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Formulating retirement targets and the impact of time horizon on asset allocation

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Abstract

This paper looks at standard retirement targets such as "70@65," meaning 70% income replacement at age 65, and reconsiders them in a probabilistic setting. The paper uses a chance constrained programming model, supplemented with Monte Carlo simulation, to extend the target to "70% of 70@65" meaning a 70% chance of meeting the target. One implication of the paper is that asset mix is a function of the investment horizon. This conflicts with the constant portfolio result of Samuelson et al. but supports the standard "your age in bonds" rule of thumb of financial planning professionals.

Jel Classification: D14; G23

Keywords: Retirement; Financial planning; Asset allocation

1. Introduction

Standard goals in personal financial planning are often represented as "70 at 65"; meaning that the retirement target is 70% of preretirement income at age 65.[1] A financial planner will then solve any problems this target creates by, for example, altering the individual's savings rate, their retirement date, or their asset mix until an acceptable strategy has been developed. This paper is concerned primarily with the latter choice, the optimal asset-mix, and analyzes how this choice affects an individual's retirement target.

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There are two key problems to be solved: what is the appropriate asset mix at a *specific* point in time; and how should it *vary* with the investment horizon and by implication the individual's age.[2] Conceptually, we know that the asset mix is related to the individual's risk aversion. However, the impact of age and time horizon remains contentious. Samuelson (1994) indicated that "it is an exact theorem that investment horizons have no effect on your portfolio proportions." However, the standard "rule of thumb" among financial planners is "your age in bonds" rule; that is at age 30, 30% should be in bonds and 70% in equities, while at age 60, 60% should be in bonds and 40% in equities.

This article re-examines these issues in the context of a chance constrained programming model and reaches two main conclusions. First, a standard retirement target like "70 @ 65" should be replaced by something like "70% of 70 @ 65," meaning a 70% chance of getting 70% of your preretirement income at age 65. Second, in the context of a chance constrained programming model, age is important for the asset mix, and standard rules of thumb, while somewhat conservative, have a role to play.

Section 2 discusses the broad nature of the retirement problem and introduces the concept of a probability constraint as one way of formulating the uncertainty of meeting retirement targets. Section 3 discusses how a probability constraint interacts with asset allocation for a fixed amount of investment. Section 4 uses the probability constraint to determine the minimum amount of investment necessary to meet the retirement objective and Section 5 adds some conclusions.

2. The approach

The optimal portfolio problem has been studied extensively in two different literatures: a theoretical literature and an applied literature. The theoretical literature has as its foundation the Samuelson (1969) and Merton (1969) multiperiod consumption-investment problem. In this literature, an individual works back from their termination (death) date and solves all of the prior investment and consumption decisions by maximizing the expected utility of per period consumption. This extension of the life cycle approach to savings solves the retirement problem simultaneously with the consumption problem. It is an intimidating literature that solves the portfolio problem dynamically each period; so much so that Samuelson once had to summarize its results in a paper written in words of one syllable! However, apart from its intimidating rigor, the approach suffers from two weaknesses.

First, Rabin (2001) points out that risk aversion in respect of small gambles implies unrealistic risk premiums for investment style "gambles." He concludes, "aversion to modest-stakes risk has nothing to do with diminishing marginal utility of wealth." However, it is this very change in marginal utility of wealth that drives the multiperiod consumption-investment literature. Rabin suggests that investors are "loss-averse" rather than risk-averse. This is not a mere semantic distinction, but recognizes that risk is relative to the investor's current status, not some absolute level.

Second, the multiperiod consumption-investment framework is so general that it fails to induce interesting results. For example, the Samuelson-Merton result of age invariance reflects the assumption of a constant opportunity set, such that the multiperiod problem collapses into an equivalent series of identical single period problems; a result that Mossin (1968) characterizes as myopic. As the assumptions of the Samuelson-Merton model are relaxed the optimal decision rules become complex and lack an immediate interpretation. Roy (1952) critiqued similar results by pointing out that "in calling in a utility function to our aid, an appearance of generality is achieved at a cost of a loss of practical significance and applicability in our results. A man who seeks advice about his actions will not be grateful for the suggestion that he maximize expected utility." Roy put his finger on the critical issue, which is to solve practical problems more structure (assumptions) is usually needed to generate usable decision rules.

In contrast to the theoretical literature, an applied literature has developed based on Monte Carlo simulation. The great advantages of Monte Carlo simulation are that it is cheap, efficient, and relatively easy to use. For example, Hickman et al., (2001) use Monte Carlo simulation to analyze the "your age in bonds" rule; Pye (2000) to analyze withdrawal rates from a retirement portfolio; Spitz et al., (1994) different asset mixes for endowment funds, and Milevsky (2001) to look at the probability of outliving your money in retirement. In each case, Monte Carlo simulation based on actual data are used to benchmark the materiality of different strategies.

However, the great "ease of use" advantage of Monte Carlo simulation frequently leads to an emphasis on more simulations, when there is a more informative analytic solution.

