	100	100	39	83
L	0	0	0	0	0	23	0	17	0	0	0	**0**	0	100	2	39
M	0	0	0	60	0	88	4	50	0	3	0	100	**0**	100	9	77
N	0	0	0	0	0	0	0	1	0	0	0	0	0	**0**	0	9
O	0	0	21	94	42	100	0	88	0	7	61	98	91	100	**0**	100
P	0	0	5	21	6	42	0	0	0	0	17	61	23	91	0	**0**

A	Simple Will	**I**	Simple Will, IRA, and Mortgage
B	Exemption Trust Will	**J**	Exemption Trust Will, IRA, and Mortgage
C	Simple Will and Gifts	**K**	Simple Will, IRA, Mortgage, and Gifts
D	Exemption Trust Will and Gifts	**L**	Exemption Trust Will, IRA, Mortgage, and Gifts
E	Simple Will, Gifts, and Whole Life	**M**	Simple Will, IRA, Mortgage, Gifts, and Whole Life
F	Exemption Trust Will, Gifts, and Whole Life	**N**	Exemption Trust Will, IRA, Mortgage, Gifts, and Whole Life
G	Simple Will, Gifts, and Term Life	**O**	Simple Will, IRA, Mortgage, Gifts, and Term Life
H	Exemption Trust Will, Gifts, and Term Life	**P**	Exemption Trust Will, IRA, Mortgage, Gifts, and Term Life

is shown to be inferior to all other plans under consideration. That is, while the *Simple Will* does represent the lowest point on the efficient frontier, it is so low that on a NPVH basis the 100 NPVHs of the *Simple Will* estate plan, when compared to the NPVHs of all other plans, is always inferior. While a financial planner, who, when counselling a very risk averse investor, could recommend the *Simple Will* estate plan because it has the lowest Standard Deviation, that financial planner would have to explain very carefully that while the risk is low, so is the size of the

TABLE 2.
E-V Analysis—Male and Female, both Age 45—Various Estate Plans—4%

Discounted at 4%

	Net Present Value to Heirs	Standard Deviation
N	$1,585,299	$295,925
L	$1,534,620	$291,595
P	$1,517,893	$263,083
F	$1,503,692	$253,611
D	$1,453,013	$250,388
H	$1,436,286	$219,805
M	$1,433,727	$249,824
K	$1,383,048	$258,847
O	$1,366,320	$204,016
E	$1,366,195	$219,191
C	$1,315,516	$227,866
G	$1,298,789	$174,094
J	$1,183,677	$218,924
B	$1,111,999	$188,194
I	$1,003,471	$159,056
A	$936,039	$127,849

Discounted at 4 %

	Net Present Value to Heirs	Standard Deviation	Coefficient of Variation
L	$1,534,620	$291,595	0.19001121
K	$1,383,048	$258,847	0.18715692
N	$1,585,299	$295,925	0.18666826
J	$1,183,677	$218,924	0.18495248
M	$1,433,727	$249,824	0.17424796
P	$1,517,893	$263,083	0.17332118
C	$1,315,516	$227,866	0.17321416
D	$1,453,013	$250,388	0.1723233
B	$1,111,999	$188,194	0.16923936
F	$1,503,692	$253,611	0.16865887
E	$1,366,195	$219,191	0.16043903
I	$1,003,471	$159,056	0.15850583
H	$1,436,286	$219,805	0.15303707
O	$1,366,320	$204,016	0.14931788
A	$936,039	$127,849	0.13658512
G	$1,298,789	$174,094	0.13404333

The E-V Frontier
(NPV verus Standard Deviation)

N	$1,585,299	$295,925
L	$1,534,620	$291,595
P	$1,517,893	$263,083
F	$1,503,692	$253,611
D	$1,453,013	$250,388
H	$1,436,286	$219,805
O	$1,366,320	$204,016
G	$1,298,789	$174,094
I	$1,003,471	$159,056
A	$936,039	$127,849

The E-V Frontier
(NPV verus Coefficient of Variation)

N	$1,585,299	$295,925	0.18666826
P	$1,517,893	$263,083	0.17332118
F	$1,503,692	$253,611	0.16865887
H	$1,436,286	$219,805	0.15303707
O	$1,366,320	$204,016	0.14931788
G	$1,298,789	$174,094	0.13404333

estate being passed to heir(s). [Note that this problem can be eliminated if the Coefficient of Variation is used to select the efficient frontier.]

Another Simulation—An Elderly Couple

This simulation is similar to the one above, but the couple is older and wealthier. Assume the following estate: Husband—Separate Assets—$500,000,

TABLE 3.
E-V Analysis—Male and Female, both Age 45—Various Estate Plans—5%

Discounted at 5%

	Net Present Value to Heirs	Standard Deviation
N	$1,068,795	$124,754
L	$1,033,812	$121,463
P	$1,025,226	$112,099
F	$1,014,970	$100,728
D	$979,988	$97,761
H	$971,401	$88,728
M	$966,728	$96,267
K	$931,746	$107,250
O	$923,159	$69,305
E	$922,170	$79,525
C	$887,188	$90,202
G	$878,601	$54,004
J	$799,565	$99,456
B	$752,250	$85,785
I	$677,506	$56,412
A	$633,028	$39,252

