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Asset Allocation, Life Expectancy and Shortfall

Kwok Ho
Moshe Arye Milevsky
Chris Robinson

An analytical model provides a solution to the retirement problem of how to allocate investment between risky and risk-free assets. The objective is to minimize the probability that the retiree will be unable to consume at the desired level over his/her expected lifetime. The procedure incorporates mortality tables, real or nominal rates of return, initial wealth, and desired consumption levels. Numerical examples using standard mortality tables, historic rates of return on Canadian equity and treasury bills, and a range of realistic values for wealth and consumption show that equity should play a much bigger role in retirement portfolios than other writers advise.

I. INTRODUCTION

How does a person who is retired invest his or her wealth to maximize the probability of a secure and sufficient income? This asset allocation decision is very critical, because the person no longer has the opportunity or time to recover from mistakes with increased earnings from work. As an increasing proportion of the population of the developed nations enters retirement years, this personal finance problem is becoming of particular interest.

The retiree faces several issues in making the investment decision:

1. How much annual income does one need to provide the desired standard of living?
2. How long does the money have to last? Another way to put this is to ask how the retiree balances lower consumption against running out of money before death.
3. How does the decision incorporate inflation?
4. How should the investment be allocated among the various classes available—shares, bonds, etc.[1]

Kwok Ho, Moshe Arye Milevsky, and **Chris Robinson,** Faculty of Administrative Studies, Atkinson College, York University, North York, CANADA M3J 1P3.

Basic financial planning answers the first question and we assume the income figure required is known. The third question involves using either real returns and constant dollars or nominal returns and nominal dollars throughout the analysis. The approach we use in this paper works equally well with either one, although for convenience we use real returns and constant dollars in the numerical examples.[2]

This paper provides an analytic solution to questions two and four, under reasonable assumptions. We incorporate standard mortality tables into the decision to arrive at an expected rate of return needed to finance future consumption, with each year's consumption weighted by the probability of survival, and given the initial wealth available to generate the income. The approach is perfectly generalizable to any mortality schedule or to any individual's preferred risk schedule. For example, an individual may decide that he or she wants to be sure of consuming until age 90, and weight each year at 100 percent.[3]

Malkiel (1990) in his chapter on the life cycle guide to investing provides an explicit answer to the allocation question without the same analytic process:

> As investors age they should start cutting back on the riskier investments and start increasing the proportion of the portfolio committed to bonds. By the age of fifty-five, investors should start thinking about the transition to retirement and moving the portfolio toward income production. . . . In retirement, portfolio mainly in a variety of intermediate-term bonds (five to ten years to maturity) and long-term bonds (over ten years to maturity) is recommended. The small proportion of stocks is included to give some income growth to cope with inflation. (pp. 356–7)

In the graphs that follow the chapter, he recommends investors in the late sixties and beyond hold 60 percent bonds, 30 percent equity and 10 percent in a money market fund. Investors in their mid-fifties are recommended to have 50 percent in stocks, 45 percent in bonds. We compare a numerical example generated in our model with Malkiel's advice.

The investment allocation in this paper incorporates the required rate of return to minimize the probability of failing to meet that rate of return on average over the weighted lifespan remaining to the person. This implied utility function of minimizing shortfall is somewhat similar to the approach taken by Leibowitz and Kogelman (1991). They "measure risk by the "shortfall probability" relative to a minimum return threshold." A fund manager can choose any combination of minimum return and probability and allocate the assets between a risk and risk-free asset to attain a desirable position. Their procedure does not endogenize the time horizon of the investor, since fund managers do not necessarily have a specific time constraint. They do observe that for longer time horizons, the proportion invested in equity rises.

Many researchers have considered the general question of which investment horizon to use and what effect different horizons have on how we view risk and return. In general, they find that risk declines if assets are held without trading for long periods. Different assets perform better in shorter periods of time so the benefits of changing portfolio composition are considerable if the investor times successfully.[4] The conclusion for asset allocation is that you should use more equity for longer horizons. Lloyd and Modani (1983) conclude:

> In general, the usefulness of time diversification is more evident for portfolios containing common stock. Further, the riskiness of any portfolio position is unclear unless the number of time periods the portfolio will be held is also considered. (p. 11)

Butler and Domian (1993) use a simulation to find that equity is almost certain to be superior to bonds for holding periods exceeding 10 years, and is likely to be better for shorter holding periods. Since we are solving the problem for an individual retiree, we incorporate this time dimension explicitly. In addition, we require annual consumption from the portfolio, which does not appear in other researchers' treatments of this problem. Substitution of standard Canadian mortality tables and reasonable estimates of return and variance for Canadian T-bills and equity provides surprising results. Only at quite high wealth levels or well into retirement do the portfolios contain less than 100 percent equity. Not surprisingly, 100 percent equity is optimal for women at an older age than men, since women have a longer expected lifespan to finance. This result highlights the contradiction in the observation that women are generally seen to invest in less risky portfolios than men do. The unrecognized risk for retirees is the risk of living too long.

In the rest of the paper we proceed as follows. The next section formulates and solves the retiree's asset allocation problem. Most of the mathematical details are left to an Appendix. The following section provides the numerical results. We then examine the problem when 100 percent equity is insufficient, and provide an heuristic solution to the question of optimal leverage on personal (margin) account. We discuss the implications of our results for retirement planning. Finally, we conclude with a brief discussion of possible improvements and extensions.

II. DEVELOPING A SOLUTION

Formulation of the Problem

We wish to solve the problem of how a retiree should allocate his/her wealth between a risky and a default risk-free asset. We consider how age, mortality rates (or equivalently, life expectancy of a person at any given age), initial wealth, and the desired level of consumption affect the allocation decision.

