

## Quantifying Time Value Errors

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*Time valuation of cash flows is an essential part of personal financial planning and management. Many financial arrangements are priced according to a cash-flow valuation model. Expected cash flows associated with a stock or bond are discounted at an appropriate risk-adjusted rate in order to determine the fair value of the financial asset. Home mortgage loans are priced according to the discounted value of the future principal and interest cash flows. Yet, despite the importance of the discounted cash flow methodology in pricing assets, computational errors are often made when discount factors are not calculated precisely. This article attempts to quantify the magnitude of the error when the mathematical function for present value is ignored and interpolation is used instead to determine the discount factor.*

### I. INTRODUCTION

Even though cash-flow valuation is an essential ingredient for financial planning and decision making, many borrowers are unfamiliar with time value mathematics. Individual borrowers tend to rely on their bankers for accurate calculation of amortization schedules. However, bank errors do occur, especially with adjustable rate mortgages (ARMs), which require frequent recalculation. According to a study published in 1993, the Federal Reserve uncovered 881 faulty loans at six of 44 banks it scrutinized. An earlier Government Accounting Office report quoted authorities as estimating that as many as 30% of ARM adjustments are incorrect.

Adjustment errors result from three mistakes—using the wrong index to determine the reset rate, adjusting loans on the wrong day, and rounding off numbers incorrectly. The problem is so large that several banks have incurred penalties for their goofs and have had to settle with borrowers. Moreover, some banks are being investigated by the Federal Trade Commission for misleading practices in lending or sued for their malfeasance. For example, the use of an incorrect computation method on the part of a large California

bank consistently led customers to think that borrowing rates were lower than they really were. Class-action lawsuits against mortgage lenders are becoming more commonplace.

When banks discover errors, they try to correct the problem with the borrower. In the case where the error leads to an underpayment of that period's mortgage amount, the bank will attempt to extend the loan payback period or obtain a retroactive adjustment. An overpayment can be corrected by rebating the excess amount to the borrower. Still, the customer has incurred opportunity costs by having paid more than was required. The ARM market is large at over \$14 billion. ARM payment miscalculation is a serious issue.

Because of the prevalence of bank errors, consumer advocates are urging borrowers to verify the accuracy of their mortgage payments. Borrowers have several choices. They can hire a financial advisor to audit the payment schedule. Alternatively, they can render their own calculations to corroborate the payment schedule. Aids such as the "ARM Check Kit" from HSH Associates provide instructions and tables for computing a mortgage amortization schedule. However, the verification process itself is fraught with difficulty. Reliance on time value tables can lead to error for several reasons.

First, table factors are often rounded to four decimal places, which in and of itself leads to inaccuracy. Second, time value functions are not linear functions. This means that the approximation of a time value factor based on linearly interpolating between two time value factors when there is no table value available for the rate in question is incorrect.

This article develops a formula to measure the error created when factors based on a linear interpolation are used in lieu of computing the actual factor value. The conditions under which the error is most acute are identified. Specifically, the authors examine the sensitivity of error to levels of interest rates, the length of the time valuation period, and the difference between the two interest rates that are being used in the linear interpolation process. For ease of exposition, only the present value of a single cash-flow function assuming discrete compounding is discussed. However, the general idea of quantifying time valuation errors can easily be extended to problems involving either multiple cash flows, continuous compounding, and/or the future value function.

To clearly see that the present value function is nonlinear, one need only plot the function as its inputs—time and periodic interest rate—change. Figure 1 shows that for both 5% and 10%, the graph of the present value function for a single cash flow of \$1.00 is curvilinear. A numerical example can also be used to emphasize the nonlinearity of the present value function. Suppose that a financial planner wants to compute the present value of \$500,000 for a client who is trying to assess how much must be deposited today in order to reach this objective in 20 years assuming an annual rate of return of 7.5%.