Discounted at 5 %

	Net Present Value to Heirs	Standard Deviation	Coefficient of Variation
J	$799,565	$99,456	0.12438764
L	$1,033,812	$121,463	0.11749041
N	$1,068,795	$124,754	0.11672397
K	$931,746	$107,250	0.11510648
B	$752,250	$85,785	0.11403789
P	$1,025,226	$112,099	0.10934077
C	$887,188	$90,202	0.1016718
D	$979,988	$97,761	0.09975734
M	$966,728	$96,267	0.09958023
F	$1,014,970	$100,728	0.09924234
H	$971,401	$88,728	0.09134024
E	$922,170	$79,525	0.08623681
I	$677,506	$56,412	0.08326421
O	$923,159	$69,305	0.07507374
A	$633,028	$39,252	0.06200674
G	$878,601	$54,004	0.0614659

The E-V Frontier
(NPV verus Standard Deviation)

N	$1,068,795	$124,754
L	$1,033,812	$121,463
P	$1,025,226	$112,099
F	$1,014,970	$100,728
D	$979,988	$97,761
H	$971,401	$88,728
O	$923,159	$69,305
G	$878,601	$54,004
A	$633,028	$39,252

The E-V Frontier
(NPV verus Coefficient of Variation)

N	$1,068,795	$124,754	0.11672397
P	$1,025,226	$112,099	0.10934077
F	$1,014,970	$100,728	0.09924234
H	$971,401	$88,728	0.09134024
O	$923,159	$69,305	0.07507374
G	$878,601	$54,004	0.0614659

with a taxable 10% expected rate of return and a standard deviation of 5%; Wife—Separate Assets—$500,000, with a taxable 7% expected rate of return and a standard deviation of 3%; and Joint Assets (a house) of $200,000, with a non-taxable expected rate of return of 5% and an expected variance of 1%. Assume that both husband and wife are age 70. Assume a 100 trial simulation.

After the initial eight plans (*Simple Will* through *Exemption Trust Will*, Gifts, and Term Life) have been evaluated, the estate assets are reallocated, holding

TABLE 4.
E-V Analysis—Male and Female, both Age 45—Various Estate Plans—6%

Discounted at 6%

	Net Present Value to Heirs	Standard Deviation
N	$727,526	$46,812
L	$703,108	$37,552
P	$699,198	$54,271
F	$691,701	$37,336
D	$667,283	$25,535
H	$663,373	$48,524
M	$658,108	$20,031
K	$633,690	$31,872
O	$629,780	$18,276
E	$628,447	$15,602
C	$604,303	$25,140
G	$600,120	$23,263
J	$545,417	$53,471
B	$513,903	$52,215
I	$461,866	$12,100
A	$432,271	$14,193

Discounted at 6 %

	Net Present Value to Heirs	Standard Deviation	Coefficient of Variation
B	$513,903	$52,215	0.10160478
J	$545,417	$53,471	0.09803691
P	$699,198	$54,271	0.07761893
H	$663,373	$48,524	0.07314738
N	$727,526	$46,812	0.06434409
F	$691,701	$37,336	0.05397708
L	$703,108	$37,552	0.05340858
K	$633,690	$31,872	0.05029589
C	$604,030	$25,140	0.04162045
G	$600,120	$23,263	0.03876391
D	$667,283	$25,535	0.03826712
A	$432,271	$14,193	0.03283357
M	$658,108	$20,031	0.03043725
O	$629,780	$18,276	0.02901966
I	$461,866	$12,100	0.02619807
E	$628,447	$15,602	0.02482628

The E-V Frontier
(NPV verus Standard Deviation)

N	$727,526	$46,812
L	$703,108	$37,552
F	$691,701	$37,336
D	$667,283	$25,535
M	$658,108	$20,031
O	$629,780	$18,276
E	$628,447	$15,602
I	$461,866	$12,100

The E-V Frontier
(NPV verus Coefficient of Variation)

N	$727,526	$46,812	0.06434409
L	$703,108	$37,552	0.05340858
D	$667,283	$25,535	0.03826712
M	$658,108	$20,031	0.03043725
O	$629,780	$18,276	0.02901966
E	$628,447	$15,602	0.02482628

consumption constant. A $100,000 mortgage is taken out on the house. Assume a 7% mortgage rate with a 5% standard deviation, and assume an offsetting investment of $100,000 with an expected taxable rate of return of 8% and with a 4% standard deviation. Note that there is, subject to random deviations, positive leverage; that is, the expected rate of return on the invested assets is 1% higher than the borrowing rate. IRAs, TSAs, and 401(k)s are not used since the couple is assumed to be retired.

The results of the simulation (a total of 1600 separate estate outcomes) are summarized in Tables 5 through 8. In Table 5, it is interesting to note that estate plan A (*Simple Will*) is no longer absolutely inferior. In two of the 100 trials, the *Simple Will* estate plan was superior to estate plan G (*Simple Will*, Gifts, and Term Life). In Table 6, the 4% discount based table, on a Coefficient of Variation basis, the 16 plans can only be reduced to 10 plans, and the individual investor would need to consider 10 alternate plans, rank order:

N. Exemption Trust Will, Mortgage, Gifts, and Whole Life;
F. Exemption Trust Will, Gifts, and Whole Life;
P. Exemption Trust Will, Mortgage, Gifts, and Term Life;
J. Exemption Trust Will and Mortgage;
B. Exemption Trust Will;
M. Simple Will, Mortgage, Gifts, and Whole Life;
E. Simple Will, Gifts, and Mortgage;
O. Simple Will, Mortgage, Gifts, and Term Life;
I. Simple Will and Mortgage; and
A. Simple Will.

At 5%, the Coefficient of Variation based efficient frontier is:

N. Exemption Trust Will, Mortgage, Gifts, and Whole Life;
L. Exemption Trust Will, Mortgage, and Gifts;
D. Exemption Trust Will and Gifts;
K. Simple Will, Mortgage, and Gifts; and
C. Simple Will and Gifts.