Assume that, at the point of retirement, the individual of n years of age has wealth of W dollars. Assume that he has no other source of income so that his current and future consumption is entirely financed from this sum and earnings on it. He will invest W in a portfolio of risky and risk-free assets in order to support the level of desired consumption until death. Let C_t be this desired annual consumption in nominal dollars.[5] We do the analysis in before-tax dollars, because the details of tax rules are too difficult to incorporate.

Let ${}_1P_n$ be the probability that the individual aged n will survive one year to age n + 1. For the first year after retirement, the expected consumption is then ${}_1P_n \cdot C_1$. For the second year after retirement, the expected consumption is ${}_2P_n \cdot C_2$. If the mortality table ends at age T, the expected consumption time path after retirement will be $\{{}_1P_n \cdot C_{1,2} P_n \cdot C_2, \ldots, {}_{T-n}P_n \cdot C_{T-n}\}$. The probabilities and the life expectancy for any given age can be found in standard mortality tables. Letting d be one plus the minimum rate of return necessary to support the expected consumption, we have:

$$W = \frac{{}_1P_n \cdot C_1}{d} + \frac{{}_2P_n \cdot C_2}{d^2} + \ldots + \frac{{}_{T-n}P_n \cdot C_{T-n}}{d^{T-n}}. \qquad (1)$$

Given the wealth, consumption and life expectancies, there is an unique solution for d, which is the level of return required to avoid disaster. That is, d is one plus the minimum rate of return that an individual with initial wealth W must earn to have enough to consume C_t per annum, given the average mortality rate.

Although we will examine this more formally later, we note that the larger the value of n, the lower the value d for a given W and C_t. In other words, older individuals may earn less in order to maintain their consumption because they have fewer years to live. This is consistent with the observation that older investors usually invest more in 'safer' assets, which provide lower rates of return. Our analysis provides an explicit way to determine when they should switch to 'safer' assets.

If we perform the analysis in real dollars, which is equivalent to assuming that the level of inflation is certain, then C_t is a constant, C.[6] We can simplify equation (1) for computation purposes to:

$$\frac{W}{C}=\frac{{}_1P_n}{d}+\frac{{}_2P_n}{d^2}+\ldots+\frac{{}_{T-n}P_n}{d^{T-n}}. \tag{2}$$

The solution d is now in real terms. We use constant dollars and real rates of return in our numerical illustrations in a later section for ease of exposition, but the theoretical development is the same.

Without loss of generality, we assume that there are two assets: treasury bills (T-bills) and a diversified equity portfolio. The individual allocates W between the two. T-bills are free from default risk, but not from interest-rate risk in the long-run. A security is completely risk-free only if it pays off a known and certain amount of consumption at exactly the date required by the investor. An important point to note is that T-bills are risky, in the sense that they have a standard deviation in either real or nominal returns. A person who holds a T-bill until maturity will get exactly the promised rate of return, but if inflation changes during the period, the return is risky in terms of the consumption it permits.

Empirically, we observe that the time series of real T-bill rates has significant variability. We treat the T-bill rate of return as a random variable, and hence even a portfolio invested 100 percent in T-bills has some risk.

The investor must redo the calculations and rebalance the portfolio periodically because the required d changes as one ages. In practical terms, annual rebalancing seems reasonable, since mortality tables report one year age differences.

An Analytic Solution

The individual's problem is to allocate W between T-bills and shares so as to minimize the probability of failing to earn the minimum gross rate of return d on average over the remaining years of one's life. We assume:

1. Rates of return on equity and treasury bills, are normally (as opposed to lognormally) distributed. This assumption is not crucial for optimal results, however it enables us to secure an analytic solution to our problem.
2. Rates of return on each asset are serially uncorrelated. Thus, we consider a series of decisions in a static framework, without the dynamic consideration of what they will do each year when they come to rebalance their portfolios.
3. Returns on each asset are uncorrelated with the other. This assumption can be relaxed, and a solution is given in the appendix.

Let us use the following notation and terminology:

- α is the proportion of W invested in T-bills
- μ_{tr} is the average annual one plus rate of return on treasury bills, (or any other relatively safe investment .)
- σ_{tr}^2 is the variance of the annual rate of return on treasury bills.
- μ_{eq} is the average annual one plus rate of return on equity, (or any other relatively risky investment.)
- σ_{eq}^2 is the variance of the annual rate of return on equity.
- Denote by:

$$\mu_p(\alpha) = \mu_{tr} \cdot \alpha + \mu_{eq} \cdot (1 - \alpha) \tag{3}$$

$$\sigma_p^2(\alpha) = \sigma_{tr}^2 \cdot \alpha^2 + \sigma_{eq}^2 \cdot (1 - \alpha)^2 \tag{4}$$

Which represents the mean and variance of the rate of return (which is normally distributed), of the investor's portfolio, assuming that he has placed a proportion α, of his wealth, in treasury bills, and a proportion $1 - \alpha$ in equity.

For a given μ_{tr}, μ_{eq}, σ_{tr}^2, σ_{eq}^2, we are looking for an asset allocation proportion α^* that will minimize the probability of earning an annual rate of return that is less than the required rate d. Thus, we are trying to solve the following stochastic optimization problem:

$$min \;\; B(\alpha) \equiv \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}\sigma_p(\alpha)} exp\left\{-\frac{1}{2}\left(\frac{x - \mu_p(\alpha)}{\sigma_p(\alpha)}\right)^2\right\}dx$$

$$s.t. \quad 0 \le \alpha \le 1 \tag{5}$$

Note that the one plus rate of return cannot fall under zero. Nevertheless, we integrate from $-\infty$ for completeness. The solution of this equation for the optimal α^* is detailed in the Appendix. The result is:

$$\alpha^* = \frac{\sigma_{eq}^2 \cdot (\mu_{tr} - d)}{\sigma_{eq}^2 \cdot (\mu_{tr} - d) + \sigma_{tr}^2 \cdot (\mu_{eq} - d)} \tag{6}$$

This fairly simple result yields quite intuitive properties. We can see under what conditions the retiree picks a corner solution. If $d \ge \mu_{tr}$ then they must 'gamble' by placing 100 percent in equities. If $\sigma_{tr} = 0$ and $d < \mu_{tr}$ then the optimal solution is entirely T-bills, because they will guarantee the minimum required consumption. In between these extremes, higher returns on either asset or lower variances on either asset drives the solution towards investment in that asset, just as we would expect.