For example, Hickman et al., point out that, "wealth relative distributions are positively skewed." Yet it is a property of the central limit theorem that a *product* of independent random variables approximates a lognormal distribution, just as the *sum* of independent random variables approximates a normal distribution. Unlike the normal, the lognormal distribution is a skewed distribution, so that the mean is higher than the median. Consequently, as the investment horizon is lengthened, the distribution of terminal wealth more closely approximates a lognormal, that is, a skewed distribution.[3]

This article combines some of the advantages of both the theoretical and the applied literature. One of the features ignored in the multiperiod consumption-savings model is that much of personal savings is contractual, where the rate is beyond the control of the individual and cannot be varied with consumption. Furthermore, the basic framework is difficult to reconcile with the way that retirement problems are actually solved. Financial planners generally solve the retirement problem in the way discussed in introductory finance textbooks. First, the retirement age is set, say 65. Next, the retirement needs are determined, usually relative to preretirement income. Finally, the terminal wealth is determined. It is then a question of working backwards to find the current wealth needed to generate that wealth and allocating that requirement to current wealth and future savings.

It is conventional to specify retirement targets as something like "70 @ 65," meaning 70% of preretirement income at age 65. For example, Greninger et al., (2000) in a Delphi study of financial planners indicated that the average ratio of replacement income was 74% of preretirement income at age 62. However, what is ignored in this retirement target is the probability of actually achieving it. Noticeably, in the Greninger study, not one of the twenty overall financial concepts considered dealt explicitly with risk, not even in the assessment of future rates of return. Instead of explicit adjustments, risk was handled implicitly through heuristics, such as the "your age in bonds" rule of thumb and the assumption that individuals would live longer than their actuarial life expectancy.

To incorporate risk, I extend the standard retirement problem by including the following probability constraint:

$$Pr((1 + ROR)^T < (1 + K)^T)) < a \tag{1}$$

where *ROR* is the uncertain annual rate of return on the portfolio and *K* is the target or expected rate of return. With fixed initial wealth, the probability constraint simply states that the probability of falling below the target level has to be less than the predetermined cut off rate *a*. This formulation of the impact of uncertainty is an example of Charnes and Cooper's (1959) chance constrained programming problem. This framework is also related to value at risk (Var). However, unlike Var the chance constraint is directly incorporated into the objective function, and the focus is not on day to day fluctuations in value and the "tail" of the distribution. Instead, the focus is on achieving a stated objective over a long-time horizon. In this sense, as I will show, the optimal asset allocation is a variation on standard mean variance results.

The key contribution of the model is the standard retirement target of say "70 @ 65" is extended to "70% of 70 @ 65," meaning a 70% chance of getting 70% of preretirement income at age 65. For convenience only, I take the risk of shortfall as being 30%, this value like the retirement age and replacement level of income, are determined by the individual. The important point is to show how this risk can be incorporated into the retirement target.

3. The problem

Suppose, for example, an individual (man) is 35 years old and earns $60,000 a year with current wealth of $50,000. If he wants to retire at age 65 a financial planner can look up their age in standard mortality tables, and assuming good health, estimate their expected life at 76 years.[4] With the "70@65" rule of thumb he needs 70% of preretirement income in retirement ,or $42,000. The question is then a simple one: can current wealth of $50,000 invested for 30 years generate $42,000 a year in retirement for 11 years? This is a straightforward present value problem. If all values are in constant dollars, the answer is that the portfolio has to earn a real rate of return of 6.43%. This is the critical target or required rate of return.

Whether or not the retirement targets are feasible depends on the level of long-run rates of return. The recent introduction of inflation-indexed bonds means that a real yield can be locked in, suppose, for example, the 30-year strip from an indexed bond is 4.0%. Clearly, investing in the real return bond at 4.0% will not meet the retirement targets, since it is less than the target rate of return; some allocation to equities is needed. Booth (1999) showed that the average annual real rate of return on equities from 1871 to 1997 was 9.0%. If we take 9.0% as the real equity return, it would seem that a portfolio of 50% equities and 50% inflation indexed bonds with an expected return of 6.50% would meet the retirement target: problem solved!

The parameters of the problem were set up in part to generate a problem that could seemingly be solved simply by moving the equity allocation to a balanced portfolio of 50%

equities and 50% bonds. However, this solution denies the fact that the real equity return from 1871 to 1997 also had an annual variability of about 19.25%. Whereas the real return strip can lock in the retirement target, the equity return is just "expected." There were many subperiods from 1871 to 1997, some of them decades long, when the stock market significantly under performed expectations. The problem is to incorporate this uncertainty in a meaningful way.

Future wealth is the product of the successive future annual rates of return. Regardless of the distribution of these annual rates of return, the central limit theorem applied to products of independent random variables indicates that in the limit the terminal wealth is *lognormally* distributed. Furthermore, many researchers assume that annual prices (one plus the annual rate of return) follow a lognormal distribution, since equity prices cannot be negative. In this case, terminal wealth is distributed exactly lognormal. For this reason, I assume that the natural logarithm of one plus the annual rate of return, the instantaneous compound rate of return, is normally distributed. With this assumption, the expected instantaneous compound rate of return equivalent to Booth's simple annual rate of return of 9.0% is approximately 7.0% with a standard deviation of 18.50%.[5]

With these estimates, the distribution of terminal wealth is,

$$LnW_T \sim \phi[LnW_0 + (r - 0.5*\sigma^2)T, \sigma\sqrt{T}] \tag{2}$$

This simply states that the natural logarithm of terminal wealth at time T (LnW_T) is normally distributed with mean equal to the natural logarithm of starting wealth at time zero (LnW_0) plus the continuously compounded rate of return times the time horizon (T), with a standard deviation equal to the standard deviation of the annual returns times the square root of the time horizon.

