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An Application of Fuzzy Set Theory to the Individual Investor Problem

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This study reviews the problem of the individual investor and applies to it a methodology based on fuzzy sets and the theory of possibility. The investment decision is characterized by uncertainty, imprecision and complexity, which lessen the effectiveness of conventional calculus and probability tools. In contrast, fuzzy set theory and its modeling language provide objects of analysis and algebra that are well suited to this problem. New concepts such as "fuzzy portfolio weights" are introduced. The result of our research is a qualitative, general, and practical model for individual investors' decision making, which is based on Smith's (1974) asset-mix model.

I. INTRODUCTION

Financial planning and investing have not become any easier. The stakes seem higher today because "normal" standards of living require more funds than ever. There are uncertainties regarding social security and the health system, and insecurity regarding job permanency. Therefore, individuals must assume greater responsibility for their future.

With respect to previous research, the first model developed to assist investors in security selection is owed to Markowitz (1959, 1991) and is part of what is known as portfolio management. Considerations regarding transaction costs and minimum budgets needed to achieve diversification made portfolio management most useful to large or institutional investors. The literature on limited diversification tries to adapt portfolio theory to small investors as in, for example, Jacob (1974) and Brennan (1975).

In spite of their practicality, interest devoted to limited diversification models waned with the arrival of mutual funds. Shortcomings of the methodology employed also exacted a high price in terms of analytical and computational cost, as in the model by Pogue (1970), and precluded further advancement. State-preference models were developed to cope with uncertainty regarding the environment where investment decisions take place, Hirshleifer (1965), Myers (1968), and Kraus and Litzenberger (1975). State-preference models cannot be used in actual investment decisions because they are too precise, use utility functions,

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and require perfect knowledge (including returns on contingent securities that are not risky themselves). Lastly, they require the user to input the very key item the model should provide: the optimal weights for the portfolio.

Zadeh (1965) is the seminal study on fuzzy sets. The inability of the conventional mathematical apparatus to assist in explaining systems that are characterized by uncertainty, complexity, and vague or imprecise concepts and variables motivated this researcher. Financial systems have these qualities, perhaps because they are ever-changing. These systems call for a different methodology and different research goals, as stated in Zadeh's principle of incompatibility: "As the complexity of the system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics" (Zimmermann, 1991, p. 3).

Ever since Zadeh's (1965) publication, there has been an explosion of papers, monographs and applications, mostly in the areas of information theory, computer sciences, and engineering. Fuzzy set-based research has provided concepts, objects and frameworks of analysis for including imprecision, complexity and uncertainty, and it is the only formal methodology which permits linguistic variables and approximate reasoning to be used (Zadeh, 1979, 1975a, 1975b, 1975c, 1973).

Why aren't there more contributions of fuzzy set theory in economics and finance? Kickert notes that applications in fuzzy sets often are designed to highlight the abilities of fuzzy methodology rather than first selecting the critical problem and then applying the methodology. Kickert also indicates the preference of researchers for what is known as "fuzzyfying" a problem, which is extending standard models into fuzzy versions (Kickert, 1978, especially pp. 2, 22).

What is gained in the application of fuzzy sets to the individual investor problem? In brief, fuzzy sets provide the algebraic tools and operations to permit the asset allocation problem to be solved in a logically proper manner. One cannot apply arithmetic and calculus to variables that are only known approximately. Moreover, the methodology presented is also closer to the type of reasoning which is actually used by investors themselves.

II. SMITH'S QUALITATIVE MODEL

Smith (1974) indicates that his main motivation in developing his model is to study the asset allocation problem, which was probably the most important problem faced by individual investors, and yet had received little attention in the literature. To develop his model, Smith breaks away from convention and develops a qualitative, but practical,

Inf. about assets ---------> Investment attributes <--------- Inf. individual

↓

Multiasset portfolio decisions

Figure **1.**
The Elements of The Investment Decision.

model. Above all, he concentrates on the essential elements of the problem: the types of assets, the most salient properties of these assets, and the characteristics of investors as Figure 1 shows.