III. NUMERICAL RESULTS

Q: A 65 year old Canadian male has retired with an initial wealth of $850,000. He plans to consume $65,000 per year (in real terms). How much should he allocate, at this time, to treasury bills, and how much should he allocate to equity, to minimize the probability of outliving his money?

Assume that the average real return from treasury bills is two percent per year with a volatility (standard deviation) of 3.5%, and that the average real return on equity is eight percent per year with a volatility (standard deviation) of 17.5 percent.

These values are derived from Hatch and White (1988) using exponential smoothing to get a continuous function that places somewhat more weight on recent observations.

A: Using a Canadian actuarial mortality table we can find the required rate of return d that the individual must earn per year so as to equate the present value of his consumption requirements to his current wealth, that is, we solve equation (2). Thus, the individual's wealth should be the equivalent of a whole life annuity immediate at the required rate of return d.

In this problem $d = 1.01$, thus, the individual must earn at least one percent per year effective. Computationally, $\mu_{tr} = 1.02$ and $\mu_{eq} = 1.08$ likewise $\sigma_{tr} = 0.035$ and $\sigma_{eq} = 0.175$. Plugging the above values into equation (6) we obtain that $\alpha^* = 0.7813$ and thus the individual should place 78.13 percent of his current wealth (or \$664,100) in treasury bills, and 21.87 percent of his current wealth (or \$185,900) in equity. Note that even though $d < \mu_{tr}$, $\sigma_{tr} > 0$; so the optimal allocation includes equity. The risk-free rate is only free of *default* risk.

Each year the above computation must be done anew, (i.e., the portfolio must be re-balanced once a year) because the individual's d, one plus the required rate of return, will change as time progresses.

To generalize the picture, we calculate a range of results for variations in initial wealth, desired consumption, age, and sex. We combine the mortality rates for females and males at various ages with wealth and consumption in constant dollars to obtain d in real terms. Using the same returns and variances as in the example, we obtain Table 1. The value of d, one plus the required rate of return, are shown in Table 2.

Table 1 has a block of Es in the upper left denoting all equity portfolios, which are preferred whenever the required rate of return equals or exceeds the T-bill rate (we will explain shortly). Below them are a few bold-face numbers ranging from 0.171 to 0.849. These are interior optima where the required rate falls between zero and the T- bill rate. Finally, the lower part of the table has values of α ranging from 0.865 to 0.955. These are portfolios where $d < 1$ (see Table 2). That is, the portfolio need not earn positive returns, but must not lose more than a very small percentage of its value. Regardless of how secure the consumption seems to be, the optimal portfolio includes some equity.

The extent to which all equity portfolios dominate is quite surprising at first glance. Equity is always characterized as the riskiest security, even in a portfolio. In fact, the greatest risk for a retiree is outliving the available wealth, and given a relatively long lifespan, high risk/high return investments are necessary to minimize this risk. Thus we see that for a reasonable range of wealth/consumption ratios, an all-equity allocation is preferred into normal retirement years, and is essential for early retirees, even if they have very substantial wealth. Numerically, the upper limit of equation (6) is $\alpha = 0.96$ for the returns and standard deviations in the example. As a practical matter, an $\alpha > 0.9$ is essentially all T-bills.

We can draw more specific observations from Table 1:

1. The equity requirement is greater for women than for men. We show only five year intervals, and women should invest in all equity until they are about five years older than men with the same wealth-to-consumption ratios.
2. Women with quite low wealth to consumption ratios—seven or less—should invest in all equity as late as 80 years of age.

TABLE 1.
Optimal Allocation Between T-Bills and Equities

A: Women

Wealth to Consumption Ratio

Age	*6*	*7*	*8*	*9*	*10*	*10.5*	*11*	*11.5*	*12*	*12.5*	*13*	*13.5*	*14*	*14.5*	*15*	*16*
50	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
55	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
60	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
65	E	E	E	E	E	E	E	E	E	E	E	E	E	E	**0.171**	**0.710**
70	E	E	E	E	E	E	E	E	E	**0.386**	**0.690**	**0.780**	**0.823**	**0.849**	0.866	0.886
75	E	E	E	E	**0.512**	**0.766**	**0.833**	0.864	0.882	0.894	0.902	0.908	0.913	0.917	0.920	0.924
80	E	E	**0.795**	0.891	0.914	0.920	0.924	0.928	0.930	0.932	0.934	0.935	0.937	0.938	0.939	0.940
85	0.862	0.918	0.932	0.938	0.941	0.942	0.944	0.944	0.945	0.946	0.946	0.947	0.947	0.948	0.948	0.949
90	0.943	0.947	0.949	0.950	0.951	0.951	0.952	0.952	0.952	0.952	0.953	0.953	0.953	0.953	0.953	0.954