This distribution of terminal wealth is estimated both directly as well as indirectly through a Monte Carlo simulation with constant parameter values. In this case, 1000 random variables are used to replicate each of 50 years of continuously compounded annual returns with a mean of 7.0% and a standard deviation of 18.50%. The 50,000 random numbers were taken from Excel's normal distribution function with a seed value of 5 and with these same random numbers used in each simulation.

To illustrate the distribution of terminal wealth, Fig. 1 gives the analytic probability distribution around a mean of 1.0 with 100% invested in equities, for investment horizons starting at year 10 and then successively increasing by 10 years. Notice that all the distributions are skewed, with a long tail to the right and the most likely values (the peak) to the left of the mean. However, after 10 years, the most likely value is only slightly below the mean, but as the investment horizon lengthens the peak moves further to the left and the distribution becomes more skewed. What is happening is simply that the possibility of a sequence of high annual returns, although extremely unlikely, requires that the most likely values get smaller to "maintain" the constant mean.

The natural question to ask is how likely is it that the expected value can be obtained?

Clearly from Fig. 1, with a skewed distribution the probability of meeting or exceeding expectations is not 50%. In the first year, the analytic probability of getting or exceeding the expected value is 46.3%, whereas only 45.0% of the 1000 randomly drawn values from the

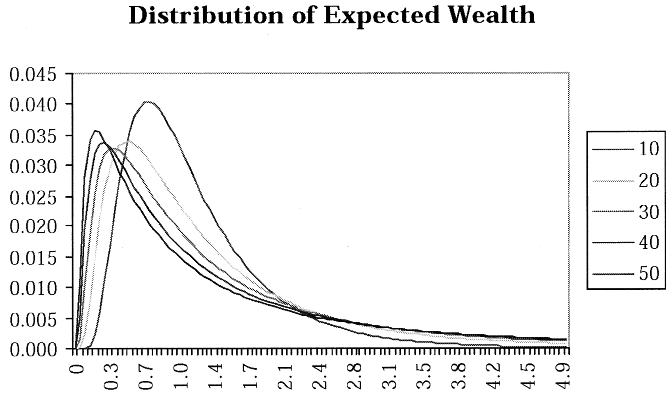

Fig. 1.

simulated distribution exceeded the mean. This difference is simply the error introduced by using this particular sample of random variables. Fig. 2 graphs this probability as the time horizon is lengthened. For a 50-year horizon, the analytic probability of meeting or exceeding expectations drops to only 25.7% while the simulated probability is 26.0%.

The implications of Fig. 2 are important for retirement planning, since it indicates that we need to be aware of the distribution of terminal wealth. This means being aware of the full distribution: not just the mean and standard deviation, but also the skewness.[6] It is of little value to be counseled that, "if things go as planned you should be OK," when the plan also includes a probability that there is only a 26% chance of things going as planned.

What causes the skewness in terminal wealth is both the time horizon and the uncertainty in the annual rates of return. Fig. 3 shows the impact of changing the standard deviation (volatility) of the annual returns. For the benchmark, historic, volatility there is a 26% chance of meeting expectations after 50 years, while for 27% volatility this probability drops to 17%, and for 10% volatility it increases to 36%. As the annual volatility increases, terminal wealth

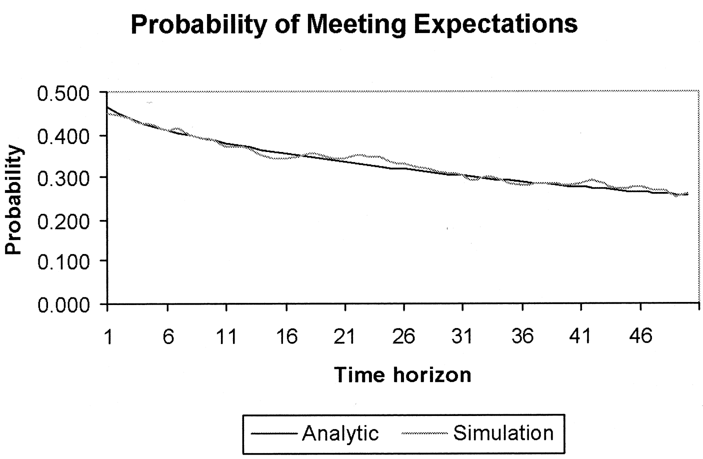

Fig. 2.