At 6%, the Coefficient of Variation based efficient frontier is N (*Exemption Trust Will*, Mortgage, Gifts, and Whole Life), and L (*Exemption Trust Will*, Mortgage, and Gifts). See Figures 6, 7, and 8 for the Coefficient of Variation based efficient frontiers for discount rates of 4, 5, and 6%.

An Analysis of the Efficient Frontiers

In the elderly couple simulation at 6% (above paragraph), estate plans N and L define the efficient frontier; however, estate plan D (*Exemption Trust Will* and Gifts) is almost the equivalent of L (*Exemption Trust Will*, Mortgage, and Gifts). It is only in the fourth digit of the Coefficient of Variation that L outranks D. Since in estate plan L the cost of obtaining the mortgage was not considered, it is possible that D could also be on the efficient frontier if the cost of the loan were high enough such that the NPVH of L dropped below $938,202 (the NPVH of D). The efficient frontier would then be N, D, L.

More important is the assumed 1% favorable interest rate differential between the borrowing rate and the investing rate. [Note that if negative leverage were

TABLE 5.
Comparative Analysis—Male and Female, both Age 70—Various Estate Plans

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
A	**0**	100	100	100	100	100	98	100	100	100	100	100	100	100	98	100
B	0	**0**	16	100	2	100	3	98	0	100	17	100	2	100	5	98
C	0	84	**0**	100	69	100	61	97	0	85	100	100	71	100	62	97
D	0	0	0	**0**	0	69	0	61	0	0	0	100	0	71	0	62
E	0	98	31	100	**0**	100	49	99	0	99	35	100	100	100	52	99
F	0	0	0	31	0	**0**	0	49	0	0	0	35	0	100	0	52
G	2	97	39	100	51	100	**0**	100	2	97	40	100	53	100	100	100
H	0	2	3	39	1	51	0	**0**	0	2	3	40	1	53	0	100
I	0	100	100	100	100	100	98	100	**0**	100	100	100	100	100	98	100
J	0	0	15	100	1	100	3	98	0	**0**	16	100	2	100	3	98
K	0	83	0	100	65	100	60	97	0	84	**0**	100	69	100	61	97
L	0	0	0	0	0	65	0	60	0	0	0	**0**	0	69	0	61
M	0	98	29	100	0	100	47	99	0	98	31	100	**0**	100	49	99
N	0	0	0	29	0	0	0	47	0	0	0	31	0	**0**	0	49
O	2	95	38	100	48	100	0	100	2	97	39	100	51	100	**0**	100
P	0	2	3	38	1	48	0	0	0	2	3	39	1	51	0	**0**

A	Simple Will	I	Simple Will and Mortgage
B	Exemption Trust Will	J	Exemption Trust Will and Mortgage
C	Simple Will and Gifts	K	Simple Will, Mortgage, and Gifts
D	Exemption Trust Will and Gifts	L	Exemption Trust Will, Mortgage, and Gifts
E	Simple Will, Gifts, and Whole Life	M	Simple Will, Mortgage, Gifts, and Whole Life
F	Exemption Trust Will, Gifts, and Whole Life	N	Exemption Trust Will, Mortgage, Gifts, and Whole Life
G	Simple Will, Gifts, and Term Life	O	Simple Will, Mortgage, Gifts, and Term Life
H	Exemption Trust Will, Gifts, and Term Life	P	Exemption Trust Will, Mortgage, Gifts, and Term Life

assumed, then, even ignoring the cost of obtaining the mortgage, D would be preferable to L.] Ceteris paribus, the positive leverage tilted the analysis in favor of any estate plan of which the mortgage was a part. That is, if one compares

I. (Simple Will and Mortgage) to A (Simple Will);
J. (Exemption Trust Will and Mortgage) to B (Exemption Trust Will);
K. (Simple Will, Mortgage, and Gifts) to C (Simple Will and Gifts);

TABLE 6.
E-V Analysis—Male and Female, both Age 70—Various Estate Plans—4%

Discounted at 4%

	Net Present Value to Heirs	Standard Deviation
N	$1,357,978	$110,815
F	$1,353,610	$110,176
P	$1,344,510	$105,833
H	$1,340,142	$105,524
L	$1,326,149	$105,432
D	$1,321,781	$104,455
J	$1,203,295	$92,147
B	$1,199,131	$91,539
M	$1,115,582	$41,743
E	$1,111,526	$40,978
O	$1,102,114	$38,619
G	$1,098,057	$38,833
K	$1,083,752	$72,239
C	$1,079,696	$71,300
I	$950,373	$28,922
A	$946,353	$27,967

Discounted at 4 %

	Net Present Value to Heirs	Standard Deviation	Coefficient of Variation
N	$1,357,978	$110,815	0.08160294
F	$1,353,610	$110,176	0.0813942
L	$1,326,149	$105,432	0.07950238
D	$1,321,781	$104,455	0.07902595
H	$1,340,142	$105,524	0.07874091
P	$1,344,510	$105,833	0.07871492
J	$1,203,295	$92,147	0.07657889
B	$1,199,131	$91,539	0.07633778
K	$1,083,752	$72,239	0.06665639
C	$1,079,696	$71,300	0.06603711
M	$1,115,582	$41,743	0.03741814
E	$1,111,526	$40,978	0.03686643
G	$1,098,057	$38,833	0.0353652
O	$1,102,114	$38,619	0.03504084
I	$950,373	$28,922	0.03043226
A	$946,353	$27,967	0.0295524

The E-V Frontier
(NPV verus Standard Deviation)