B: Men

Wealth to Consumption Ratio

Age	*6*	*7*	*8*	*9*	*10*	*10.5*	*11*	*11.5*	*12*	*12.5*	*13*	*13.5*	*14*	*14.5*	*15*	*16*
50	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
55	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
60	E	E	E	E	E	E	E	E	E	E	E	E	E	**0.298**	**0.623**	**0.784**
65	E	E	E	E	E	E	E	E	**0.254**	**0.670**	**0.774**	**0.821**	**0.848**	0.865	0.877	0.893
70	E	E	E	E	**0.651**	**0.795**	**0.845**	0.871	0.886	0.896	0.904	0.909	0.914	0.917	0.920	0.924
75	E	E	**0.760**	0.883	0.910	0.916	0.921	0.925	0.928	0.930	0.932	0.934	0.935	0.936	0.937	0.939
80	**0.743**	0.903	0.924	0.933	0.937	0.939	0.940	0.941	0.942	0.943	0.944	0.944	0.945	0.945	0.946	0.947
85	0.933	0.940	0.944	0.946	0.948	0.948	0.949	0.949	0.949	0.950	0.950	0.950	0.951	0.951	0.951	0.951
90	0.949	0.950	0.952	0.952	0.953	0.953	0.953	0.954	0.954	0.954	0.954	0.954	0.954	0.954	0.954	0.955

Notes: This table presents the fraction of investment capital at retirement to be allocated to T-bills (*alpha* from equation (6)). These estimates use the historical returns in Hatch and White (1988) and the required returns (d) from Table 2. The values of *alpha* in the table must be interpreted carefully.

The table shows *alphas* for different retirement ages and levels of wealth relative to the consumption in constant dollars to be funded by the wealth. For example, a ratio of 10 could be \$40,000 desired consumption in real terms to be funded by \$400,000 in savings. The 'E' entries are all equity portfolios. The non-E values are optimal portfolios containing both T-bills and equity. The values in bold-face are portfolios where the value of d lies between zero and the T-bill rate, where the allocation decision is particularly significant. If the required real rate of return is non-positive, the procedure produces allocations ranging from about 86 percent (if the rate is zero) to 96 percent. Here the allocation decision is not too significant, although some equity is always desired. The risk of failing to earn enough is quite low, and changing to a 100 percent T-bill portfolio would not raise the risk significantly.

TABLE 2.
Required Rates of Return Weighted by Survival Probabilities
(For Different Wealth/Consumption Ratios)

A: Women

	Wealth/Consumption Ratio															
Age	*6*	*7*	*8*	*9*	*10*	*10.5*	*11*	*11.5*	*12*	*12.5*	*13*	*13.5*	*14*	*14.5*	*15*	*16*
50	1.159	1.134	1.116	1.101	1.088	1.083	1.078	1.074	1.070	1.066	1.062	1.059	1.055	1.052	1.050	1.044
55	1.155	1.130	1.111	1.096	1.083	1.078	1.073	1.068	1.064	1.060	1.056	1.052	1.049	1.046	1.043	1.037
60	1.149	1.123	1.104	1.088	1.075	1.069	1.064	1.059	1.055	1.051	1.047	1.043	1.040	1.036	1.033	1.028
65	1.139	1.113	1.093	1.077	1.063	1.057	1.052	1.047	1.042	1.038	1.034	1.030	1.026	1.023	1.019	1.013
70	1.123	1.097	1.076	1.059	1.045	1.039	1.033	1.028	1.023	1.018	1.014	1.010	1.006	1.003	0.999	0.993
75	1.098	1.071	1.049	1.032	1.017	1.011	1.005	0.999	0.994	0.990	0.985	0.981	0.977	0.973	0.969	0.962
80	1.060	1.031	1.009	0.991	0.976	0.969	0.963	0.957	0.952	0.946	0.942	0.937	0.933	0.929	0.925	0.918
85	1.000	0.971	0.948	0.929	0.913	0.906	0.899	0.893	0.887	0.882	0.877	0.872	0.868	0.863	0.859	0.851
90	0.904	0.873	0.849	0.829	0.812	0.804	0.797	0.790	0.784	0.778	0.772	0.767	0.762	0.757	0.753	0.745

B: Men

	Wealth/Consumption Ratio															
Age	*6*	*7*	*8*	*9*	*10*	*10.5*	*11*	*11.5*	*12*	*12.5*	*13*	*13.5*	*14*	*14.5*	*15*	*16*
50	1.153	1.128	1.109	1.093	1.081	1.075	1.070	1.066	1.061	1.057	1.053	1.050	1.047	1.043	1.040	1.035
55	1.146	1.120	1.100	1.085	1.072	1.066	1.061	1.056	1.052	1.047	1.044	1.040	1.036	1.033	1.030	1.024
60	1.134	1.108	1.088	1.072	1.059	1.053	1.048	1.043	1.038	1.034	1.030	1.026	1.022	1.019	1.016	1.010
65	1.118	1.091	1.071	1.054	1.041	1.035	1.029	1.024	1.019	1.015	1.010	1.007	1.003	0.999	0.996	0.990
70	1.094	1.067	1.046	1.029	1.015	1.009	1.003	0.998	0.993	0.988	0.984	0.980	0.976	0.972	0.969	0.962
75	1.060	1.033	1.011	0.994	0.980	0.973	0.967	0.962	0.957	0.952	0.947	0.943	0.939	0.935	0.932	0.925
80	1.012	0.985	0.963	0.945	0.931	0.924	0.918	0.912	0.907	0.902	0.897	0.893	0.889	0.885	0.881	0.874
85	0.946	0.918	0.896	0.878	0.863	0.856	0.850	0.844	0.839	0.834	0.829	0.824	0.820	0.816	0.812	0.805
90	0.851	0.823	0.799	0.780	0.764	0.757	0.750	0.744	0.738	0.733	0.727	0.722	0.718	0.713	0.709	0.701

Notes: These values are 1 + rate of return = d based upon standard Canadian mortality tables. Equation (2) is solved for the rate of interest that equates a constant dollar value for consumption, weighted by probability of living to the end of each year, with the current investable wealth of the individual. Since the consumption is a constant (see Equation 2), wealth and consumption can be summarized in a single ratio. For example, an individual with $400,000 to invest who wishes to consume $40,000 p.a. in constant dollars has a wealth/consumption ratio of 10. This yields the same d as if one had $600,000 to invest and wished to consume $60,000 p.a.

