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A Simplified Approach to Measuring Bond Duration

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Because interest rates vary over time, the realized return on a fixed–income investment will depend on the price at which the instrument is ultimately liquidated and the rate at which interim cash flows are reinvested. This variation in realized return, known as interest–rate risk, should be addressed by both individual and institutional investors. Tools for measuring the impact and adjusting for the effects of interest rate changes on fixed–income instrument performance have long been available with duration and its companion adjustment factor, convexity. In this article, a simplified alternative to the traditional complex duration calculation is developed and demonstrated. Thus, anyone who can calculate a bond price can quickly estimate the interest rate risk associated with a bond as well as calculate the expected bond price change for a given change in market yield–to–maturity.

I. Introduction

Because interest rates vary over time, the realized return on a fixed–income investment will depend on the price at which the instrument is ultimately liquidated and the rate at which interim cash flows are reinvested. This variation in realized return, known as interest–rate risk, should be addressed by both individual and institutional investors. Tools for measuring the impact and adjusting for the effects of interest rate changes on fixed–income instrument performance have long been available with duration and its companion adjustment factor, convexity.[1] Although the rigorous calculation of these measures is intellectually appealing to academics, and easy to employ in a high–tech institutional setting, such calculations can be daunting to individual investors. In this paper a different, yet simple, approach for measuring duration and convexity is developed and demonstrated.

II. Prior Approaches to Computing Duration

Duration was originally derived as a superior way of capturing in a single measure the impact of the pattern of a bond's cash flows on its sensitivity to interest rate changes. Two bonds

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with the same maturity and yield–to–maturity (yield) but different periodic coupon cash flows can result in very different levels of terminal wealth for their owners. As market interest rates fluctuate following the acquisition of a bond, the rate at which interim cash flows are reinvested will vary. Also, if the bond is liquidated prior to maturity, the price received will depend on market interest rates at that time. The decrease (increase) in liquidation value seldom exactly offsets the increase (decrease) in reinvestment income that results from changing market interest rates. One way to avoid such terminal wealth risk is to hold only zero–coupon bonds that have a maturity equal to the planned holding period. However, because of a shortage of a range of "zeros" with desired risk/return relationships as well as certain tax implications associated with holding zeros, other strategies for addressing interest rate risk are necessary.

In a book written for the National Bureau of Economic Research in 1938, Frederick Macaulay first developed the concept of bond duration. Duration basically measures the weighted average amount of time it takes to receive the present value of cash flows from a bond.

$$Duration = [\sum tC_t / (1 + i)^t] / [\sum C_t / (1 + i)^t]$$

where C_t is the cash payment received at time t and i is the yield–to–maturity. Reilly (1989) shows that the duration of a 4% annual coupon bond that matures in 10 years with a yield–to–maturity of 8% is computed as follows:

(1) Year	*(2) Cash Flow*	*(3) PV @8%*	*(4) PV of CF*	*(5) PV as % of Price*	*(6) (1) × (5)*
1	40	.9259	37.04	.0506	.0506
2	40	.8573	34.29	.0469	.0938
.	..	.....	.....	.....	.....
9	40	.5002	20.02	.0274	.2466
10	1040	.4632	481.73	.6586	6.5850
			731.58	1.0000	8.1193

Duration = (6) ÷ (5) = 8.12 *Years*

As noted by Reilly (1989), generally:

1. Bond duration is less than term to maturity;
2. There is an inverse relationship between duration and coupon rate;
3. There is an inverse relationship between duration and yield–to–maturity; and
4. Bond price movements vary proportionally (linearly) with duration

The estimated bond price change resulting from a change in yield is:

$$\Delta P \cong -D_{mod} \times \Delta i \times P$$

where: ΔP = the change in bond price caused by Δi

P = the beginning price of the bond

$-D_{mod}$ = modified duration in years = duration / $(1+i/n)$ where n is the number of coupons per year

Δi = the change in yield–to–maturity

Thus, for the above bond, if the market yield–to–maturity declined by 75 basis points, the change in price would be:

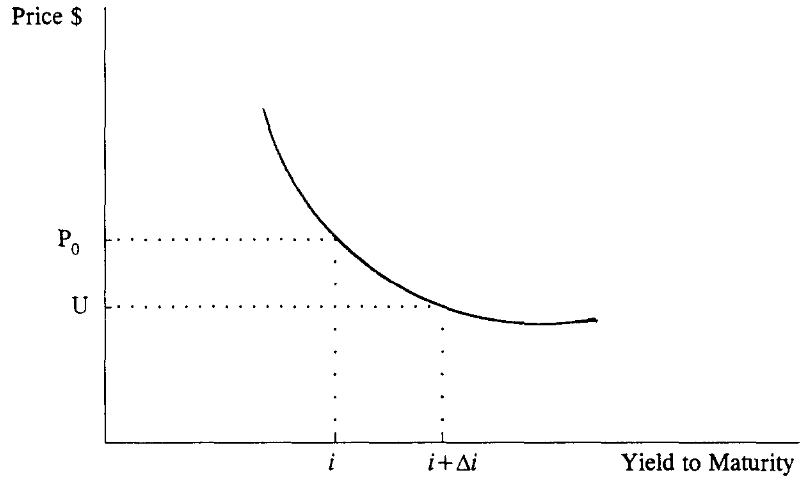

Figure 1. Price/Yield curve

$$\Delta P \cong (-8.12/1.08) \times (-.0075) \times (731.58) = 41.25$$

This indicates that if the market yield–to–maturity declines from 8% to 7.25%, the bond price would increase to:

$$\$731.58 + 41.25 = \$772.83$$

Duration can be viewed as the slope of a straight line tangent to the price/yield curve. Figure 1 presents the price/yield curve for the 4 percent coupon bond in the preceding example. Points along the curve represent prices for differing yields–to–maturity. If the yield changes from i to $i + \Delta$, the price of the bond would change from point P_0 to point U. The slope of the tangent line at point P_0 estimates the change in the bond price that would occur given a change in the yield. Because the curve is convex, the accuracy of the estimate of price change depends on that degree of convexity. A convexity correction factor is often used to adjust the price change estimated by using the bond duration. That factor is given by:

$$\begin{aligned} \text{Convexity} = CVX &= (\delta^2 P / \delta i^2) / P \\ &= [1/(1+i)^2] \times [\textstyle\sum C_t (t^2 + t) / (1+i)^t] \end{aligned}$$

The total change in price resulting from a change in yield is therefore:[2]

$$\begin{aligned} \Delta P &\cong \text{Duration change} + \text{Convexity change} \\ &= -D_{mod} \times \Delta i \times P + 0.5 \times P \times CVX \times (\Delta i)^2 \end{aligned}$$

From the above calculations, we observe that estimating the expected change in bond price for a given change in yield–to–maturity is quite complex and time consuming. However, duration has several valuable uses for investors, including individual investors.

First, duration serves as a measure of interest rate risk. The prices of bonds with greater durations are more sensitive to interest rate changes. Thus, knowledge of a bond's duration provides a useful benchmark for comparing the riskiness of alternative bonds. Second, investors with a specific investment planning horizon can select those bonds having durations most closely matching their planning horizon. Interest rate risk is minimized for bonds whose duration matches the planning horizon. Third, because the duration of a portfolio is the weighted average of the individual bond durations, investors can readily compute their total interest rate risk exposure from durations of the component bonds.

Numerous attempts have been made to simplify the estimation of Macaulay's duration measure. Jess Chua (1984) derived a closed-form formula that enables faster computation of duration. Chua's formula is:

$$D = \frac{C[\{(1+y)^{M+1} - (1+y) - yM\}/y^2(1+y)^M] + (F)(M)/(1+y)^M}{B}$$

where: D = duration in periods
M = maturity in periods
C = coupon in dollars per period
F = face value
y = yield to maturity per period
B = value of the bond at y

Caks, Lane, Greenleaf, and Joules (1985) (hereafter, CLGJ) utilized the linearity of duration to calculate duration for a coupon bond as a combination of the interest and maturity payments:

$$D = N - (C/Py)\,[N - (1+y)\,A_N]$$

where: D = duration in years
N = years to maturity
C = yearly coupon payment
P = market price of bond
y = yield to maturity in annual terms (all periods in CLGJ are annual periods)
A_N = present value of an annuity yielding y for N years