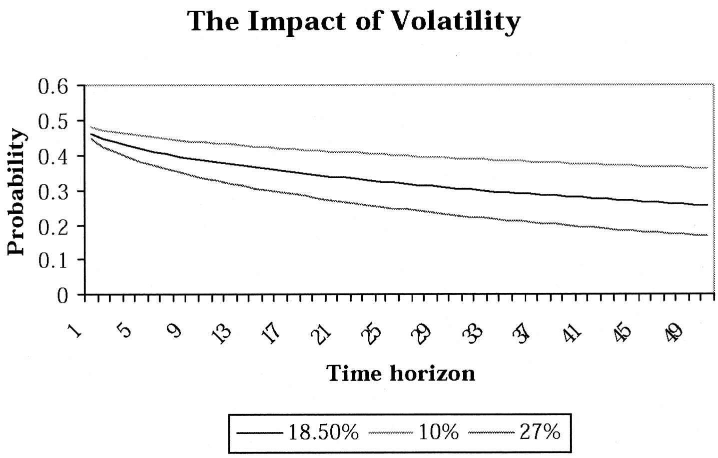

Fig. 3.

becomes more skewed, so that the most likely result gets further and further away from that expected. Since retirement targets are by nature long term, it is clear that the problem of failing to meet expectations becomes more severe as the asset allocation moves more heavily into riskier equities and away from bonds.

The insights from Figs. 1–3 allow me to re-evaluate the solution to the pension problem. Earlier, I showed that moving the retirement portfolio to 50% equities would meet the retirement target. This was based on the expected portfolio rate of return of 6.5% beating the required rate. However, as I have shown above, the probability of meeting expected values is not the 50% one might assume. Consequently, the important question is: what is the probability of meeting the target?

To examine this, note that terminal wealth is distributed

$$LnW_T \sim \phi[LnW_0 + (\alpha r + (1 - \alpha)R - 0.5{*}\alpha^2\sigma^2)T, \alpha\sigma\sqrt{T}] \tag{3}$$

where α is the equity weight and R the certain return on the real return bonds. The important point is that the weight on equities increases the expected rate of return (assuming a positive risk premium), while also increasing the volatility of the annual rate of return, which as I have shown will decrease the probability of meeting that expectation.

Using the approximate 50% equity weight solution for our retirement problem generates the set of probabilities in Fig. 4.

With the 30-year time horizon, the probability of getting the expected terminal wealth, needed for the "70 @ 65" target, is 40%. This is a higher probability than that in Fig. 2 for the 100% equities case, since with a 50% equity weight, the portfolio's standard deviation is approximately 9.25%. Consequently, the portfolio is less risky and compounding through time creates less skewed terminal wealth. However, what is more important is that the "70@65" retirement target is more accurately described as "40% of 70@65," that is, a 40% chance of getting 70% of preretirement income at age 65. Phrased in this way, many would not regard the 50% equities solution to our retirement problem as a solution.

Other useful information can also be calculated. For example, the median income level is

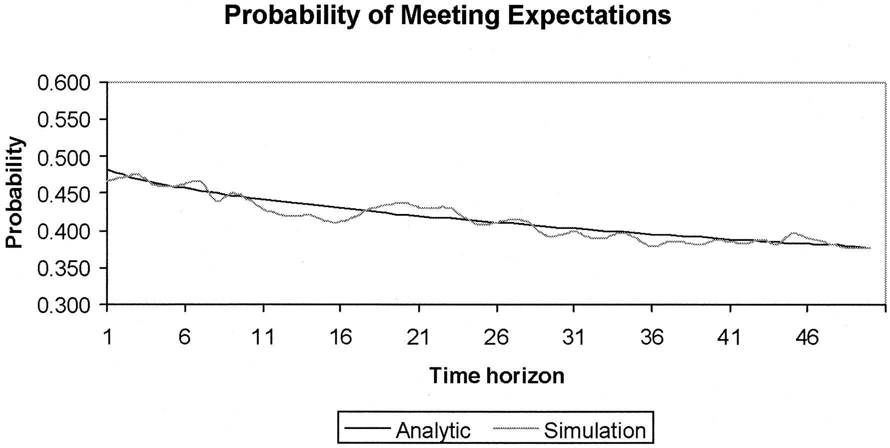

Fig. 4.

that for which there is a 50% probability. The median does not take into account the size of the deviations above or below the median. In this way, it is not affected by the very slim chance of earning very high rates of return for many years in the same way as the expected value is. Since the terminal wealth is skewed, the median, or 50% level of income, is lower than the expected level. In our case, the median is 88% of the expected value, so another way of stating the retirement target is "50% of 62@65," that is there is a 50% chance of getting 62% of the preretirement income at age 65. This is an alternative description of the same investment decision as "40% of 70@65." All else constant, as the target retirement income goes down, the probability of getting it goes up.[7]

Most people implicitly assume that there is a 50% chance of getting more, and 50% chance of getting less, than the expected amount, that is, that risk is symmetric. However, the example emphasizes that planning on the basis of expected values overestimates the chances of achieving the retirement target due to the skewed nature of terminal wealth. This raises the question can "50% of 70@65" be achieved? The answer is yes, but the equity allocation has to be increased to 61%. In this case, the expected level of income is now higher than the target level by 21%, but due to the increased volatility the probability of actually getting this expected amount drops from 40% to 38%.

Dealing with medians, rather than expected values, handles most people's conception of risk, since it expresses the problem in terms of "what is most likely to happen." However, there is nothing special about a probability of 50%. Many would be just as unhappy to be told that they only have a 50% chance of meeting their retirement target with a 61% equity allocation as they were with the original 40% chance. This naturally leads into the relationship between the probability target and portfolio composition. I have already shown that increasing the equity allocation from 50% to 61% increases the probability of meeting the retirement target from 40% to 50%, the natural question is what happens with even more equities?