The planner uses a discounting table with rate increments of 5%. Using the linear interpolation method to compute the present value interest factor for 7.5%, the investor should make a deposit today in the amount of \$131,375.<sup>1</sup> The exact present value is \$500,000 times  $(1.075)^{-20}$ , or \$117,706.57. The magnitude of the error is \$13,668.43, or 11.61% of the correct deposit amount. Admittedly, this is an extreme example to illustrate the point inasmuch as table values are usually given in increments of 1% up to 10% and in increments of 2% for higher rates.

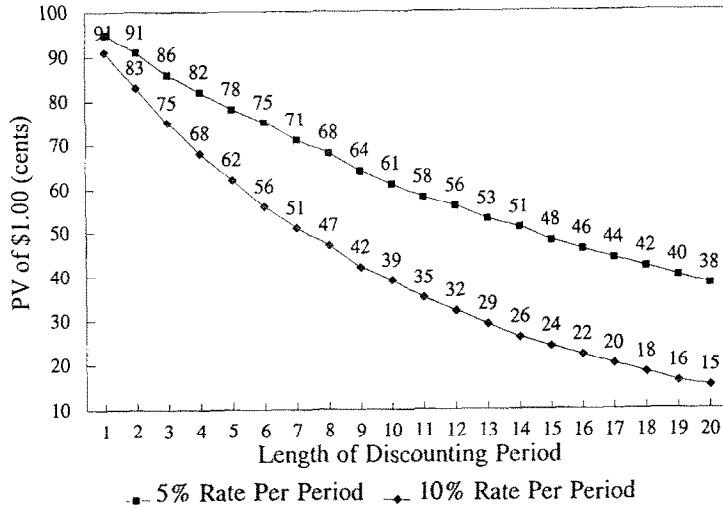


Figure 1. Present value of \$1.00 at 5% and 10%, respectively

## II. A GENERAL MODEL

Consider the general model in which the present value error is expressed as the linearly interpolated present value factor minus the actual present value factor.

$$E = \{ [(r_2 - r^*) (r_2 - r_1)^{-1}] \times [(1 + r_1)^{-n} - (1 + r_2)^{-n}] + (1 + r_2)^{-n} \} - [(1 + r^*)^{-n}] \quad (1)$$

- where:
- $r^*$  = the rate in question (not found in the table)
  - $n$  = length of the period
  - $r_1$  = lower bound of the table rate interval
  - $r_2$  = upper bound of the table rate interval.

Redefining  $r_2$  as  $r_1 + .01$  reflects the assumption of table factors being provided for 1% interest intervals. Later, this interval will be generalized and referred to as  $D$ . The distance between the rate in question and the lower boundary of the rate interval,  $r^* - r_1$ , is redefined as  $\alpha$ . Equation (1) can thus be simplified as:

$$E = \{ [(0.01 - \alpha) / .01] \times [(1 + r_1)^{-n} - (1.01 + r_1)^{-n}] + (1.01 + r_1)^{-n} \} - [(1 + r_1 + \alpha)^{-n}]. \quad (2)$$

Note that when  $r^*$  equals  $r_1$  or  $r_2$ , the only error present is that which results from using a rounded factor. No linear interpolation error exists when  $r^*$  equals either the upper or lower boundary of a rate interval. To maximize the error  $E$  with respect to the distance between  $r^*$  and  $r_1$ , it is necessary to find the first derivative  $d(E)/d\alpha$  and set this to zero:

$$0 = \{ -(1/.01) \times [(1 + r_1)^{-n} - (1.01 + r_1)^{-n}] \} + [n \times (1 + r_1 + \alpha)^{-(n+1)}] \quad (3)$$

Letting  $k$  equal  $[100 \times (1 + r_1)^{-n}] - [100 \times (1.01 + r_1)^{-n}]$ , Equation (4) gives a value for the distance between  $r^*$  and  $r_1$  which corresponds to the maximum error made when relying on linear interpolation between table present value factors for a periodic interest rate,  $r_1$ , and that same interest rate plus 1%.<sup>2</sup>

$$\alpha = (n/k)^{1/(n+1)} - (1 + r_1) \quad (4)$$

Recall that the minimum interpolation error is zero, which occurs when  $r^*$  equals either  $r_1$  or  $r_2$ . Therefore, it is not necessary to take the second derivative of the error function to verify that  $\alpha$  is associated with the maximum error.