Moser and Lindley (1989) adapted the CLGJ formula to the case of multiple coupons per year with the following:

$$D = N - C/Py\,[N - (1 + y/k)]A_N / k]$$

where y/k is the rate per period and kN is the total number of periods. Benesh and Celec (1984) offered the following simplified formula with the alternative assumption that annual yields are arrived at by compounding periodic yields rather than the usual assumption that annual yields are computed by multiplying the periodic yield by the number of periods per year:

$$D = 1/m + c/k[(ACF_n - \mathrm{n}) + (n-1)] / [cACF_n + m]$$

where: $ACF_n = (m/k)[(1 + k/m)^n - 1]$
c = annual coupon rate
k = annualized yield to maturity from compounding periodic YTM

m = number of payments per year
n = total number of payments remaining until maturity

The choices for calculating duration boil down to the original procedure involving numerous weighted present value calculations or the above "simplified" formulas. All of the choices are daunting for the typical individual investor. To address this problem, the following section develops a simplified procedure for calculating duration that is "truly" simple.

III. Derivation of Simplified Duration Formula

Most individual investors have access to a financial calculator and are capable of calculating a bond price. If not, learning to do so is quite simple. The derivation of the simplified formula is based on Figure 2 and the definition of duration as the negative interest rate elasticity of the bond price. At the market yield–to–maturity, i, a bond would be priced at P_0. The modified duration for a bond is related to the slope of the tangent line at M. However, for small changes in i, the slope of that tangent line at point M is equal to the slope of the line drawn from point L to point M. Thus, it follows:

$$\text{Slope of tangent line} = (P_+ - P_-)\,/\,(2\Delta i) = (P_0 - P_-)\,/\,\Delta i = (P_+ - P_0)\,/\,\Delta i$$

and

$$\text{Slope of line from } L \text{ to } M = (M - L)\,/\,\Delta i$$

Therefore:

$$\frac{\lim\limits_{\Delta i \to 0} (L-M)/\Delta i}{P_0} = \frac{\lim\limits_{\Delta i \to 0} (M-U)/\Delta i}{P_0} = \frac{\lim\limits_{\Delta i \to 0} (P_- - P_+)/\Delta 2i)}{P_0} = \text{Modified duration}$$

Therefore:

$$D_{\text{mod}} = \frac{(L-M)/\Delta i}{P_0} \tag{1}$$

Also, convexity is proportional to the change in slope at point M:

$$\text{Convexity} = \frac{(\text{Slope}_\text{U} - \text{Slope}_\text{L})/\Delta i}{P_0} \tag{2}$$

and, from Figure 2, we see that:

$$\text{Convexity} = \frac{-(M-U)/\Delta i - [-(L-M)]/\Delta i}{P_0} = \frac{L+U-2M}{P_0 \Delta i} \tag{3}$$

Duration is reported in terms of a number of periods while the units of convexity are periods squared. With semi-annual coupons, duration would be in terms of half-years, while convexity is in terms of half–years squared. Thus, with semi-annual coupons, semi-annual duration is divided by 2 to arrive at duration in years while convexity is divided by 4.