3. Virtually all women should invest in all equity at age 65 or earlier.
4. Men with quite low wealth should invest in all equity as late as 75 years of age.
5. Virtually all men should invest in all equity at age 60 or earlier.

The specific values of alpha derived from this procedure must be interpreted with some caution, which is why we have shown '*E*' instead of the specific values. The definition of the problem requires that $0 \leq \alpha \leq 1$. This is the same as saying that the required d cannot exceed one plus the treasury bill rate. As soon as it does, we would want no treasury bills in the portfolio.

The intuition is that you cannot minimize the probability of falling below a rate of return by including in the portfolio any asset which is expected to earn less than that rate of return. Including a high risk, high return asset like equity may yield a greater loss on some occasions, but the probability of earning more than the required minimum is still higher. Given enough years of returns, the long-run return will converge to the expected return. Since so many people are in a position where they need more return to minimize shortfall risk than 100 percent equity will provide, we model borrowing in the next section.

IV. OPTIMAL MARGIN POSITION

As long as the borrowing rate is less than the return on equity, borrowing to buy more equity provides a higher rate of return than 100 percent equity, but it is also more risky. Persons normally borrow on margin or demand loans, which charge floating rate interest. Therefore, although the equity returns will fluctuate in real terms, the interest expense is essentially fixed in real terms.

The investor is faced with the annual (one plus) rate of interest charged on margin loans denoted by r_b together with the previously-mentioned μ_{eq}, σ_{eq}. The choice variable is q which represents the proportion of wealth that the investor should borrow on margin. That is, $q = 0.5$ means that the investor should borrow 50 percent of his current liquid wealth W_0 and invest the proceeds in the equity market. Note that we are assuming:

$$r_b < d$$

$$r_b < \mu_{eq}.$$

The interest rate on margin is less than the minimal required rate of return and is less than the expected rate of return on equity. Note that no assumptions are being made about the relationship between the variables μ_{eq} and d; hence d can indeed be larger than μ_{eq}.

We provide the mathematical solution in Appendix 2. It turns out that there is no global minimum probability. Intuitively, if the equity rate exceeds the borrowing rate, you can always get a little lower probability of failing to consume a given level of income by borrowing more and investing it.

The objective function does have a well-defined asymptotic value, which can be obtained by taking the limit as $q \to \infty$. In other words, the asymptotic minimum is the probability of failing to earn the required rate of return as the individual's margin approaches infinity. We derive a reasonable and practical heuristic solution to calculate the margin, $q°$, that is required for the individual to get 'very close' to the asymptotic minimum. We denote this distance to the asymptotic minimum by ε, which can be any arbitrarily-small fraction.

The probability approaches the asymptotic minimum fairly quickly, and so values of the order of one to five percent give reasonable answers.

What follows is a table of values of q° for various values of d with an $\varepsilon = 0.02$ under the assumption that one plus the margin interest rate is $r_b = 1.03$ while one plus the expected rate of return on equity is $\mu_{eq} = 1.08$ and the standard deviation is $\sigma_{eq} = 0.175$. These values are again all in real terms.

d	q°
1.040	0.1019
1.045	0.6527
1.050	1.2037
1.055	1.7547
1.060	2.3056
1.070	3.4071
1.080	4.5094
1.100	6.7132

This illustrates the fact that once the required rate of return on investors' wealth d exceeds the margin interest rate r_b, by more than a little, the actual 'optimal' margin rate q° becomes unreasonably high, and on the practical level a lottery ticket.

V. RETIREMENT PLANNING IMPLICATIONS

The most important implication is that retirees should be holding a lot of high risk, high return assets in their portfolios at much later ages than most financial planners and authors have recommended. Equity and treasury bills are merely representative of classes of assets and the implications extend to all situations.

There are more portfolios with 100 percent equity for women than men, because they have longer life expectancy. We have not seen any personal finance source advising women that equity is even more important for them than for men—the implicit assumption is that the allocation should be the same for both sexes. If the unsystematic comments of investment advisers are valid, it appears that women are less likely than men to buy shares. The evidence in this paper shows that this tendency is riskier, not safer, since women thus have an even higher probability of outliving their capital.

Suppose we consider Malkiel's advice, as quoted earlier, to move into long-term bonds as retirement nears. Let us assume that long-term bonds are expected to yield a five percent real rate.[7] As long as $\sigma_{bonds} < \sigma_{eq}$, the standard deviation is irrelevant in determining whether or not equity dominates. For every case in which $\mu_{bonds} \leq d$, the probability of not having enough to consume is minimized (as much as is feasible) by an all-equity portfolio. Inspection of Table 2 reveals that a five percent required rate would turn 15 of the all-equity cases into internal optima with a mixture of bonds and equity. A greater number of all-equity cases remain, and they can occur as late as age 75 for men and age 80 for women. Thus, Malkiel's advice is not appropriate for all retirees.

Finally, borrowing does not seem to be a good plan to cover shortfall in most cases. A person who is short a lot must borrow an unreasonably large amount to reach desired consumption. Someone close to the line should be able to reduce current consumption, rethink planned future consumption, and/or defer retirement date.

VI. CONCLUSION

We have developed a procedure for a retiree to estimate how to allocate his/her investable wealth between any two classes of investment assets. This procedure combines mortality tables, desired consumption, initial investable wealth, and the age of the retiree in a single analytic result. Numerical examples using realistic values show that if the risky asset is equity and the safe asset Treasury Bills, the allocation favors all equity in a large number of cases, even well into retirement years. A similar result will hold for long-term bonds. The procedure is easily adaptable to any mortality schedule, initial wealth, desired consumption, and set of returns and standard deviations.