### III. NUMERICAL VERIFICATION OF $\alpha$

Let  $r_1$  equal .10. Thus,  $r_2 = .11$ . When the time interval consists of two periods ( $n = 2$ ),  $d(E)/d\alpha = -1.482 + 2(1.10 + \alpha)^{-3}$ . Setting the first derivative to zero, the result is that  $1.482 = 2/(1.10 + \alpha)^3$ . Solving for  $\alpha$  results in a value of .00498. This means that the maximum interpolation error occurs when  $r^*$  is almost half way between  $r_1$  and  $r_2$ .

Consider the case where  $n = 20$  ceteris paribus. The variable  $k$  takes on a value of 2.460972090.<sup>3</sup> Using Equation (4),  $\alpha$  equals .004917 or  $(20/2.4609 \dots)^{1/21} - 1.1$ . When  $n$  is changed to 1 and no other variables change values,  $k$  equals .819.<sup>4</sup> Thus,  $\alpha$  equals .0049886, or  $(1/.819)^{1/2} - 1.1$ . When  $n$  is changed to 100,  $\alpha$  still takes on a value of .00499. Changing the numeric values of  $r_1$  and  $r_2$  still results in an unchanged value of  $\alpha$ . Consider the case when  $r_1$  equals .50 and  $r_2$  equals .51. For a one-period time interval,  $\alpha$  still takes a value of .00499.<sup>5</sup>

Importantly, however, it turns out that  $\alpha$  does change value as the model inputs— $r_1$  and  $n$ —change. Moreover,  $\alpha$  is affected by relaxing the assumption that table values always represent periodic interest rates, which differ by 1%. The effects of these three model parameters on the magnitude of  $\alpha$  are discussed in Section IV.

### IV. SENSITIVITY OF $\alpha$

$\alpha$  is a function of three parameters—the percentage increment underlying table factors ( $D$ ), the interval in which the target rate falls ( $r_1$ ), and the number of periods which define the investment horizon ( $n$ ):

$$\alpha = n^{1/(n+1)} \{D^{-1} \times [(1 + r_1)^{-n} - (1 + r_1 + D)^{-n}]\}^{1/(n+1)} - (1 + r_1) \quad (5)$$

Note that when  $D$  equals 0.01, Equation (5) becomes Equation (4). For example, the  $\alpha$  associated with finding the present value of funds for a 20-year period ( $n = 20$ ) discounted at a rate of 8.25% (i.e.,  $r^* = 0.0825$ ) will depend not only on  $n$  equal to 20 and  $r_1$  equal to 0.08 but also on  $D$ —that is, whether one is linearly interpolating between an 8% factor and a 9% factor versus interpolating between an 8% factor and a 10% factor. Likewise,  $\alpha$  will change if one lengthens or shortens the discounting period,  $n$ .

Using MATLAB, a program was written to compute  $\alpha$  which is associated with the largest mathematical error as a function of  $D$ ,  $r_1$ , and  $n$ . The  $r_1$  used to generate Figures 2 and 3 equals 0%. In Figure 2, however,  $D$  equals 1%, while in Figure 3,  $D$  equals 5%. Both figures demonstrate that the error grows without bound as the  $n$  increases. The obvious ramification is that financial planners or lenders who erroneously use interpolated discount factors falling between zero percent and low interest rates are not only wrong in doing so but are especially wrong when working with long-term transactions.