The answer is that as the asset allocation moves more towards equities, although the expected portfolio rate of return goes up, so too does its volatility, which increases the

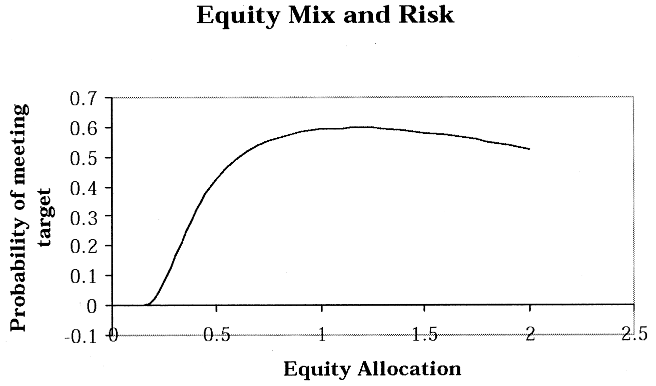

Fig. 5.

skewness of terminal income. These two effects work in opposite directions. Fig. 5 graphs the probability of meeting the "70@65" target as the equity allocation increases. The starting point is 0% equities, for which the probability of meeting the target is zero. This is because the real return bond is assumed to yield 4%, so there is a zero chance of earning the target rate of return of 6.5% without equities. As the equity allocation increases, there is initially little impact on the probability, simply because the expected portfolio rate of return is neither high enough nor the returns volatile enough to ever reach the target. However, starting at about 25% equities, there is a (slim) chance of meeting the target; at 50% equities the probability increases to the original 40% probability (expected income); it then increases at a slower rate to 61% equities with a 50% probability (median income) and reaches a maximum probability of 60% with an equity allocation of 120%, after which the probability declines as more equities are added.

Fig. 5 illustrates the two effects of changing asset allocation: increasing the expected rate of return, while also increasing the annual volatility and the skewness of terminal income. Initially, the higher expected rate of return outweighs the increase in volatility, but after about 120% equities the increased volatility generates so much skewness in terminal income that the probability of meeting the target starts to fall, despite the increasing level of expected income. The paradox is that expected terminal income always increases with the equity allocation, but past a certain point, the probability of meeting a specific target starts to fall. Equity risk does not disappear in the long run even though the investor expects to be wealthier with more equities.

An additional insight from Fig. 5 is that if an investor is simply interested in meeting a target level of income then the maximum asset allocation is 120% equities. In this case, there is a 60% chance of getting "70@65." However, in many of these cases where the target is achieved, retirement income will be vastly in excess of the 70% target; for example, if expectations are met retirement income will exceed the target by 183%. This is because the expected rate of return on the portfolio of 10.2% is significantly higher than the target rate of return of 6.43%. If this excess is not valued, then it raises the possibility of selling it and using the proceeds to bolster the amount invested. This is equivalent to a covered call writing strategy.

For example, at the target rate of return each dollar invested is expected to grow to \$6.426 by year 30. The Black-Scholes value for a 30-year European call on \$1 with an exercise price of \$6.426 and a standard deviation of 18.5% is \$0.20. Consequently, selling a \$6.426 call for every dollar invested boosts current wealth by 20% at the cost of losing the upside beyond the target level of income. If a 100% equities strategy is adopted the probability of meeting the "70@65" target is 59.3%, but if a call at the target rate is sold and the proceeds invested in equities, this probability increases to 66%, which is very close to the target of "70 of 70@65."

With the current development of the options market, selling 30-year calls by individuals is not practicable. However, pension funds should think in terms of the probability of meeting their liabilities to individuals from their investment portfolio. For example, the "70@65" target can be thought of as the defined benefit plan liability for someone with a 35-year working life and a standard 2% per year accrual, with \$50,000 as the current asset value of the fund. In this case, using a 50% equity allocation and a 6.5% target rate of return may classify the plan as fully funded using standard actuarial and accounting assumptions, even though there is only a 40% chance of generating the return necessary to fund the liability. For a pension fund concerned about meeting its liabilities, a covered call writing strategy significantly increases this probability.

A further important implication follows: more equities often mean lower risk. This stands traditional financial planning models on their head, but the reason is simple. In traditional models, the financial decision simply allocates wealth between different points in time; there is no constraint that the future level of wealth even reaches a subsistence level. In contrast, a "70 @ 65" target specifies risk not in absolute terms but relative to the target. In this case, a conservative investment strategy that preordains poverty is exceedingly risky! Machol and Lerner (1969) made a similar point: "First, risk is reduced by buying securities which have a high expected value, since this reduces the probability that the portfolio value will drop below the ruin level." If we replace their concern with ruin, with our concern for failing to meet the retirement target the analogy is complete.

Required levels of wealth

The prior discussion indicates that \$50,000 is not enough, even with call writing, to allow our hypothetical individual to meet their retirement target of "70% of 70 @ 65." The obvious question is how much is needed and how does the portfolio composition change as a result? To answer this, I explicitly incorporate the probability constraint into the objective. For convenience, I continue to assume that the objective is "70% of 70 @ 65," that is, a 70% probability constraint.