The procedure is generalizable to situations where the individual's initial wealth includes one or more pension plans, sponsored by government, employer or self. Defined contribution plans simply provide a part of the initial wealth to be allocated. The pension provided by the defined benefit plan can be deducted from the desired consumption to arrive at a shortfall between retirement income and consumption. This shortfall is treated as the consumption value to be financed by the non-pension investments.[8]

The procedure described in the previous paragraph is technically correct. Our model uses minimization of shortfall instead of a particular form of utility function, as is commonly done elsewhere. This increases the generalizability of the results, since a person can choose various levels of desired consumption retirement and see what the investment implications are. However, the following problem arises.

Two persons faced with the same shortfall may assess it very differently if one depends on investment for all their income, and the other depends on a large part from, say, an employer pension plan. The person with the separate employer plan is much more secure, and would presumably have a different utility for the extra consumption than would someone for whom missing it is a serious risk. We make that tradeoff in order to maintain a paper that provides practical solutions, since derivation of specific utility functions is quite problematic. The procedure applies to a wide variety of problems, but there are some improvements which would make it more valid.

In principle, it should handle the problem of a retirement couple as well, but we have not analyzed the practical difficulties. All the changes occur in the determination of d. Since there is now a joint probability of either one or two persons consuming an amount which is dependent on who survives how long, the mechanics would be much more complex. Once a d is estimated the procedure continues as for a single person. Since the probability is good that one (unspecified) member of the couple will live to a substantial age, the requirement for a high return portfolio increases.

The derivation assumes no serial correlation of returns. If year-to-year returns are serially correlated, the minimization of the probability of not earning enough over the entire lifetime is a much more complicated procedure which may or may not prove tractable mathematically. Serially-uncorrelated returns allow minimization of the product of the probabilities, and a closed-form solution.

We do not consider the dynamic behavior of the retiree faced with this problem each year. Every year the retiree can rebalance the portfolio or change consumption. They face a new Table 1, and can choose to adjust accordingly. For example, a bad year may induce lower consumption, and this affects all future probabilities of failing to earn enough.

Although we do not give any details, in the appendix we do present the analytical solution to the problem if the returns on the two assets are correlated (see equation (25)). We do not provide numerical examples.

APPENDIX 1

Solving for the Optimal Asset Allocation

The solution to equation (5) is the optimal proportion that is to be invested in treasury bills, and should be expressed as the following function:

$$\alpha^* = f(d, \mu_{tr}, \mu_{eq}, \sigma_{tr}^2, \sigma_{eq}^2) \tag{7}$$

To simplify the notation:

- Denote the probability density function (p.d.f) of the standard normal distribution, (with mean zero and variance one.) by:

$$\varphi(y) = \frac{1}{\sqrt{2\pi}} exp\ \{-\frac{1}{2}y^2\} \tag{8}$$

- Denote the cumulative density function (c.d.f) of the standard normal distribution[9] by:

$$\Phi(u) = \int_{-\infty}^{u} \varphi(y)\, dy \tag{9}$$

Let us make the following substitution in the integral of equation (5) so as to simplify the integrand.

$$y = \frac{x - \mu_p(\alpha)}{\sigma_p(\alpha)} \tag{10}$$

The above implies that:

$$dy = \frac{1}{\sigma_p(\alpha)} dx \tag{11}$$

Thus we can substitute and transform the stochastic optimization problem in equation (5) to the elementary representation:

$$min\ \ B(\alpha) \equiv \int_{-\infty}^{\frac{d-\mu_p(\alpha)}{\sigma_p(\alpha)}} \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{1}{2}y^2\right\} dy \tag{12}$$

$$s.t.\ \ 0 \leq \alpha \leq 1$$

Which can be expressed, using equation (8), as:

$$\min \quad B(\alpha) \equiv \int_{-\infty}^{\frac{d-\mu_p(\alpha)}{\sigma_p(\alpha)}} \varphi(y)\, dy \tag{13}$$

$$s.t. \quad 0 \le \alpha \le 1$$

However, using the notation introduced in equation (9), we can rewrite the above as:

$$\min \quad B(\alpha) \equiv \Phi\left(\frac{d-\mu_p(\alpha)}{\sigma_p(\alpha)}\right)$$

$$s.t. \quad 0 \le \alpha \le 1 \tag{14}$$

In order to find an α^* that will minimize the above, we must take the derivative of the objective function, $B(\alpha)$, with respect to α and set it equal to zero. That is, the first order condition (F.O.C.) is:

$$\frac{\partial B(\alpha)}{\partial \alpha} = 0 \tag{15}$$

Or we can write explicitly:

$$\frac{\partial \Phi\left(\frac{d-\mu_p(\alpha)}{\sigma_p(\alpha)}\right)}{\partial \alpha} = 0 \tag{16}$$

Taking derivatives with respect to α by using the chain rule, we derive that the first order condition for optimality can be expressed as:

$$\Phi'\left(\frac{d-\mu_p(\alpha)}{\sigma_p(\alpha)}\right) \times \left(\frac{-\mu_p'(\alpha)\cdot\sigma_p(\alpha) - \sigma_p'(\alpha)\cdot(d-\mu_p(\alpha))}{\sigma_p^2(\alpha)}\right) = 0 \tag{17}$$

The objective now is to find an α^* that will solve the above equation.