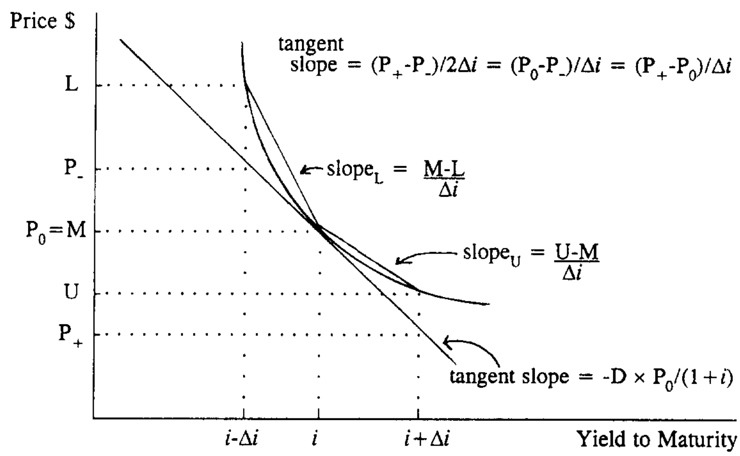

Figure 2. Price/Yield curve

It should be noted that the above calculations for duration and convexity hold for general conditions about the shape of the yield curve. In particular, Macaulay's duration assumes that the yield curve is flat and that changes in the level of interest rates result in parallel shifts in the yield curve. The analytic "simplifications" described by others also rely upon these restrictive assumptions. Our methodology is more general because we compute duration directly from the prices implied by any yield curve. Thus, our method is not only simpler but in general more accurate than previous methodologies.

A. Example of Duration and Convexity Calculations

The simplicity of these newly derived duration and convexity calculations is best demonstrated through an example. Consider an 18–year to maturity, 12% coupon bond, which is selling to yield 9% (Reilly, 1989, p. 432). Using the traditional procedures, we find the following:

$$\text{Price}^3 = 1265$$

$$\text{Modified duration} = 8.38 \text{ years}$$

$$\text{Convexity} = 107.7$$

$$\Delta i = 100 \text{ basis points}$$

Using the Macaulay Steps:[4]

Duration Calculation:

$$
\begin{array}{llll}
\Sigma\ [60 \times (1+.045)^{-1} \times 1 & + 60 \times (1+.045)^{-2} \times 2 & + 60 \times (1+.045)^{-3} \times 3 & + 60 \times (1+.045)^{-4} \times 4 \\
+ 60 \times (1+.045)^{-5} \times 5 & + 60 \times (1+.045)^{-6} \times 6 & + 60 \times (1+.045)^{-7} \times 7 & + 60 \times (1+.045)^{-8} \times 8 \\
+ \ldots\ldots\ldots & + \ldots\ldots\ldots & + \ldots\ldots\ldots & + \ldots\ldots\ldots \\
+ 60 \times (1+.045)^{-33} \times 33 & + 60 \times (1+.045)^{-34} \times 34 & + 60 \times (1+.045)^{-35} \times 35 & + 60 \times (1+.045)^{-36} \times 36 \\
+ 1000 \times (1+.045)^{-36} \times 36] \div 1265 \div 2 & & &
\end{array}
$$

$$\text{Duration} = 8.76$$
$$\text{Modified Duration} = 8.76/1.045 = 8.38$$

Convexity Calculation:

$$\begin{array}{llll}
\Sigma\ [60 \times (1+.045)^{-1} \times 2 & + 60 \times (1+.045)^{-2} \times 6 & + 60 \times (1+.045)^{-3} \times 12 & + 60 \times (1+.045)^{-4} \times 20 \\
+\ 60 \times (1+.045)^{-5} \times 30 & + 60 \times (1+.045)^{-6} \times 42 & + 60 \times (1+.045)^{-7} \times 60 & + 60\ \ddot{o}\ (1+.045)^{-8} \times 72 \\
+\ \ldots\ldots\ldots\ldots & + \ldots\ldots\ldots\ldots & + \ldots\ldots\ldots\ldots & + \ldots\ldots\ldots\ldots \\
+\ 60 \times (1+.045)^{-33} \times 1092 & + 60 \times (1{+}.045)^{-34} \times 1190 & + 60 \times (1+.045)^{-35} \times 1260 & + 60 \times (1+.045)^{-36} \times 1332 \\
+\ 1000 \times (1+.045)^{-36} \times 1332] \times [1/(1.045)^2] \div 1265 \div 4 & & &
\end{array}$$

$$\text{Convexity} = 107.70$$

For a 100 basis point decrease in yield, the predicted price change would be:

$$\text{Predicted price change} \cong \text{Duration change} + \text{Convexity change}$$

$$= -D_{mod} \times \Delta i \times Price + .5 \times Price \times Convexity \times \Delta i^2$$

$$= -8.38 \times (-.01) \times 1265 + .5 \times 1265 \times 107.70 \times (-.01)^2$$

$$= 112.82$$

$$\text{Predicted new price after } \Delta i = 1265 + 112.82 = 1377.82$$

The actual bond price with a yield to maturity of 8% would be $1,378.17. Thus, using the usual measures of duration and convexity to predict the new bond price results in an error of 35 cents.