The terminal wealth is simply

$$W_T = W_0(1 + ROR)^T, \tag{4}$$

so taking natural logarithms gives,

$$LnW_T = LnW_0 + TLn(1 + ROR) \tag{5}$$

where the assumption of log-normality means that the logarithm of one plus the rate of return (*ROR*), the continuously compounded rate of return, *r*, is normally distributed.

The probability constraint is

$$PR(Tr \ < \ Tk) < 30\% \tag{6}$$

Note that starting wealth cancels out, so the only concern is for the difference between the continuously compounded rate of return rate, *r*, and the target rate of return, *k*. Subtracting the expected rate of return from both terms inside the probability function and dividing by the standard deviation of the rate of return gives

$$N\left(\frac{Tk - TR(r)}{\sigma\sqrt{T}}\right) < 30\% \tag{7}$$

Where *N(.)* is the standard unit normal density function. With the 70% probability target the constraint imposes a value of −0.5244. This states that 70% of the probability mass of a unit normal density function is between −0.5244 standard deviations from the mean and infinity, or conversely that 30% is below −0.5244.

This constraint can be rewritten as

$$T(k - (T(E(r) - 0.5244\sigma_{\,p}T^{-0.5})) = 0 \tag{8}$$

where $E(r)$ is the expected continuously compounded rate of return and σ_p the standard deviation of the portfolio's rate of return. Eq. (8) is the deterministic equivalent of the probability constraint.

We already know that the maximum probability of meeting the target of "70 @ 65" is 60%, which is with 120% equities. Consequently, to increase to the 70% probability level means increasing the amount invested. In terms of the constraint this means reducing the target rate of return (k^*) until this lower rate of return on a larger initial wealth meets the target, that is,

$$LnW_* = LnW_0 + T(k - k^*) \tag{9}$$

Using the mean and standard deviation of the portfolio's rate of return from Eq. (3) in the probability constraint gives the minimum level of wealth as a function of the asset allocation:

$$Ln(W_*) = Ln(W_0) + T(k - (R \ + \ \alpha(r - R) \ - \ .5^*\alpha^2\sigma^2) - .5244\alpha\sigma T^{-0.5} \tag{10}$$

All values except the equity mix (α), are known, so we can solve for the minimum wealth level as the equity mix is varied, subject to the 70% probability constraint.

The optimal asset allocation is the asset allocation that meets the constraint with minimum wealth. For the 30-year retirement horizon, Fig. 6 graphs the minimum required level of wealth as the equity allocation increases.

A portfolio with no equities, that is, 100% invested in the risk-free bond clearly meets the 70% constraint. However, this requires minimum wealth of $99,064, since at the assumed 4.0% risk-free rate this is what generates the target terminal wealth at retirement. As the equity share increases the required level of wealth falls. It reaches a minimum of about

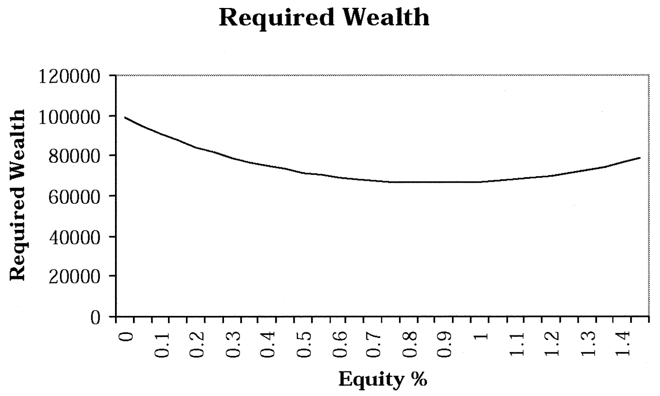

Fig. 6.

\$66,470, where the equity allocation is about 90% in equities. The 90% equity allocation has a portfolio expected rate of return of 8.59% and a 70% chance of meeting the target rate of return of 5.40% on \$66,470. As the equity share increases above 90%, the increased variability in the portfolio return causes the skewness of the terminal wealth to increase sufficiently to lower the probability of meeting the target. This in turn causes the minimum amount of wealth to increase.

The exact solution from fig. 6 can be determined by choosing the equity allocation to minimize the following:

$$Ln(W_*(\alpha)) = Ln(W_0) + T\,(k - (R + \alpha(r - R) - .5^*\alpha^2\sigma^2) - .5244\alpha\sigma T^{-0.5})) \tag{11}$$

This has the optimal solution

$$\alpha = \frac{(r - R) - 0.5244\sigma T^{-0.5}}{\sigma^2} \tag{12}$$

With the assumed parameter values the exact solution for the 30-year time horizon is 88.2% equities.[8]

If this optimal equity allocation is substituted back into the minimum wealth equation we get

$$Min\ Ln(W_*(\alpha)) = Ln(W_0) + T\,(k - (R + ((((r - R) - .5244\alpha\sigma T^{-0.5})^2)/2\sigma^2))) \tag{13}$$

With the assumed parameter values the minimum wealth is \$66,429.