Smith uses four asset categories, (a) savings accounts, (b) corporate bonds, (c) common stock, and (d) real estate, with four asset properties, (a) liquidity (L), (b) income (I), (c) appreciation (A), and (d) safety (S). He also classifies investors by age into five groups. These variables are divided into two matrices: one relating different assets to their properties, and another relating asset properties to different age groups. The coefficients in each matrix are derived from experience and empirical observation, as well as from normative prescriptions regarding investors' holdings at different stages in their life cycle.

Using matrix operations, Smith obtains a weighted suitability index, which can be used to allocate wealth. Reading the product matrix by columns, suitable allocations for each age group can be seen in Table 1. For example, the second column in the product matrix shows recommended allocations for the 35-year-old: 19%, 16.5%, 32%, 32.5% in a savings account, corporate bonds, common stocks, and real estate, respectively (these numbers appear boldfaced). Reading the matrix by rows, it is observed how holdings of each asset may evolve over time in each investor's portfolio. The distribution of assets over time is consistent with investment advice based on the life-cycle hypothesis.

Smith indicated his model had the following limitations: (a) it needed clarification concerning how to obtain probability assignments for matrices; (b) it had no sense of optimality; (c) it did not include issues bearing on the management of security holdings over time, for example, transaction costs; (d) it held suitability measures for asset types constant for all age levels.

Smith's departure from conventional modeling must be appreciated. It represents a "back to basics" approach and, to a certain extent, rescues qualitative analysis from oblivion. The following sections will show the fertility of Smith's departure when the adequate tools and framework are employed.

TABLE 1
Smith's Model

Matrix 1. Assets-properties

	L	I	A	S
Sav.	0.6	0.3	0.0	0.4
Cb	0.2	0.5	0.1	0.3
Cs	0.2	0.1	0.4	0.2
Re	0.0	0.1	0.5	0.1

Matrix 2. Properties-Ages

	25	35	45	55	65
L	0.3	0.15	0.1	0.1	0.0
I	0.1	0.00	0.1	0.2	0.6
A	0.0	0.60	0.7	0.5	0.0
S	0.6	0.25	0.1	0.2	0.4

Product matrix:

Assets\Age:	25	**35**	45	55	65
Savings	0.45	**0.190**	0.13	0.20	0.34
Cbonds	0.29	**0.165**	0.17	0.23	0.42
Cstocks	0.19	**0.320**	0.33	0.28	0.14
Restate	0.07	**0.325**	0.37	0.29	0.10
Totals	1.00	**1.00**	1.00	1.00	1.00

III. RELATIONAL EQUATIONS-POSSIBILISTIC MODELING

"Relational equations-possibilistic modeling" (REP) is the modeling language of fuzzy sets and uses algebra instead of calculus. When problems and variables cannot precisely be reduced to numbers, algebra tools and the concept of a partial ordering still choices to be made, for example, by comparing and ranking the alternatives. In this context, REP represents a problem-determined bridge between exactness and inexactness, which also integrates qualitative and quantitative information.

REP is an example of logical analysis. It begins by defining a problem and then identifies relevant elements (categories), relationships, and variables (Langer, 1967). In the algebra of sets, the problems and the elements of analysis are known as the universe of discourse, and the categories are referred to as classes, the group of objects that share something in common, and sets. Sets are composed of elements, which roughly correspond to the variables of calculus.

A. Fuzzy Sets Concepts

The area of fuzzy sets can be regarded both as a formal theory and as the modeling language aforementioned (Zimmermann, 1991, p. 6). The formal theory is based on algebra of fuzzy sets, which is, in turn, built upon the concept of a fuzzy set. A *fuzzy set* is a set in the universe of discourse defined by a graded membership function. For example, the sets "young" (Y) and "old" (O) in the universe "active investors" (I) are fuzzy sets. These sets can be described as $Y = \{x, \mu y\ (x)\} = \{(10, 0.6), (20, 0.8), (30, 1), (40, 0.8), (50, 0.6)\}$, and $O = \{x, \mu o\ (x)\} = \{(40, 0.6), (50, 0.8), (60, 1), (70, 0.8), (80, 0.6)\}$, and $Y, O \subset I$.