For  $r_1$  greater than zero percent, the maximum interpolation error occurs sooner in the time period for larger  $r_1$  values. Figure 4 highlights this trend. When  $r_1$  equals 5% and  $r_2$

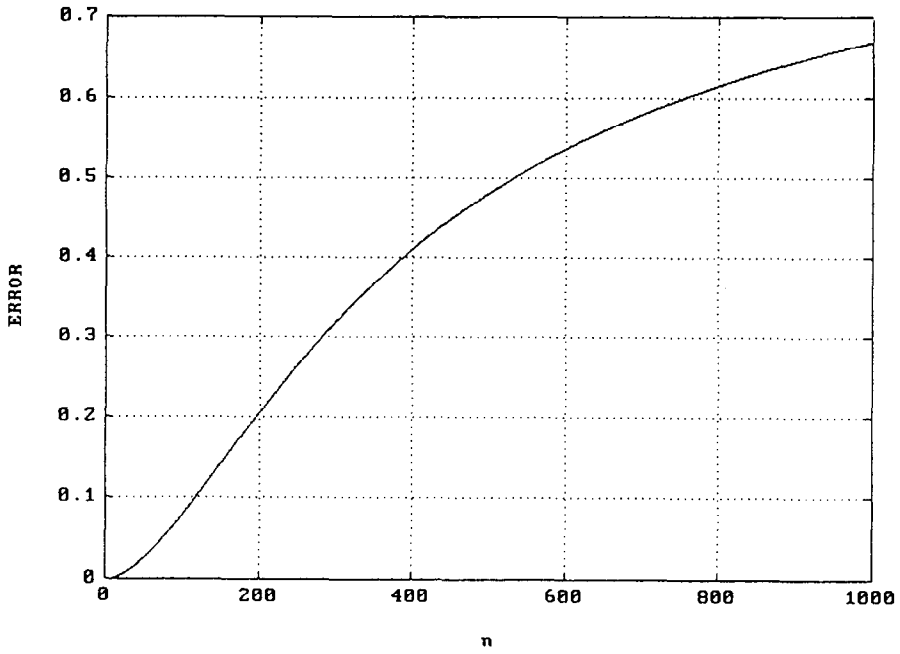


Figure 2. Error due to interpolation  
( $r_1 = 0\%$ ;  $d = 1\%$ )

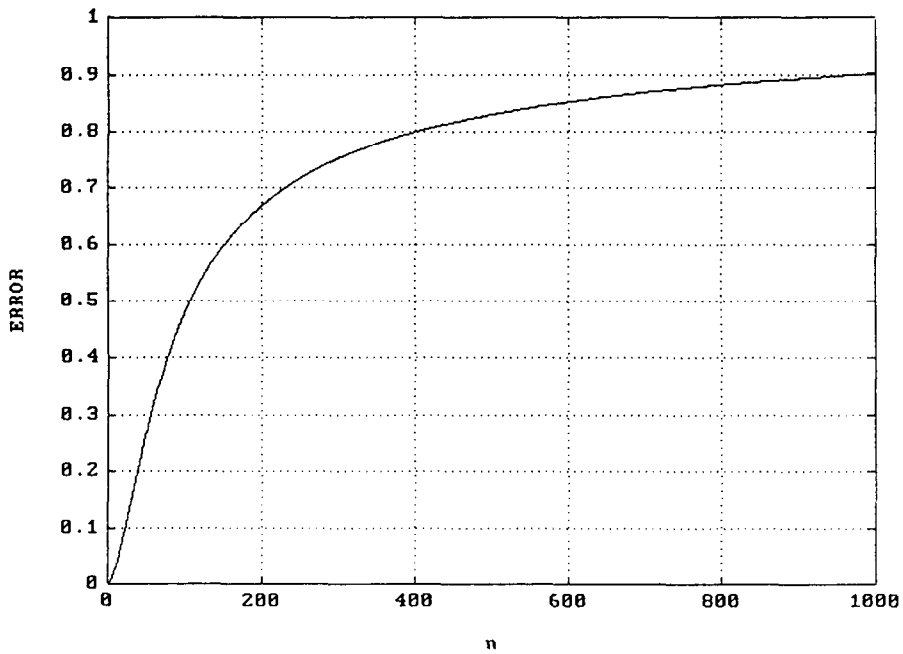
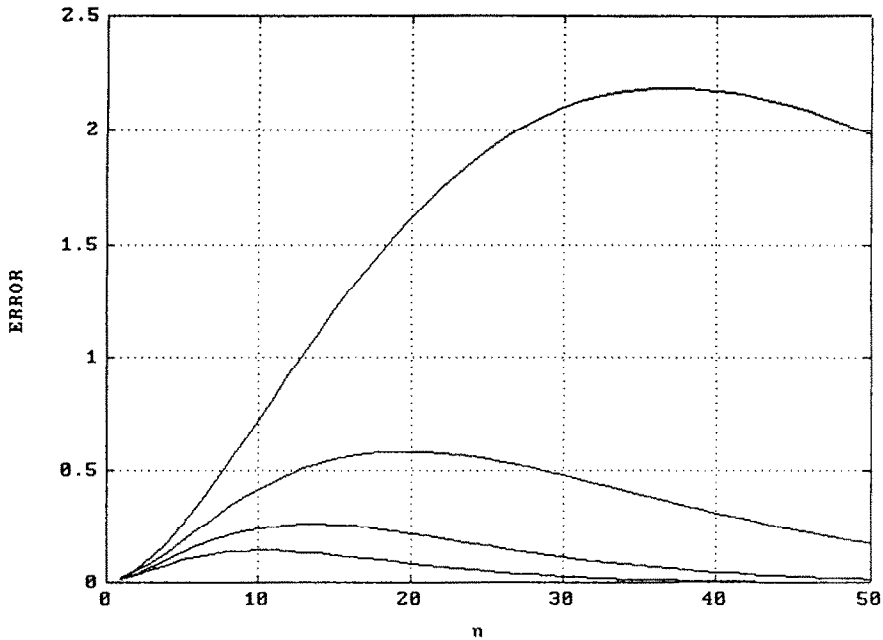