The optimal solution indicates that if the investor accepts a target of "70% of 70 @ 65" then they will have to increase their invested wealth from \$50,000 to \$66,429. If this portfolio is invested 88% in equities then the retirement target can be met. Alternatively, the initial wealth can be partly met by selling calls at the target level of retirement wealth.

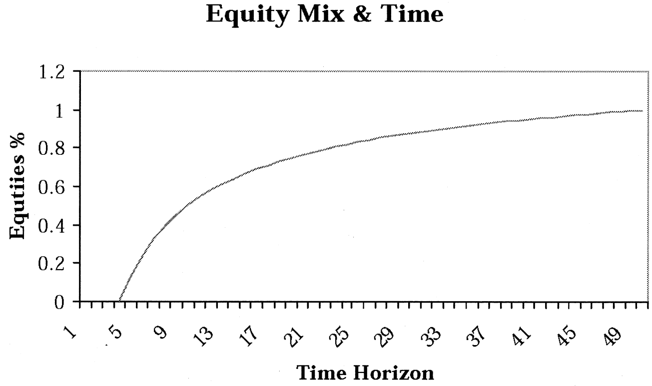

Fig. 7.

Regardless the solution indicates that meeting realistic retirement goals is considerably more complex than simply compounding at the expected rate of return!

Finally, we can consider how the portfolio should be structured according to the individual's age and investment horizon. To keep the nature of the probability constraint constant we continue to assume that the target or required rate of return of 6.43% exceeds the 4.0% risk free rate and that the volatility of the equity market is constant. All that changes is the individual's age and thus their investment horizon. Essentially, this means that over the individual's lifetime, wealth grows according to expectations. Fig. 7 graphs the optimal asset allocation that results from varying the time horizon.

For individuals with less than a 5-year time horizon, the optimal portfolio is 100% risk free bonds. To explain this, consider the 1-year time horizon, here the investor is assumed to need the target rate of 6.43% to retire in 1 year. However, suppose they invest 100% in equities, in this case with a continuously compounded rate of return of 7.0% and a standard deviation of 18.5% the 70% probability constraint means a minimum rate of return of −2.7%. This is because 70% of the probability mass lies between −2.7% and +infinity. A minimum rate of return of −2.7% requires invested wealth of $54,665 to meet the 70% retirement target. In contrast, $51,152 invested in the risk free bond provides 100% guarantee of meeting the target, so investing in the risk-free bond is optimal. With these parameters the optimal portfolio allocation is 100% risk-free bonds for all investment horizons less than 5 years. With a longer time horizon the equity allocation increases with the time horizon.[9]

The reason for this result is the structure of the optimal equity allocation in Eq. (12). Note that the risk premium (*r-R*) is reduced by the effect of the 70% probability constraint. This reduction is the constant coefficient −0.5244 times the portfolio's standard deviation times the inverse of the square root of the time horizon. The constant -0.5244 simply comes from the 70% constraint and specifies the percentage of the distribution that has to be above the minimum rate of return. The remainder specifies how the uncertainty evolves over time and is a function of the lognormal distribution, where variances are additive over time. Consequently, the effect of the probability constraint increases at a *decreasing* rate with the time horizon. For 1 year, 0.5244 of the standard deviation is subtracted from the risk premium, whereas for 4 years only half this is subtracted. As a result, there is less chance of meeting

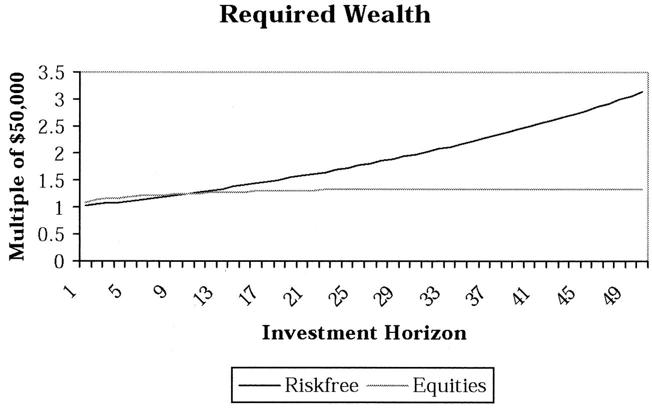

Fig. 8.

the probability constraint by using equities in the short run than there is in the long run. In the limit with an infinite horizon the asset allocation is the same as that for the growth optimal portfolio.

Another way of looking at this is to think in terms of the multiple of current wealth needed to meet the target. This helps abstract from the notional $50,000 of the example.

Fig. 8 shows these multiples with two extreme choices of 100% risk-free bonds and 100% equities. For the 1-year horizon, the 100% risk free bond strategy needs $51,152/$50,000 or a multiple of 1.023. In contrast, the 100% equities strategy needs $54,665/$50,000 or 1.093. Consequently, it is optimal to use a risk-free bond strategy. However, the additional wealth required to meet the target with a 100% risk-free bond strategy increases at an increasing rate. This is due to the constant shortfall between the risk-free rate and the target rate. In contrast, the higher expected rate of return from equities in part offsets their risk, so that the increased wealth requirement increases at a decreasing rate and peaks at just over 1.3.