Each set is described by an ordered pair: the first element indicates the age, the second element the grade of membership to the set, $\mu y(x)$ is the *membership function*. For example, the pair (40, 0.8) in the subset "young" can be read as, a 40-year-old is only a 80% member of this group. The same 40-year-old would be a 60% member of the set "old." The best representatives of the (fuzzy) adjectives "young" and "old" are 30 and 60 years, respectively. Note that there is some grading, as well as an overlap, between a fuzzy set and its complement. This is different from the sets employed in Boolean algebra, in which an investor would belong to either the "young" or the "not young" set, but never to both (law of the excluded middle). The selection of elements above a given level of membership (say $\mu(x) > 0.6$) is called an *alpha-cut*. Memberships can be normalized to 1 and can be constructed to be symmetric, but these are nonessential properties. The values for the membership function used in the set young (Y), above, have a possibility meaning (Zadeh, 1978). These values have a meaning of compatibility, which can be of a linguistic, or economic nature (a savings account is 60% liquid, common stock is 20% safe, at age 65 income and safety replace appreciation as the most important attribute of securities, etc.), as in Smith's model.

Y, O, and I are examples of fuzzy sets, with fuzzy memberships and a fuzzy universe definition. The same types of fuzziness and sets appear when assets are grouped into classes (fixed income versus variable income) or into subclasses (growth versus income stocks) and in the labeling of economic situations (recessions, transitions, and periods of prosperity). Note that no amount of additional data can eliminate the uncertainty regarding these sets.

These constructs are used in financial counseling because they are useful. For example, a brokerage service may want to learn how many of their clients are really "active investors." The adverb "really" is another fuzzy qualifier that is useful to communicate. Fuzzy set theory has been developed to provide support to everyday reasoning. Rather than purifying concepts so that they conform to the Boolean framework, fuzzy set theory conforms itself to individuals.

Major operations are defined in fuzzy sets such as the Cartesian product, the algebraic product, etc. and in set-theoretic operations such as union and intersection, the max-operator (logical "or") and the min-operator (logical "and"). For example, if there is the perception that something is missing between "young" and "old," the intersection of these two fuzzy sets would provide us with a third set that could be called "mature" and which fills the logical void, that is, "mature" = $Y \cap O = M = \{(40, 0.6), (50, 0.6)\}$.

B. Fuzzy Relationships and Their Composition

Relationships are generalizations of functions. Functions assign a unique object (value of the function) for each object in their domain (argument). Functions pair values of variables, while relationships pair variables or sets themselves. Much of the scientific effort is aimed at uncovering and discovering such relationships because they lend a degree of permanency to our knowledge.

A *fuzzy relationship* is a relationship defined over fuzzy sets. With discrete supports, that is, numbers, matrices express relationships. Strength is the most important characteristic of a relationship, and it is represented by the elements in the relational matrix.

In order to study the composition of fuzzy relationships, let us first establish two fuzzy relationships, for example R1 = "security Xi is a close substitute for Xj" and R2 = "performance in economic cycles." Let R2 have the following membership function defined over the elements of the class ECONOMY = {Prosperity-Z1, Transition-Z2, Recession-Z3}, which is represented in Table 2.

The *composition of fuzzy relationships* amounts to compiling or merging the information from each of the relationships being composed. The max-min composition is frequently used to compose fuzzy relationships and is also a fuzzy set defined by a membership function:

$$R1 \bullet R2 = \{[(x,z), \max\{\min\{\mu R1\,(x,z), \mu R2\,(x,z)\}\}] \mid x \in X, z \in Z\}.$$

TABLE 2
First Example of Composition

	R1					R2		
	X1	X2	X3			Z1	Z2	Z3
X1	1.0	0.8	0.4		X1	1.0	0.6	0.4
X2	0.8	1.0	0.2		X2	0.6	0.8	0.4
X3	0.4	0.2	1.0		X3	0.2	0.6	1.0
			1.0	0.8	0.4			
	R3 = {R1 • R2} =		0.8	0.8	0.4			
			0.4	0.6	1.0			

The composition of the two relationships into R3, which is shown in Table 2, could be interpreted as securities that are close substitutes and perform well in economic cycles.