The next step is to realize that the first term in equation (17), namely Φ', is the derivative of the cumulative distribution function. Thus, using the fundamental theorem of calculus and the definition in equation (9), we can rewrite the first order condition as:

$$\varphi\left(\frac{d-\mu_p(\alpha)}{\sigma_p(\alpha)}\right) \times \left(\frac{-\mu_p'(\alpha)\cdot\sigma_p(\alpha) - \sigma_p'(\alpha)\cdot(d-\mu_p(\alpha))}{\sigma_p^2(\alpha)}\right) = 0 \tag{18}$$

Let us analyze equation (18). It should be obvious that φ, which is a probability density function, will never take on the value zero. Thus, the only way that the left hand side of equation (18) can evaluate to zero, is by the following being satisfied:

$$-\mu_p'(\alpha)\cdot\sigma_p(\alpha) - \sigma_p'(\alpha)\cdot(d-\mu_p(\alpha)) = 0 \tag{19}$$

In order to solve the above equation for α (or isolate α), we must compute $\mu_p'(\alpha)$ and $\sigma_p'(\alpha)$ by taking the derivatives of equations (3) and (4) to obtain:

$$\mu_p'(\alpha) = \mu_{tr} - \mu_{eq} \tag{20}$$

$$\sigma_p'(\alpha) = \frac{(\sigma_{tr}^2 + \sigma_{eq}^2) \cdot \alpha - \sigma_{eq}^2}{\sigma_{tr}^2 \cdot \alpha^2 + \sigma_{eq}^2 \cdot (1-\alpha)^2} \tag{21}$$

Substituting equations (3) and (4) together with equations (20) and (21) *into* the first order condition of equation (19), we can rewrite equation (19) as:

$$(\mu_{eq} - \mu_{tr}) \sqrt{\sigma_{tr}^2 \cdot \alpha^2 + \sigma_{eq}^2 \cdot (1-\alpha)^2} = \left[\frac{(\sigma_{tr}^2 + \sigma_{eq}^2) \cdot \alpha - \sigma_{eq}^2}{\sigma_{tr}^2 \cdot \alpha^2 + \sigma_{eq}^2 \cdot (1-\alpha)^2} \right] \times (d - \mu_{tr} \cdot \alpha + \mu_{eq} \cdot (1-\alpha)) \tag{22}$$

Solving for α (with some tedious algebra and the help of the symbolic computational language MAPLE V), we find that there is *only one* value of α that will satisfy equation (22) and it is equation (6):

$$\alpha^* = \frac{\sigma_{eq}^2 \cdot (\mu_{tr} - d)}{\sigma_{eq}^2 \cdot (\mu_{tr} - d) + \sigma_{tr}^2 \cdot (\mu_{eq} - d)}$$

Checking second order conditions (S.O.C.) one can verify that the second derivative of $B(\alpha)$ with respect to α is positive at α^*. (Based on intuition, one could assert without resorting to S.O.C. that α^* must be a minimum because of diversification arguments.) Hence, α^* is a unique (global) minimum, and we have obtained a closed form solution for the proportion α^* that must be allocated to treasury bills so as to minimize the probability of earning a rate of return that is less than the required amount.

Let us perform some comparative statics by inspecting equation (6), we can recognize that *ceteris paribus* as σ_{tr}^2 decreases α^* will increase. Intuitively this means that as treasury bills become less risky one should allocate a higher proportion of his wealth to treasury bills, and vice versa if the riskiness of treasury bills were to increase. Mathematically this can be represented as:

$$\frac{\partial \alpha^*}{\partial \sigma_{tr}^2} < 0$$

Likewise, *ceteris paribus* as σ_{eq}^2 increases, α^* will decrease. As equity becomes riskier one should allocate more to treasury bills. Mathematically this can be expressed as:

$$\frac{\partial \alpha^*}{\partial \sigma_{eq}^2} > 0$$

The same kind of argument will show that as μ_{tr} increases, α^* increases; and as μ_{eq} increases, α^* will decrease. Mathematically:

$$\frac{\partial \alpha^*}{\partial \mu_{tr}} > 0$$

and:

$$\frac{\partial \alpha^*}{\partial \mu_{eq}} < 0$$

Let us rewrite equation (6) as:

$$\alpha^* = \frac{d \cdot \sigma_{eq}^2 - \sigma_{eq}^2 \cdot \mu_{tr}}{d \cdot (\sigma_{eq}^2 + \sigma_{tr}^2) - \sigma_{tr}^2 \cdot \mu_{eq} - \sigma_{eq}^2 \cdot \mu_{tr}} \tag{23}$$

We see by inspection that as d increases, α^* decreases. Hence, as the required rate of return goes up, the individual must 'gamble' by investing more in equity. Mathematically:

$$\frac{\partial \alpha^*}{\partial d} < 0$$

It is important to note that by the definition of the problem α is restricted to be between zero and one. On the other hand, there exist realistic situations where plugging the numbers into equation (6) will produce an α^* that is negative or larger than one. In the event that should happen, most probably the required rate of return is *too high*; that is, there is no way to minimize the probability of default because $B(\alpha)$ is a monotonic function that has no interior minimum. Thus, one should choose the boundary value closest to the α^*. The above criteria will probably not be appropriate for individuals with either very high or very low d's.

Finally, the problem can be solved analytically for the case in which the returns on the two assets are correlated. Letting ρ be the correlation between the two assets, equation (4) becomes:

$$\sigma_p^2(\alpha) = \sigma_{tr}^2 \cdot \alpha^2 + \sigma_{eq}^2 \cdot (1-\alpha)^2 + 2\alpha(1-\alpha) \cdot \rho \cdot \sigma_{tr} \cdot \sigma_{eq}. \tag{24}$$

Proceeding to solve it in a similar fashion yields the following:

$$\alpha^* = \frac{d \cdot \sigma_{eq}^2 - \sigma_{eq}^2 \cdot \mu_{tr} - d \cdot \rho \cdot \sigma_{tr} \cdot \sigma_{eq} + \mu_{eq} \cdot \rho \cdot \sigma_{tr} \cdot \sigma_{eq}}{d \cdot (\sigma_{eq}^2 + \sigma_{tr}^2) - \sigma_{tr}^2 \cdot \mu_{eq} - \sigma_{eq}^2 \cdot \mu_{tr} + \mu_{tr} \cdot \rho \cdot \sigma_{eq} \cdot \sigma_{tr} - 2d \cdot \rho \cdot \sigma_{eq} \cdot \sigma_{tr} + \mu_{eq} \cdot \rho \cdot \sigma_{eq} \cdot \sigma_{tr}} \tag{25}$$