N	$1,357,978	$110,815
F	$1,353,610	$110,176
P	$1,344,510	$105,833
H	$1,340,142	$105,524
L	$1,326,149	$105,432
D	$1,321,781	$104,455
J	$1,203,295	$92,147
B	$1,199,131	$91,539
M	$1,115,582	$41,743
E	$1,111,526	$40,978
O	$1,102,114	$38,619
I	$950,373	$28,922
A	$946,353	$27,967

The E-V Frontier
(NPV verus Coefficient of Variation)

N	$1,357,978	$110,815	0.08160294
F	$1,353,610	$110,176	0.0813942
P	$1,344,510	$105,833	0.07871492
J	$1,203,295	$92,147	0.07657889
B	$1,199,131	$91,539	0.07633778
M	$1,115,582	$41,743	0.03741814
E	$1,111,526	$40,978	0.03686643
O	$1,102,114	$38,619	0.03504084
I	$950,373	$28,922	0.03043226
A	$946,353	$27,967	0.0295524

L. (Exemption Trust Will, Mortgage, and Gifts) to D (Exemption Trust Will and Gifts);

M. (Simple Will, Mortgage, Gifts, and Whole Life) to E (Simple Will, Gifts, and Whole Life);

N. (Exemption Trust Will, Mortgage, Gifts, and Whole Life) to F (Exemption Trust Will, Gifts, and Whole Life);

TABLE 7.
E-V Analysis—Male and Female, both Age 70—Various Estate Plans—5%

Discounted at 5%

	Net Present Value to Heirs	Standard Deviation
N	$1,142,989	$76,430
F	$1,139,353	$76,499
P	$1,133,679	$97,873
H	$1,130,043	$98,133
L	$1,114,497	$40,767
D	$1,110,861	$40,373
J	$1,012,875	$63,020
B	$1,009,409	$63,109
M	$939,846	$34,690
E	$936,473	$35,238
O	$930,536	$68,862
G	$927,162	$69,453
K	$911,355	$24,817
C	$907,980	$24,635
I	$800,718	$25,653
A	$797,375	$26,296

Discounted at 5 %

	Net Present Value to Heirs	Standard Deviation	Coefficient of Variation
H	$1,130,043	$98,133	0.08684006
P	$1,133,679	$97,873	0.0863322
G	$927,162	$69,453	0.07490924
O	$930,536	$68,862	0.07400251
F	$1,139,353	$76,499	0.06714249
N	$1,142,989	$76,430	0.06686854
B	$1,009,409	$63,109	0.06252074
J	$1,012,875	$63,020	0.06221893
E	$936,473	$35,238	0.03762842
M	$939,846	$34,690	0.0369103
L	$1,114,497	$40,767	0.03657883
D	$1,110,861	$40,373	0.03634388
A	$797,375	$26,296	0.03297821
I	$800,718	$25,653	0.0320375
K	$911,355	$24,817	0.02723088
C	$907,980	$24,635	0.02713165

The E-V Frontier
(NPV verus Standard Deviation)

N	$1,142,989	$76,430
L	$1,114,497	$40,767
D	$1,110,861	$40,373
J	$1,012,875	$63,020
M	$939,846	$34,690
K	$911,355	$24,817
C	$907,980	$24,635

The E-V Frontier
(NPV verus Coefficient of Variation)

N	$1,142,989	$76,430	0.06686854
L	$1,114,497	$40,767	0.03657883
D	$1,110,861	$40,373	0.03634388
K	$911,355	$24,817	0.02723088
C	$907,980	$24,635	0.02713165

O. (Simple Will, Mortgage, Gifts, and Term Life) to G (Simple Will, Gifts, and Term Life); and
P. (Exemption Trust Will, Mortgage, Gifts, and Term Life) to H (Exemption Trust Will, Gifts, and Term Life);

the Table 5 Comparative Analysis demonstrates that I to A is 100 to 0, that J to B is 100 to 0, . . . and that P to H is 100 to 0.

TABLE 8.
E-V Analysis—Male and Female, both Age 70—Various Estate Plans—6%

Discounted at 4%

	Net Present Value to Heirs	Standard Deviation
N	$966,829	$93,049
F	$963,790	$93,348
P	$960,640	$119,936
H	$957,601	$120,253
L	$941,241	$50,893
D	$938,202	$51,173
J	$856,823	$80,107
B	$853,924	$80,395
M	$795,736	$72,524
E	$792,917	$72,956
O	$789,542	$101,582
G	$786,728	$102,002
K	$770,149	$45,892
C	$767,330	$46,245
I	$677,976	$60,639
A	$675,185	$61,092

The E-V Frontier
(NPV verus Standard Deviation)

N	$966,829	$93,049
L	$941,241	$50,893
K	$770,149	$45,892

Discounted at 4 %

	Net Present Value to Heirs	Standard Deviation	Coefficient of Variation
G	$786,728	$102,002	0.12965345
O	$789,542	$101,582	0.1286594
H	$957,601	$120,253	0.12557735
P	$960,640	$119,936	0.1248501
F	$963,790	$93,348	0.09685512
N	$966,829	$93,049	0.09624142
B	$853,924	$80,395	0.09414772
J	$856,823	$80,107	0.09349306
E	$792,917	$72,956	0.09200963
M	$795,736	$72,524	0.09114078
A	$675,185	$61,092	0.09048187
I	$677,976	$60,639	0.08944122
C	$767,330	$46,245	0.06026742
K	$770,149	$45,892	0.05958847
D	$938,202	$51,173	0.05454369
L	$941,241	$50,893	0.05407011

The E-V Frontier
(NPV verus Coefficient of Variation)

N	$966,829	$93,049	0.09624142
L	$941,241	$50,893	0.05407011

It is critical for estate planners to determine whether or not their assumptions caused a plan to fall on the efficient frontier or whether the plan fell on the frontier on its own merits. In addition to looking at the plans which are on the efficient frontier, it is necessary for estate planners to look at the plans which fall just off that frontier (like estate plan D), and to analyze the input data, assumptions, and other factors that may cause a plan to just fall off of the efficient frontier.