B. Using the Heck/Zivney/Modani (HZM) Steps

To compute duration and convexity using our simplified procedures, calculate bond prices for a small change in *i* above and below the current semi-annual yield–to–maturity for the bond. Because of rounding in the Reilly example, our calculations will differ slightly.[5] For example:

$$\text{At } i = 4.5\%, \quad Price\ (M) = 1264.99$$
$$\text{At } i = 4.49\%, \quad Price\ (L) = 1267.11$$
$$\text{At } i = 4.51\%, \quad Price\ (U) = 1262.87$$

Thus:

$$\Delta i = .0450 - .0449 = .0001$$

$$D_{mod} = [(L-M)/M] \div \Delta i \div 2$$
$$= [(1264.99 - 1267.11)/1264.99] \div .0001 \div 2$$
$$= 8.39$$

The percentage change in price is divided by 2 in the equation for modified duration to reflect semi-annual compounding. The formula for convexity involves dividing by 2^2, or 4, to reflect the semi–annual compounding:

$$Convexity \cong (L+U-2\text{M}) \div \Delta i^2 \div M \div 4$$

$$= (1267.11 + 1262.87 - 2 \times 1264.99) \div .0001^2 \div 1264.99 \div 4$$

$$= 107.70$$

The Δi used to compute duration and convexity (generally a very small number) need not be the same as the Δi used to estimate price changes. Therefore, the predicted change in the price for a 100 basis point change in i (50 basis points each six-month compounding period) results in the following predicted price:

$$\textit{Price change} \cong \textit{Duration change} + \textit{Convexity change}$$

$$= -D_{Mod} \times \Delta i \times \text{Price} + 0.5 \times \text{Price} \times \text{Convexity} \times \Delta i^2$$

$$= -8.39 \times (-.01) \times 1264.99 + 0.5 \times 1264.99 \times 107.70 \times (-.01)^2$$

$$= 112.94$$

$$\text{Predicted price after } \Delta i = 1264.99 + 112.94 = 1377.93$$

The predicted price of \$1,377.93 is only 24 cents from the actual price of \$1,378.17 and, hence, more accurate than the traditional computations, which resulted in an error of 34 cents.

The HZM estimated bond price change calculations involve seven separate mathematical steps. To summarize those seven steps, calculate:

1. Bond price at current yield–to–maturity
2. Bond price for a small increase from current yield-to-maturity
3. Bond price for a small decrease from current yield–to–maturity
4. Change in yield–to–maturity (step 1 yield minus step 2 yield)
5. Modified duration
6. Convexity
7. Expected price change as sum of duration change and convexity change

C. Evaluating the Performance of HZM

The previous section demonstrated the ease with which the expected bond price change can be estimated using the HZM method. In the above example, the HZM and traditional Macaulay results were quite similar. The obvious question the example raises is how consistent is the accuracy of the HZM method. To address this question, a simulation was performed.

Table 1 presents some of the results of a simulation that generated predicted bond price changes for both the HZM and traditional methods under varying scenarios. Using a 10 percent coupon bond with maturities ranging from five to 30 years, predicted bond price changes were calculated for yield–to–maturity changes ranging from one to 100 basis points and for HZM Δi ranging from .01 to 100 basis points.

Results presented include yield–to–maturities of 3% to 18% in three-percent intervals, for the HZM method using Δi of .01 and one basis point as well as the traditional Macaulay method. It can be seen in each cell of Table 1 that the prediction error associated with the HZM method decreases with smaller Δi; also, for small Δi, HZM and Macaulay yield nearly identical results. The last row and column of cells compares the average error under each

TABLE 1
Actual Price Change Minus Predicted Price Change Using 100 Basis Point Change in YTM