Figure 3. Error due to interpolation  
( $r_1 = 0\%$ ;  $d = 5\%$ )



**Figure 4.** Error due to interpolation  
( $r_1 = 5\%, 10\%, 15\%, 20\%$ ;  $d = 1\%$ )

equals 6%, the maximum interpolation error occurs when  $n$  equals 37. Compare this to the timing of the maximum error which occurs when  $n$  equals 10 for an  $r_1$  of 20% and an  $r_2$  of 21%. Table 1 provides  $n$  values for various combinations of the lower and upper interest rate boundaries.

This same phenomenon is observed when the rate interval,  $D$ , is widened to 2%. Recall that time value tables typically include factors in increments of one percent for periodic interest rates of 0% to 10% and use increments of two percentage points for rates greater than 10%. For example, to find an interpolated discount factor for a 22.25% rate, one would have to use the factors provided for 22% and 24%. Figure 5 illustrates how the error function is maximized at smaller and smaller  $n$  values as the two boundary rates are increased. Notice

**TABLE 1**  
Sensitivity of the Timing of the Maximum Error to Interest Rates When  $D = 1\%$

$r_1$	$r_2$	Maximum Error	Timing of Maximum Error ( $n$ )
5%	6%	0.0021864	37
10%	11%	0.0005842	20
15%	16%	0.0002621	13
20%	21%	0.0001468	10