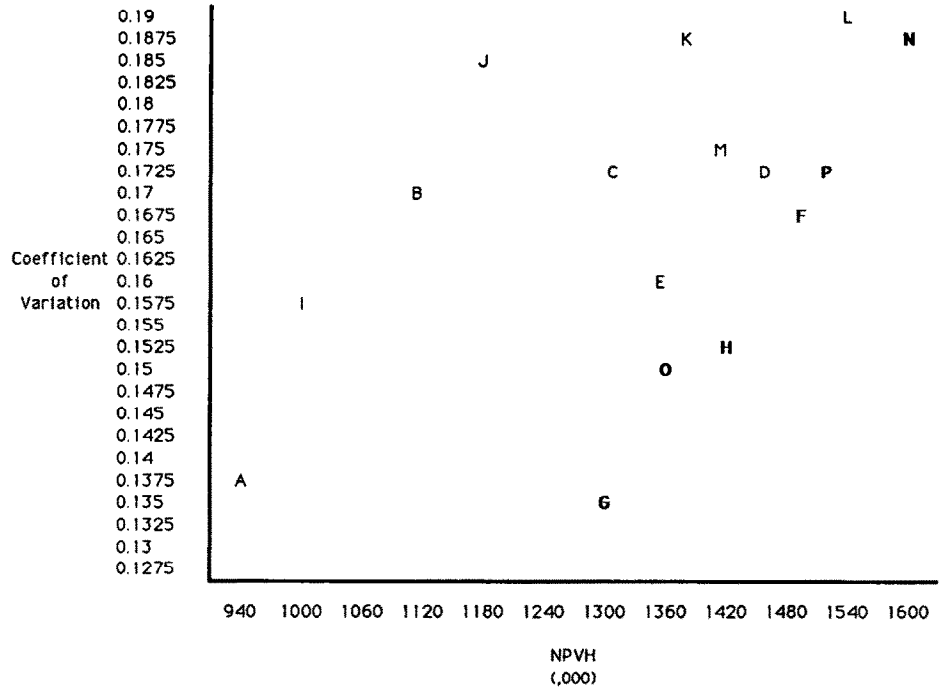

Figure 3. The Efficient Frontier (**Bold** Letters) and Ten Other Estate Plans for Age 45 at 4%

When two or more plans result in almost similar outcomes, then sensitivity analysis should be done to determine which of the nearly identical plans is the most stable (least affected by changes in assumptions and/or random variations in rates of return or borrowing rates). In the D to L comparison, D has the advantage that it is an unleveraged estate, and, as such, is not subject to interest rate variations, an important consideration for an elderly couple.

Consider the age 45 efficient frontier discounted at 5%. The order of the plans on that frontier are:

N. Exemption Trust Will, IRA, Mortgage, Gifts, and Whole Life;
P. Exemption Trust Will, IRA, Mortgage, Gifts, and Term Life;
F. Exemption Trust Will, Gifts, and Whole Life;
H. Exemption Trust Will, Gifts, and Term Life;
O. Simple Will, IRA, Mortgage, Gifts, and Term Life;
G. Simple Will, Gifts, and Term Life;

makes sense. TIAA whole life insurance, given the excellent risk characteristics of the TIAA group, should outperform commercial term life insurance, at least on an

NPVH basis (and possibly on an E/V basis as well); that is, there should exist an inherent bias in favor of TIAA whole life insurance over commercial term life insurance. Since IRAs and mortgages, especially where the mortgage has an assumed favorable leverage, serve to increase the wealth of the estate owners, estate plans that involve mortgages and IRAs should outperform those that do not. Ceteris paribus, the *Exemption Trust Will* at age 45 is absolutely superior to the *Simple Will*. If one looks at the age 45, Coefficient of Variation, 5% discounted-efficient frontier, the four plans that fall at the top of the frontier (N, P, F and H) are rank ordered first on the type of will, second on the use of tax shelter and leverage, and third on the basis of the type of insurance.

The last two plans that form the bottom of the frontier, estate plans O (*Simple Will*, IRA, Mortgage, Gifts, and Term Life) and G (*Simple Will*, Gifts, and Term Life) should, on a NPVH basis, be inferior to M (*Simple Will*, IRA, Mortgage, Gifts, and Whole Life) and E (*Simple Will*, Gifts, and Whole Life). While in Table 1 that is true, as M outperforms O, 91:9, and E outperforms G, 91:9, on the efficient frontier M is eliminated by F (*Exemption Trust Will*, Gifts, and Whole Life) and E is eliminated by O (*Exemption Trust Will*, IRA, Mortgage, Gifts and Term Life). Then, because the NPVHs of G and O are respectively about $44,000 lower than E and M, and because their standard deviations are about $26,000 lower, G and O qualify for the efficient frontier, albeit they lie on the lowest portion of that frontier. See Figure 4.

The timing of cash flows to heir(s), coupled with varying discount rates, can affect the nature of plans falling on and lying off the efficient frontier. While the NPVH rank order of the 16 plans did not change as the discount rate changed from 4% to 6%, the rank order of their Coefficients of Variation did. (See the top left rectangle of Tables 2, 3, and 4, and Tables 6, 7, and 8 for NPVH rankings, and the top right rectangle of the same Tables for Coefficient of Variation rankings.) Plans which were on the frontier dropped off, and plans which were not on the frontier appeared.