Maturity	*Method*	*Yield–to–Maturity on 10% Coupon Bond* 3%	6%	9%	12%	15%	18%	*Average Error*
5 years	HZM Δ = 1.0	0.26*	0.21	0.18	0.15	0.12	0.10	0.174
	HZM Δ = .01	0.20	0.17	0.14	0.11	0.09	0.08	0.136
	Macaulay	0.20	0.17	0.14	0.11	0.09	0.08	0.136
10 years	HZM Δ = 1.0	1.56	1.13	0.83	0.61	0.45	0.33	0.821
	HZM Δ = 0.1	1.35	0.98	0.71	0.52	0.38	0.28	0.708
	Macaulay	1.35	0.98	0.71	0.52	0.38	0.28	0.707
15 years	HZM Δ = 1.0	4.54	2.90	1.87	1.22	0.80	0.53	1.981
	HZM Δ = .01	4.09	2.60	1.67	1.08	0.71	0.47	1.775
	Macaulay	4.08	2.60	1.67	1.08	0.71	0.47	1.773
20 years	HZM Δ = 1.0	9.64	5.43	3.11	1.82	1.09	0.66	3.629
	HZM Δ = .01	8.85	4.97	2.84	1.65	0.98	0.59	3.318
	Macaulay	8.84	4.96	2.84	1.65	0.98	0.59	3.315
25 years	HZM Δ = 1.0	17.08	8.54	4.39	2.33	1.28	0.73	5.728
	HZM Δ = .01	15.88	7.92	4.05	2.14	1.16	0.66	5.307
	Macaulay	15.87	7.91	4.05	2.14	1.16	0.66	5.302
30 years	HZM Δ = 1.0	26.95	12.02	5.59	2.72	1.39	0.75	8.242
	HZM Δ = .01	25.29	11.24	5.20	2.51	1.28	0.68	7.704
	Macaulay	25.27	11.23	5.20	2.51	1.28	0.68	7.699
Average Error	HZM Δ = 1.0	10.007	5.043	2.665	1.478	0.859	0.523	3.429
	HZM Δ = .01	9.281	4.649	2.439	1.341	0.773	0.465	3.158
	Macaulay	9.273	4.645	2.437	1.340	0.772	0.465	3.155

Note: *For a 10% coupon bond maturing in 5 years, with a yield–to–maturity of 3%, and experiencing a 100 basis point change in yield, the difference between the actual change in price and the price change predicted using the HZM model (with a $\Delta = 1.0$) is 26 cents. $\Delta\, i = 1.0$ means that a one basis point change was used to compute duration and convexity with the HZM method while $\Delta\, i = .01$ means that a one–hundredth basis point change was used to compute duration and convexity.

scenario. The results suggest that not only is the HZM method easier to employ, but it yields equally accurate results when computed using small Δi.

IV. Summary

Although the derivation of the HZM simplified formulas for duration and convexity calculations is somewhat complex, the resulting duration and convexity formulas are much easier for an individual investor to use. The HZM simplified method provides estimates of price changes which differ from those produced by traditional duration measures by less than one penny on average. Anyone who can calculate a bond price (actually, three different bond prices) can now quickly estimate the interest rate risk associated with a bond as well as calculate the expected bond price change for a given change in market yield–to–maturity. Also, unlike previously derived closed–form formulas, our formulas can be used in the general cases of sloping yield curves and nonparallel shifts in yield curves.

NOTES

1. For a detailed discussion of convexity, see Riley (1989, pp. 427-431), Francis (1991, pp.407-411), Kolb (1992, pp. 245-251), and Maginn and Tuttle (1990, pp. 68-69).

2. The expression is the first two terms of a Taylor expansion series. An explanation of the origins of the .5 and $(\Delta i)^2$ terms can be found in Reilly (p. 430) and Dunetz and Mahony (1988, p. 57).

3. Reilly uses $1,265, although the actual price would be $1,264.99. The $1,265 figure is used here to maintain comparibility with the Reilly text example.

4. For semi-annual coupon bonds, the duration calculation must be divided by 2. Because convexity is the second derivative, the $\Delta i/2$ term is squared, yielding $\Delta i^2/4$. Thus, the convexity calculation must be divided by 4 for semi-annual coupon bonds.

5. Answers are shown to the nearest cent. All calculations are performed using the memory functions of the calculator to maximize the accuracy for small changes in *i*.

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