It is this tension between the higher expected rate of return and the risk of equities that causes the optimal portfolio composition to change with the investment horizon. For shorter time horizons, a retirement target like "70% of 70 @ 65" implies more risk-free bonds, than it does for longer time horizons. Although Fig. 7 is specific to these parameter values, the general result holds: whenever the target or required rate of return is above the risk-free rate some allocation to equities is needed. This may not exactly equal the industry rule of thumb of "your age in bonds," but the general pattern is the same. It is also clear that with this formulation of the retirement problem, Samuelson's constant equity percentage result does not hold.

The model has been developed in terms of a single future sum. However, the retirement life can be thought of as a sequence of retirement targets, for example "70% of 70@65," "60% of 65@66," "50% of 60@67" and so forth with the probability as well as the amount of retirement income changing with each year of the expected retirement life. There is no reason, for example, to assume that individuals want to smooth their consumption equally over their entire retirement life; furthermore, they may be inclined to take more risks with the later years of their retirement than the earlier ones. In this way, the solution developed in this paper can be applied to the wealth "strips" required for each year of the retirement life. The overall asset allocation is then the value weighted sum of the optimal asset allocations for each year of retirement.

A final caveat is that the solution to the retirement problem put forward in this paper is an applied solution, not a general one. It relies implicitly on a utility function that places no weight on income above the target level and huge disutility to income below it. However, it formulates the problem in a manner consistent with the way that financial planners tackle the problem and extends that algorithm to incorporate uncertainty. In doing so, it supports the general rule that the longer the time horizon, the greater the equity allocation and shows how these allocations can be determined.

5. Conclusions

This paper has extended standard retirement targets such as "70@65" to include the uncertainty of meeting the target. It might be assumed that using expected rates of return in formulating a "70@65" target means that there is a 50% chance of exceeding and a 50% chance of falling below the target. However, this is not the case. The wealth at the retirement date is the product of successive investment returns, so that even if these annual returns are symmetric, it is a result of the central limit theorem that the terminal wealth becomes skewed and approximates a lognormal distribution with the expected terminal wealth exceeding the median. For example, with a standard "70@65" retirement target I show, using historic parameter values, that for a 30-year investment horizon, the probability of meeting the target with a balanced portfolio is only 40%. Consequently, I argue that a better formulation of the retirement target is something like "70% of 70@65," meaning a 70% chance of getting 70% of preretirement income at age 65.

Casting the retirement problem in this way is an application of chance constrained programming with the following results:

- The probability of meeting the retirement target is a concave function of the equity allocation for positive holding of equities. Initially, the higher expected return on equities over bonds increases the probability of meeting the retirement target, but at some point the increasingly skewed nature of retirement wealth causes the probability to fall. As a result there is an optimal equity allocation.
- If the individual does not value the income in excess of the retirement target, European calls at the target rate of return can be sold to enhance the amount invested and thus the probability of meeting the target.
- To satisfy the retirement target exactly, I determine the minimum wealth required as a function of the equity allocation. For shorter time horizons, the optimal allocation is 100% in risk-free bonds, but as the time horizon is lengthened more equities are optimal. In the limit the allocation results in the standard "growth optimal portfolio."

Unlike the standard consumption-investment model the chance constrained programming model gives results that are easy to implement: Roy would be happy.

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Notes

1. See "Nifty calculator," Business Week, September 29, 2003, for a typical usage of the "70@65" goal based on models provided by T. Rowe Price or "Your Retirement Income Planner," TD Canada Trust.
2. Various aspects of the impact of the investment horizon has been looked at by several authors including Milevsky, Ho and Robinson (1994, 1996, 1997), Milevsky and Robinson (2000), Butler and Domian (1991), Thorley (1995), and Gunthorpe and Levy (1994).
3. The justification for mean-variance analysis is that with normality the true measure of risk, the risk of loss, is equivalent to the variance or standard deviation. This

assumption is weakened as the time horizon is lengthened and terminal wealth becomes skewed. This problem is hidden in continuous time models, where the focus is on myopic decision making, where the instantaneous consumption is still normal.

4. The most recent 1997 tables issued by the National Center for Health Statistics gives the male life expectancy at age 35 as 75.4 years and for a female 80.4. If the individual has more favorable personal characteristics, based, for example, on being a non-smoker, then the retirement period is lengthened, increasing the target rate of return. This requires either an increase in initial capital or future savings, or conventionally a higher equity allocation and more risk.
5. If prices are distributed exactly lognormal then the annual compound rate of return is the simple annual rate of return minus half its variance or 7.15% ($9\%–0.5*.1925^2$). The continuously compounded rate of return is the natural logarithm of 1.0715, which is 6.90%.
6. The skewed nature of multiplicative random events has been known at least since Bernouilli discussed the St Pertersburg paradox [see Bernouilli, D. (1954) Exposition of a new theory on the measurement of risk *Econometrica 22,* (reprint of his 1738 paper)].
7. There are obviously many ways of expressing the same investment plan in terms of probability targets; using the median and mean target levels of income are the most intuitive.
8. Note that without the probability constraint, the solution is the standard growth optimal portfolio that maximizes the expected compound rate of return on the portfolio. A mean variance solution would have the investor's risk aversion applied to the variance in the denominator.
9. Greninger et al. find that professional financial planners generally recommend moving to a more conservative portfolio 3–5 years prior to retirement.