For example, consider the age 45 couple simulation. On a Coefficient of Variation basis discounted at 4%, plan G (*Simple Will*, Gifts, and Term Life) is the lowest plan on the efficient frontier. While plan E (*Simple Will*, Gifts, and Whole Life) has a higher NPVH, it has a considerably higher Coefficient of Variation, and plan E, eliminated by plan H (*Exemption Trust Will*, Gifts, and Term Life), is not on the efficient frontier. At 6%, G fell off of that frontier, and the lowest plan on the efficient frontier is plan E (*Simple Will*, Gifts, and Whole Life). At age 70, at a 5% and/or 6% discount rate, estate plans L (*Exemption Trust Will*, Mortgage, and Gifts) and/or L and K (*Simple Will*, Mortgage, and Gifts) are on the efficient frontier. At 4%, neither L nor K fall on that frontier. The efficient frontier, when defined on a Coefficient of Variation basis, is very sensitive to the discount rate of the heir(s).

The estate planner must carefully consider the ages of her/his clients when recommending estate planning tools. For example, consider the *Simple Will* and Gifts estate plan and the *Exemption Trust Will* estate plan. For the age 45 couple,

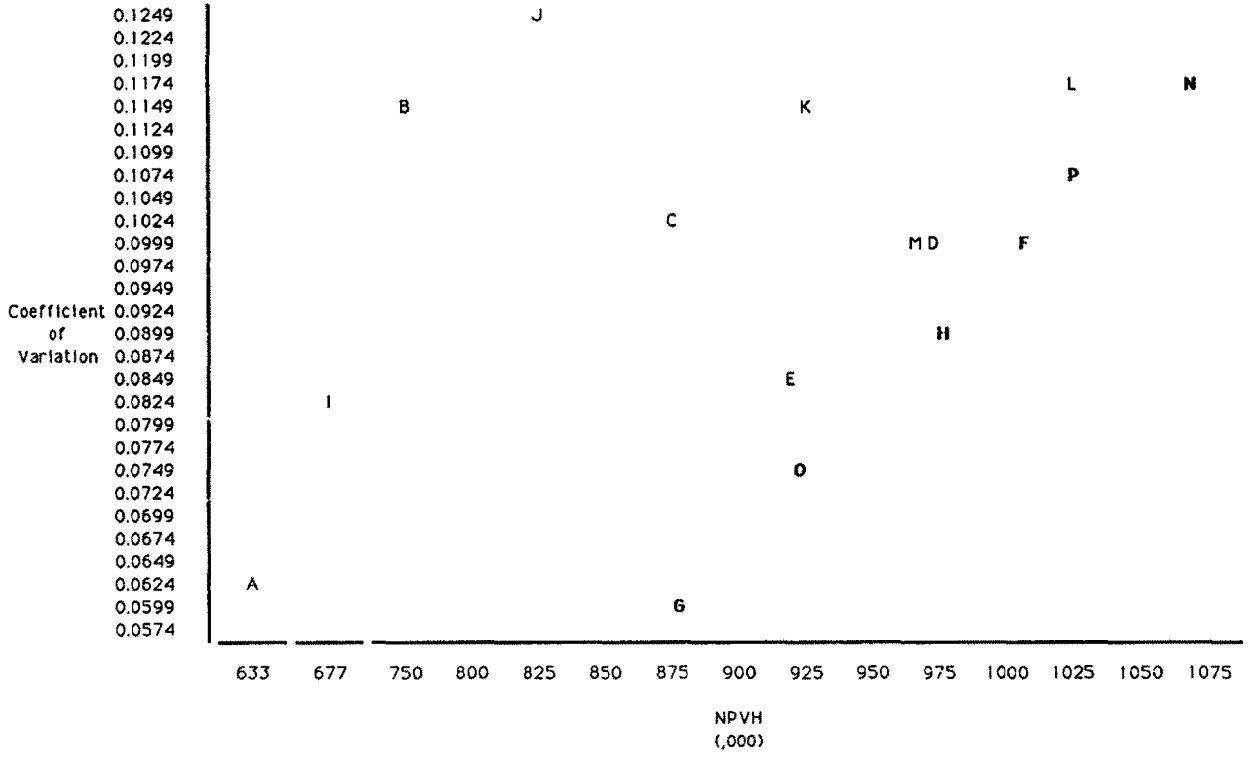

Figure 4. The Efficient Frontier (**Bold** Letters) and Ten Other Estate Plans for Age 45 at 5%