The membership functions for the composition are calculated in a manner that is similar to matrix multiplication. Merge the elements in each row in the first matrix with those in the first column of the second matrix, which results in three ordered pairs, {(1, 1), (0.8, 0.6), (0.4, 0.2)}, and select the minimum values in each of these pairs. This results in a set of three elements: {1, 0.6, 0.2}. Select the maximum element from these values, 1, which will be the first element in the first column and row in the R3 = {R1 • R2} set, and can be written as T = R3(X, Z) = {R1(X, Y) • R2(Y, Z)}.

Dubois and Prade (1980, p. 74) provide a very clear interpretation of the *max-min rule*. The membership functions in each relational matrix can be regarded as links in the chain of reasoning. This chain is only as strong as its weakest element (*min* evaluation), while the strength of the relationship between each of the sets (say, X and Y) is that of their strongest chain (*max* evaluation). The max-min composition in this study is based on partial orderings called Browerian lattices, which are a further refinement of Boolean lattices, Rutherford (1966) and Abbot (1969). Lattice algebra is a special type of algebra which emphasizes ordering.

Relationships can be established among several symbolic objects, X, Y, and Z, which provides a wider scope of analysis and brings us to the more familiar territory of simultaneous equations systems. *Fuzzy relational equations* express causality implied by the relationship among X, Y, and Z. Assume, for example, that Yi's represent the fuzzy class of "AGE" = { Young-Y1, Mature-Y2, Senior-Y3}. The left-hand-side matrix, fuzzy relation R1, could express "adequacy of security Xi for group age Yi." And the composition of these two relationships, R3 = R1 • R2, could be interpreted as "adequacy of securities Xi's for age groups Yi's through economic cycles Zi's," which is shown in the upper section (example 2) of Table 3.

The information in the composition matrix can be read as follows: "Security one is the most adequate if we expect prosperity. Securities one and two are comparable, but better than security three, in transition periods. Security three is the most appropriate if a recession is expected." These propositions would be adequate for any of the three age groups.

TABLE 3
Further Examples of Compositions

Example 2.

	Y1	Y2	Y3		Z1	Z2	Z3
X1	1.0	0.8	0.4	Y1	1.0	0.6	0.4
X2	0.8	1.0	0.2	Y2	0.6	0.8	0.4
X3	0.4	0.2	1.0	Y3	0.2	0.6	1.0

Example 3.

	X1	X2	X3		Z1	Z2	Z3
Y1	1.0	0.8	0.4	X1	1.0	0.6	0.4
Y2	0.8	1.0	0.2	X2	0.6	0.8	0.4
Y3	0.4	0.2	1.0	X3	0.2	0.6	1.0

Assume now that the left-hand-side matrix is transposed. The left-hand-side matrix (R1) could now take the meaning of "holding of securities Xi's by ages Yi's," the right-hand-side matrix (R2), could mean "performance of securities Xi's in economic cycles Zi's," and the composition R3(Y, Z) = {R1(Y, X) • R2(X, Z)} could be interpreted as "performance by age groups across economic cycles," as in example 3 in Table 3.

A hierarchy of classes can be established and they can be composed successively into two main overarching classes, which resemble the use of intermediate variables. For example, suppose that learning about classes (M, Q) requires that a knowledge tree be built, in which information is processed at different levels of analysis or layers. One may start with intermediate relations R1's in level one, compose then into R2's at level two, and finally obtain the integrated, all-encompassing R3 target relationship. Table 4 shows a diagram of this process.

The following three considerations are critical in modeling by relational equations: (a) to determine a relatively sufficient universe of discourse, (b) to determine what is known, and what is to be known, and (c) to establish proper causality.

The first element is similar to having conformable matrices. Each column in the relational matrix R1(X,Y) represents a variable (set). A corresponding variable (Yi), or link, in R2(Y,Z) is needed if it is to be composed with relation R2(Y,Z).

With respect to the second element, examples 2 and 3 show that relational compositions have different effects with respect to which variable becomes implicit in the informational structure formed by X, Y, and Z. In example two, the Y set, age groups becomes implicit, and the result is a relationship in terms of X and Z, as in the case of simultaneous equations systems. In example 3, it is X which becomes implicit, as the following compact notation shows: R3(X,Z) = {R1(X,Y) • R2(Y,Z)} and R3(Y,Z) = {R1(Y,X) • R2(X,Z)} for examples 1 and 2, respectively, shown in Table 4.