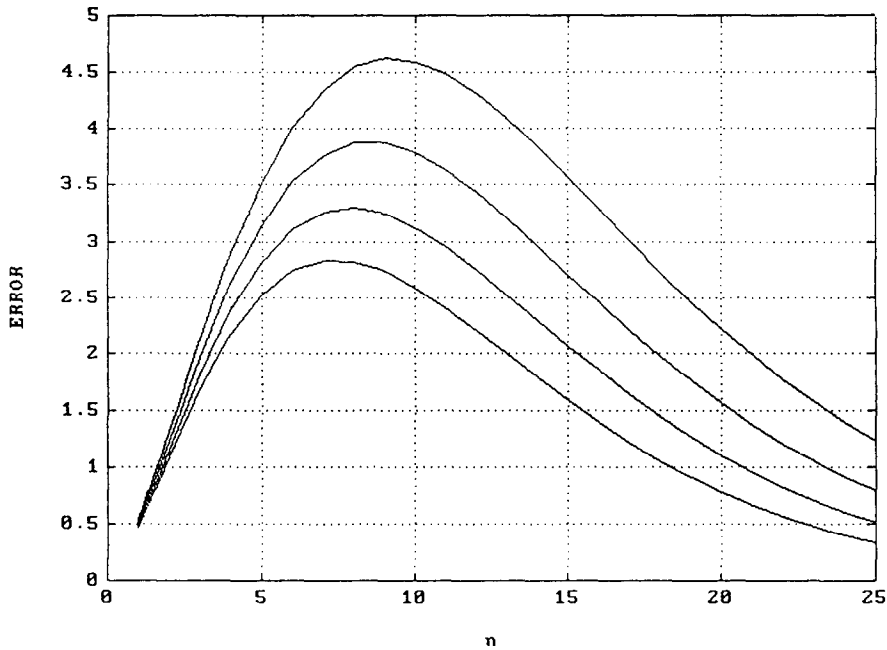


Figure 5. Error due to interpolation  
 ( $r_1 = 22\%, 24\%, 26\%, 28\%$ ;  $d = 2\%$ )

how the maximum point on each curve shifts to the left as  $r_1$  increases. Table 2 provides a summary of the rate combinations examined and the associated timing of the largest interpolation error.

Clearly, the interpolation methodology is fraught with problems. For a given lower rate boundary  $r_1$  and a specified number of periods  $n$ , the interpolation error increases as the factor increment  $D$  widens. This conclusion is intuitive and implies that any individual using a factor table is best served by using as detailed a table as possible. For a given lower rate boundary and a specified distance between  $r_1$  and  $r_2$ , the interpolation error increases steadily, reaches a peak, and then decreases as  $n$  increases. For a given number of periods and a specified interest rate interval, the interpolation error decreases as the lower rate

**TABLE 2**  
 Sensitivity of the Timing of the Maximum Error to Interest Rates  
 When  $D = 2\%$

$r_1$	$r_2$	Maximum Error	Timing of Maximum Error ( $n$ )
22%	24%	0.0004620	9
24%	26%	0.0003869	9
26%	28%	0.0003301	8
28%	30%	0.0002832	7

boundary increases. Hence, in a low interest environment, there is more at stake when relying on interpolating from a time value table rather than computing the correct factors.

Clearly, verifying one's ARM amortization schedule is most accurate when precise time value mathematics is employed.

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### NOTES

1. Using a table, the 5% PV factor for 20 years rounded to four decimal places is .3769. The 10% factor for 20 years is .1486. The linearly interpolated factor for 7.5% is  $.3769 - .5(.3769 - .1486)$ , or .26275. Multiplying .26275 by \$500,000 results in a present value of \$131,375.

2. Equation (3) can be rewritten as Equation (3'):  $[n / (1 + r_1 + \alpha)^{n+1}] = [-100 / (1.01 + r_1)^n] + [100 / (1 + r_1)^n]$  or  $[n / (1 + r_1 + \alpha)^{n+1}] = k$ . Reciprocating each side and then multiplying both sides of the resulting equation by  $n$  gives (3'')  $(1 + r_1 + \alpha)^{n+1} = n/k$ . Taking the  $(n+1)$  root of each side results in (3''')  $1 + r_1 + \alpha = (n/k)^{1/(n+1)}$ . Now it is possible to solve for  $\alpha$ .

3. The variable  $k$  equals  $100 / (1.10)^{20} - 100 / (1.11)^{20}$ , or 2.460972090.

4.  $k = 100/(1.10) - 100/(1.11)$ , or .819.

5.  $k = 100/(1.50) - 100/(1.51)$ , or .441501110.

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