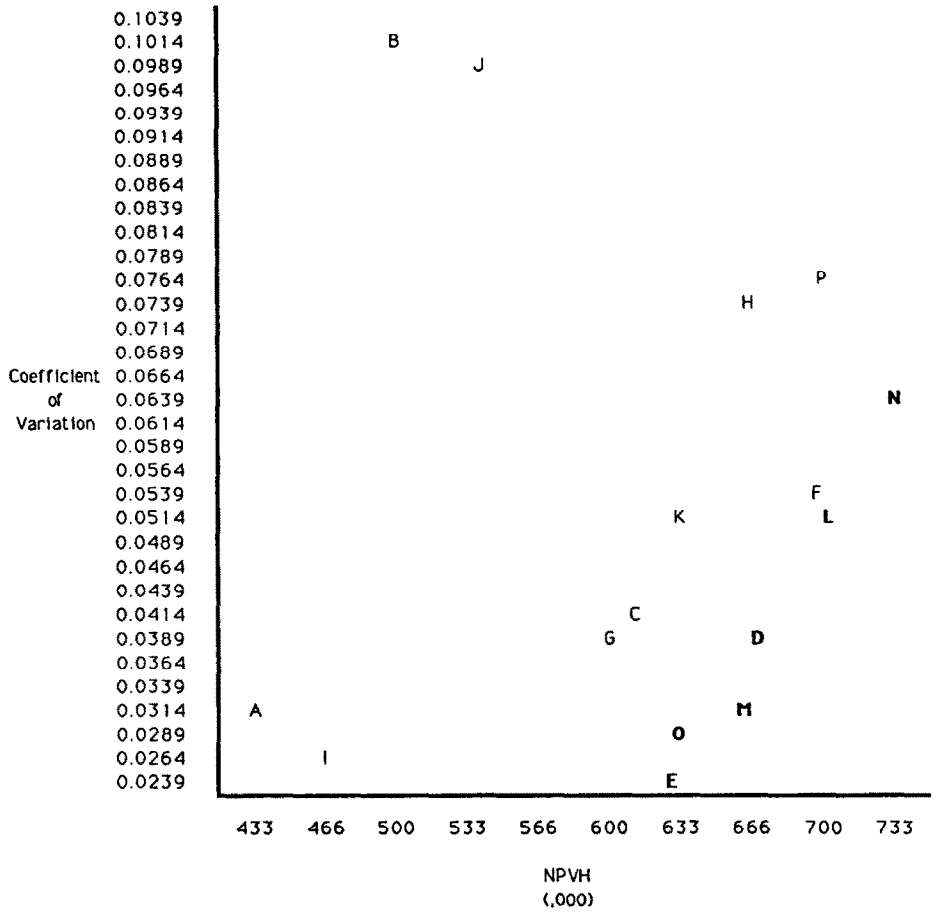

Figure 5. The Efficient Frontier (**Bold** Letters) and Ten Other Estate Plans for Age 45 at 6%

87 of the 100 NPVHs (Table 1—Comparative Analysis) of the *Simple Will* and Gifts estate plan were superior to the *Exemption Trust Will* estate plan. However, for the age 70 couple, 84 of 100 NPVHs (Table 5—Comparative Analysis) of the *Exemption Trust Will* estate plan were superior to the *Simple Will* and Gifts estate plan.

Extrapolating from these results, somewhere in between the ages of 45 and 70 there exists an age where the split would be about 50-50, and the simulation results would predict that both plans would be equally likely to produce similar results for the heir(s) of the estate owner(s). [Since the *Exemption Trust Will* and Gifts estate plan includes both gifts during the lifetimes of the estate owners as well as a $600,000 transfer at death, the *Exemption Trust Will* and Gifts estate plan is preferable to either the *Simple Will* and Gifts estate plan or the *Exemption Trust Will* estate plan; on a Comparative Analysis NPVH basis (Tables 1 and 5), the numbers

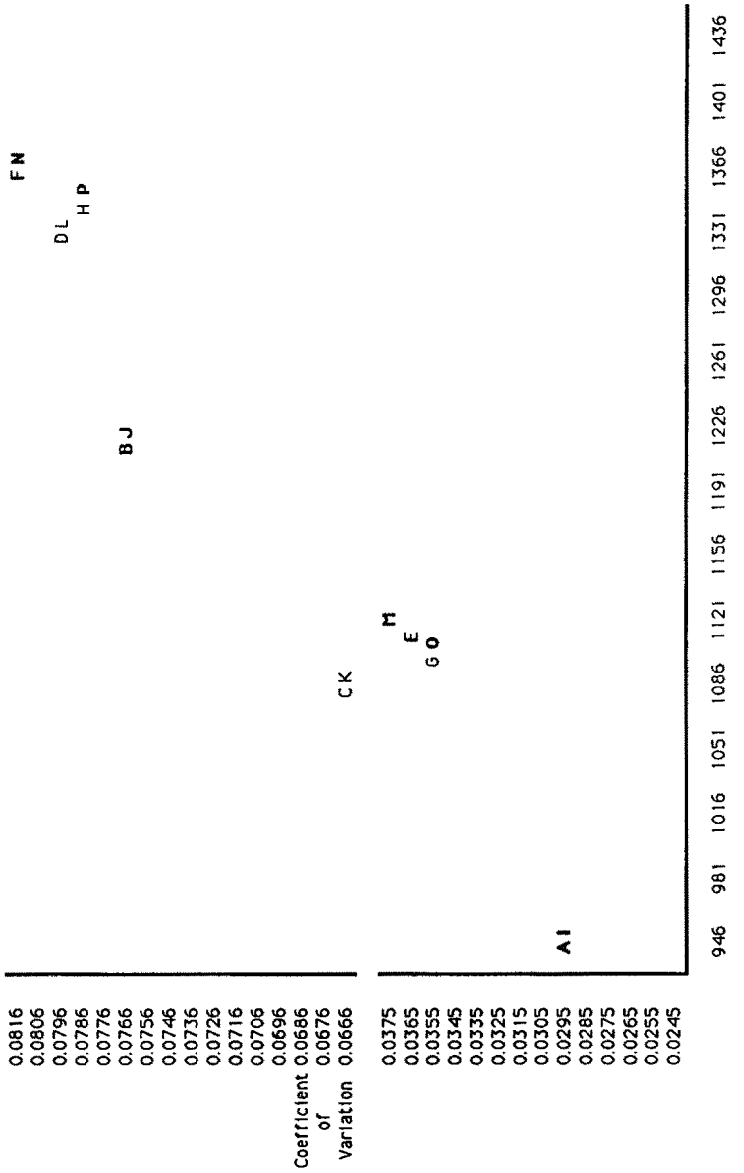

Figure 6. The Efficient Frontier (**Bold** Letters) and Six Other Estate Plans for Age 70 at 4%