The third element is significant because much of the validity of knowledge rests on the notion of causality. Two variables may have a very large correlation coefficient but that does not mean they are related in any way. A regression of the set of variables X = {X1 = x1, ..., xt, X2, X3} on the set Y = {Y = y1, ..., yt} may have a large R-squared value and yet the "X" variables-set may have nothing to do with the variable "Y". What is critical in scientific work (econometric, biomedical, etc.) is the reasoning that justifies using set X as a *basis* for Y. Recall that a relationship is a set of ordered pairs, by definition, and that this "ordering" is meant to carry causality content. In other words, *fuzzy relationships* may or may not imply causality, but *causality* is required in *fuzzy relational equations*, as in any other type of equation.

Relational equations modeling is not more difficult, and is equally (or more) rigorous from a formal viewpoint, than conventional modeling. The relational equations-possibilis-

TABLE 4
Relationships and Multilayered Compositions

Level 1:	R11(M,N)		R12(N,O)		R13(O, P)		R14(P, Q)
Level 2:		R21(M, O)				R22(O, Q)	
Level 3:				R31(M, Q)			

tic approach is simply different from the calculus-plus-probability "gold exchange," and it stresses different aspects of the decision-making process.

Readers who feel reluctant to leave the "calculus + probability" safe harbor of numerical exactness may ponder Shackle's observation: "(P)robability is a model of thought, not a character of the natural world" (1972, p. 385). As usual, logical reasoning must weave variables together if the resulting information tapestry is to make any sense and have any meaning.

III. APPLICATION TO THE INDIVIDUAL INVESTOR PROBLEM

Reformulating Smith's model into the relational equations-possibilistic setting is easy. It amounts to composing information on two relationships: (a) between securities and their associated properties, and (b) between properties of the securities and age groups. The latter one is the main "class-ification" to identify investors in Smith's model. Note that security properties are the commonality, or link, that permits the composition of the two relationships.

Smith's numbers are used in the relational compositions to facilitate comparisons. There are several critical differences:

1. *The modeling framework now is different.* The REP framework is of a wider scope than the purely numerical one, given the use of fuzzy sets (classes and variables) and membership functions endowed with linguistic and possibilistic content and truth values.
2. *The receptacles of information (relational matrices and their coefficients) are also different.* The matrices now are relational matrices, which provide information about approximate relationships among variables. The coefficients of these matrices inform us about approximate adequacy, possibility, or compatibility between the sets in each of the classes. Recall that the analysis must be endowed with flexibility to enable it to handle uncertainty, imprecision, and complexity. This flexibility is appropriate and necessary in each of the areas of the problem. For example, real estate may not appear as such a good investment when evaluated strictly in terms of returns, but all individual investors need a place to live.
3. *Causation and inference rules are different than in Smith's model.* REP modeling requires only conjectures or approximate causation and approximate inference, while Smith's methodology presumes exact causation and exact inference.

Table 5 presents the composition of the two previous fuzzy relationships and a comparison of results with Smith's, respectively. Several points regarding the results need comment.

First, the results differ from Smith's, even though the ranking of securities in each portfolio is similar. The results obtained by matrix multiplication correspond to the arithmetic product rule, while the (fuzzy) numbers in the first composition have been obtained by the max-min composition rule. There are some cases in which they may coincide; in general they will not. In some cases, the max-min rule may resemble a rounding or averaging of the arithmetic rules, but the differences are more profound than that.

TABLE 5
Individual Investor's Problem

R3 = R1 • R2 = R1 (Assets, *A-properties*) • R2(*A-properties*, Age) = R3(Assets, Age)

	L	I	A	S		25	35	45	55	65
Sav.	0.6	0.3	0.0	0.4	*L*	0.3	0.15	0.1	0.1	0.0
Cb	0.2	0.5	0.1	0.3	*I*	0.1	0.0	0.1	0.2	0.6
Cs	0.2	0.1	0.4	0.2	*A*	0.0	0.6	0.7	0.5	0.0
Re	0.0	0.1	0.5	0.1	*S*	0.6	0.25	0.1	0.2	0.4

Max-min rule:

Assets\Age:	25	35	45	55	65
Savings	0.40	0.25	0.10	0.20	0.40
Cbonds	0.30	0.25	0.10	0.20	0.50
Cstocks	0.20	0.40	0.40	0.40	0.20
Restate	0.10	0.50	0.50	0.50	0.10
Totals	1.00	1.40	1.10	1.30	1.20

Max-min rule (normalized):

Assets\Age:	25	35	45	55	65
Savings	0.40	0.18	0.09	0.15	0.33
Cbonds	0.30	0.18	0.09	0.15	0.42
Cstocks	0.20	0.29	0.36	0.31	0.17
Restate	0.10	0.36	0.45	0.38	0.08
Totals	1.00	1.00	1.00	1.00	1.00

Arithmetic product rule, Smith (1974):

Assets\Age:	25	35	45	55	65
Savings	0.45	0.19	0.13	0.20	0.34
Cbonds	0.29	0.165	0.17	0.23	0.42
Cstocks	0.19	0.32	0.33	0.28	0.14
Restate	0.07	0.325	0.37	0.29	0.10
Totals	1.00	1.00	1.00	1.00	1.00

The critical fact is that the arithmetic product rule is only valid with non-fuzzy, or precise (Boolean) "crisp" sets. When the classes, sets, indicators and inferences are not precise, the logical composition rules for fuzzy sets must be used. The individual investor's problem simply does not have the precision required by the arithmetic product rule. The rankings are similar because the "ordering" induced by the supporting lattice (a partially ordered set) is approximately the same. That is, the most heavily weighted securities in the arithmetic product rule (Smith's case) are also the most heavily weighted securities in the relational composition.

Second, the optimal weights in the relational composition do not add up to one. This is a by-product of the approximate composition rule and the use of fuzzy sets, which results in *fuzzy portfolio weights*. It is both easy and instructive to normalize fuzzy weights—divide each weight by its column cumulant—so that they can be compared to those obtained from the arithmetic product rule.

The arithmetic product weights look impressively informative. For individuals of ages 35, 45, and 55, the max-min composition does not seem to be able to determine security holdings as precisely as the product rule. However, there is a problem with the "precise" weights of the arithmetic rule: *there is not enough knowledge to guarantee their accuracy*. "Fuzzy weights" do not give such a false security.

Note that there is no limit to how many fuzzy relationships can be composed as long as they are logically integrated. That is, the lists of assets, investors' characteristics, etc. can be expanded, but the models introduced suffice to present the new methodology. Moreover, each user of this methodology is expected to develop custom-made or situation-specific models, to which the REP methodology lends itself well.

The REP model addresses some of the limitations of Smith's model. It provides alternative ways to obtain coefficients and assignments. A sense of optimizing is implicit in the model via possibility theory, since the model provides the best course of action given the information available. Suitability measures for asset types can be changed for different ages, and additional fuzzy relationships can be included. In sum, the REP model achieves what Smith set out to do in a more satisfactory in a logically proper manner.

REP modeling is inductive in that the integration of the premises permits insights to be gained that are not evident in the premises themselves. It opens a door to approximate reasoning, which is "a type of reasoning which is neither very exact, nor very inexact, ... and provides a way of dealing with problems which are too complex for precise solution" (cf. Zadeh, 1975d, p. 2).

IV. CONCLUDING COMMENTS

This study applied to the case of the individual investor's problem a new methodology based on fuzzy sets. In this problem, the classes, variables, indicators, causality, and inferences are at best approximate, which calls for solutions and reasoning which will also be approximate at best. It also requires a type of flexibility and imprecision that can not be obtained in the conventional (calculus plus probability) framework.

The relational equations-possibilistic modeling is only part of the ongoing research on "approximate reasoning," which shows considerable potential for providing solutions to investors. Research in approximate reasoning points toward the possibility of building an evolving model or automaton soon (Klir & Yuan, 1995, p. 349) which could manage investments. Before this "automatic pilot" arrives, relationships between investment professionals and their clients are likely to continue to be very important. The methodology presented can contribute to enhancing that relationship, since it properly incorporates the formality requirements of investment professionals with clients' needs and reasoning in a flexible framework that one can both relate to and understand.

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