APPENDIX 2

Determining the Optimal Margin

The (levered) investor's wealth at the end of one year will be a random variable, if one starts with W_0 to invest:

$$W_0 \times (1+q) \times \tilde{X} - W_0 \times q \times r_b, \tag{26}$$

where $\tilde{X}$ is the random variable, which represents the (one plus) rate of return from equity with mean μ_{eq} and standard deviation σ_{eq}. As previously-defined, q is the margin, expressed as a fraction of the investor's total assets and r_b is the borrowing rate. We create a new random variable $\tilde{Y}$ such that:

$$\tilde{Y} = (1+q) \times \tilde{X} - q \times r_b, \tag{27}$$

which represents the random *levered* (one plus) rate of return from the margined position.

$$E[\tilde{Y}] = (1+q) \times \mu_{eq} - q \times r_b; \tag{28}$$

likewise

$$S.D.[\tilde{Y}] = (1+q) \times \sigma_{eq}. \tag{29}$$

This represents the higher expected return, as well as higher risk, in the margined position. In addition $\tilde{Y}$ is normally distributed by simple additivity. As before, we seek the optimal q^* that will minimize the probability of earning a (one plus) rate of return that is under d.

Mathematically:

$$\min_q \; p[\tilde{Y} \le d] = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}\sigma_{eq} \times (1+q)} exp\left\{ -\frac{1}{2}\left(\frac{y - (1+q) \times \mu_{eq} + q \times r_b}{(1+q) \times \sigma_{eq}} \right)^2 \right\} dy \tag{30}$$

Proceeding with the derivation and conditions as in Appendix 1, we are seeking to minimize the function:

$$\min_q \; \Phi\left[\frac{d - (1+q) \times \mu_{eq} + q \times r_b}{(1+q) \times \sigma_{eq}} \right] = \Phi\left[\frac{d - r_b}{(1+q) \times \sigma_{eq}} - \frac{(\mu_{eq} - r_b)}{\sigma_{eq}} \right]. \tag{31}$$

Setting up the first order conditions for optimality implies finding a q^* such that:

$$\Phi'\left[\frac{d - r_b}{(1+q^*) \times \sigma_{eq}} - \frac{(\mu_{eq} - r_b)}{\sigma_{eq}} \right] = 0, \tag{32}$$

which is equivalent to finding a q^* that satisfies:

$$\varphi\left(\frac{d - r_b}{(1+q^*) \times \sigma_{eq}} - \frac{(\mu_{eq} - r_b)}{\sigma_{eq}} \right) \times (-1) \times \left(\frac{d - r_b}{(1+q^*)^2 \times \sigma_{eq}} \right) = 0 \tag{33}$$

However, there is *no* positive value of q^* that will satisfy the above first order condition. In fact, by inspection, one can ascertain that *for all* values of q the derivative is negative, this in turn implies that the objective function is *decreasing* in q.

But all is not lost. The objective function has a well defined asymptotic value which can be obtained by taking the limit as $q \to \infty$. The asymptotic value of the objective function will be

$$\Phi\left[-\frac{(\mu_{eq} - r_b)}{\sigma_{eq}} \right] \tag{34}$$

So in fact, the probability of earning a rate of return that is under d, by buying on margin, will approach a limit as the amount of margin goes out to infinity.

It is important to realize, that the above mentioned probability *is not* a probability of *starvation* (or failing to meet the consumption requirement). It is the probability of earning, in one year, a rate of return that is under the required level. If indeed one earned an above average return in the following year then one could recuperate the losses and meet all future consumption requirements. The overall probability of starvation is the probability of *repeat-*

edly earning a rate of return under the required rate, over long periods of time, so as to make the required consumption infeasible. This probability is not addressed in the paper.

Let us continue with our margin discussion. In theory the asymptotic value means that there is no *optimal* margin per se. Yet, we can find a q° that will bring us within ε of the minimal probability. This q° will satisfy

$$\Phi\left[\frac{d-r_b}{(1+q^{\circ})\times\sigma_{eq}}-\frac{(\mu_{eq}-r_b)}{\sigma_{eq}}\right]-\Phi\left[-\frac{(\mu_{eq}-r_b)}{\sigma_{eq}}\right]=\varepsilon.$$

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NOTES

1. We assume throughout that investment in any one asset class is approximately diversified, perhaps through a mutual fund.

2. Gitman and Joehnk (1990, Exhibit 15.1, p. 541) and Turner, Le Rossignol, Rinfret, and Daw (1991, Form 27, p. 115) make the mistake of capitalizing a constant dollar value using a nominal discount rate,which yields too low an estimate of required savings. This error seems common among financial planners.

3. Financial planners generally seem to assume that a perpetual dollar return equal to consumption is required, explicitly ignoring the possibility of consuming capital over time. See previous note for two examples. This error overstates the savings required, and thus the two errors offset to some extent.

4. See, for example, Benari (1990), Butler and Domian (1991), Grauer and Hakansson (1982) and Lloyd and Modani (1983).

5. In retirement planning, the usual approach is to have the retiree define the lifestyle they want and then price each expenditure of that lifestyle. The total expenditure required to achieve the desired lifestyle in each year is C_t. The value of C_t can range from bare subsistence to great luxury, provided it is feasible.

6. If inflation is certain, then we can convert real to nominal returns using the Fisher equation without affecting the analysis. We do not consider uncertain inflation.

7. The long-run historical experience in Canada is actually a rate lower than the T-bill rate, but this is surely not a good basis for an expected rate.

8. If the pension is indexed, then everything can be treated in real terms, as in equation (2). If the pension is unindexed, then it pays in nominal dollars, and equation (1) is appropriate.

9. That is, the probability that a standard normal random variable will take on a value less than or equal to u. Graphically, it is the area under the p.d.f. from u to negative infinity.

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