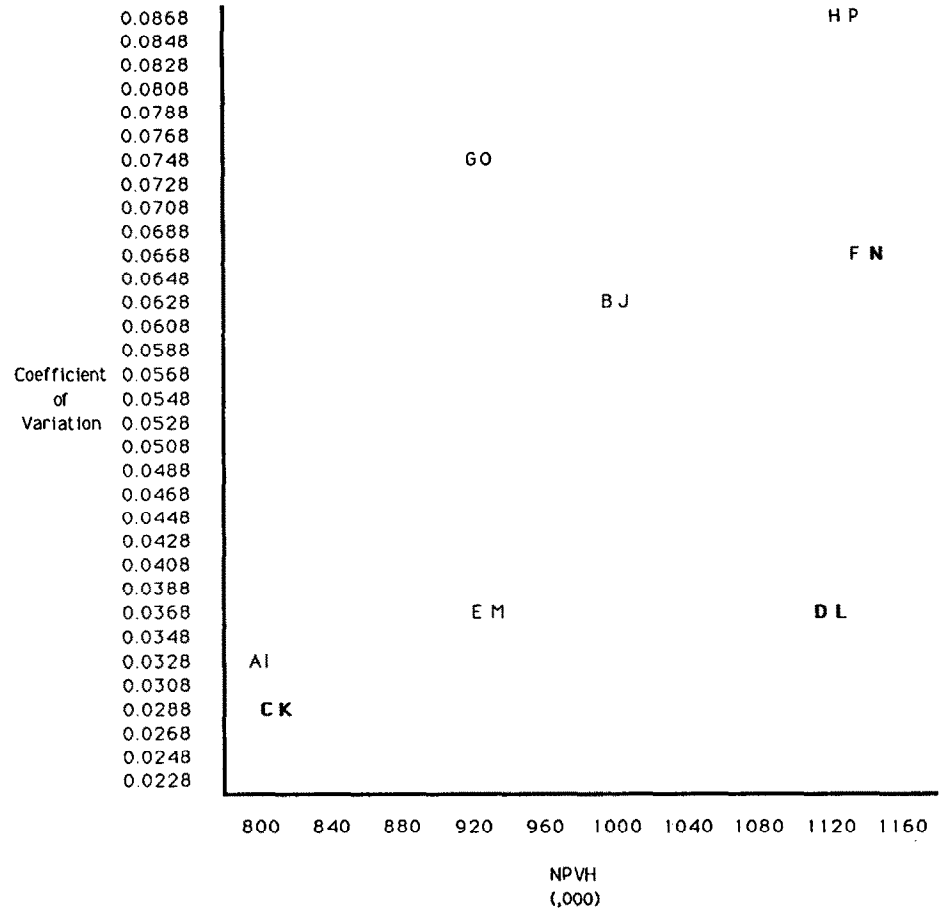

Figure 7. The Efficient Frontier (**Bold** Letters) and Eleven Other Estate Plans for Age 70 at 5%

are 100:0 for the *Exemption Trust Will* and Gifts estate plan relative to the other two estate plans.]

Finally, note that at age 45 estate plan N (*Exemption Trust Will*, IRA, Mortgage, Gifts, and Whole Life) was almost a corner-point solution on the Table 1 Comparative Analysis, recording 100:0 NPVH results against all other estate plans except for H (*Exemption Trust Will*, Gifts, and Term Life) and P (*Exemption Trust Will*, IRA, Mortgage, Gifts, and Term Life) where the results were 99:1 for N to H, and 91:9 for N to P. For both the estate planner and the estate owner, N seems to be a logical choice.

However, at age 70, the estate owners and heir(s), when trying to choose between N and P, have a difficult task, as does the estate planner who has that couple as clients. On a Comparative Analysis basis (Table 5), the N to H results were 53:47,

Figure 8. The Efficient Frontier (**Bold** Letters) and Fourteen Other Estate Plans for Age 70 at 6%

and the N to P results were 51:49. At the 4% discount rate, both N and P were on the efficient frontier, H was eliminated by P in the fifth digit of the Coefficient of Variation, and N is no longer a clearly logical choice.

III. Conclusions

The configuration of the efficient frontier is a function of three factors: the assumptions made by the estate owners (or made for them by an estate planner), the ages of the estate owners, and the discount rate(s) applied to the transferred wealth. Estate owners (and estate planners) engaged in probabilistic estate planning must be certain to address the sensitivity that this process has to those three factors, and to probe the nature of not only the plans that fall on the efficient frontier but also the nature of those plans lying close to the frontier.

As people age and the probabilities of death increase, probabilistic estate planning becomes more difficult. The near corner-point solution at age 45 becomes indeterminate at age 70. Unlike point estimate life expectancy estate planning, where risk is ignored and an "optimal" estate plan is easy to find, when the discount rate is allowed to vary, and when ages at death, rates of return on assets, and borrowing rates on debts are treated as random variables, estate planning decisions become harder to make.

While harder to make, those decisions allow the estate owner(s) and the estate planner to address the riskiness of the alternate estate plans, and provide a realism to the estate planning process. If used by an unbiased estate planner, then the probabilistic estate planning process can assist the individual estate owner(s) in choosing an efficient estate plan to maximize the expected net present value of wealth transferred to heir(s) consistent with the risk preferences of that (those) heir(s).

References

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