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Expected returns, correlations, and optimal asset allocations

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Abstract

With increasing uncertainties in financial markets, individual investors now face difficult asset allocation choices. This article provides a framework for this deliberation by examining the marginal effects of the key input variables in the asset allocation for a portfolio of stocks, bonds, and bills. Expected stock returns are positively related to the optimal weight on stocks, but we show that the magnitude of this relation depends upon both the expected stock-bond correlation and the investor's attitude toward risk. We also find that an increase in the expected stock-bond correlation leads to shifts from bonds to bills for more risk-averse investors. © 2005 Academy of Financial Services. All rights reserved.

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1. Introduction

Mean-variance analysis is one of the cornerstones of finance theory, allowing investors to select optimal portfolio combinations of assets to maximize return for given risk levels or minimize risk for given return levels. Mean-variance analysis is most useful when considering broad asset classes, such as stocks, bonds, and cash. The decision-making process requires expectations of asset class returns, standard deviations, and correlations as inputs.

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Given that we cannot predict the future with any certainty, the historical values of these input parameters are often employed as expected values. Currently, there is a particularly wide divergence of opinions regarding these mean-variance inputs and a strong sentiment that history will not repeat itself, at least in the near future. A great deal of attention has been focused specifically on stock returns and the correlation between the returns of stocks and bonds. Differences in expectations regarding these items will certainly affect asset allocation decisions, but to what degree?

Many analysts believe that the historical returns experienced with equity over the past eighty years cannot continue. Investors have become accustomed to double-digit stock returns, but several recent studies have suggested that the equity premium compared to bonds will be noticeably smaller in the coming years. Philips (1999) forecasts long-run stock returns of a mere 6.5% per year. Ilmanen (2003) offers a pessimistic forecast for equity of 5–6% per year, while his optimistic view is for equity to earn only 7–8% per year. Arnott and Bernstein (2002) show that the "normal" risk premium of stocks is only 2.4% and suggest that a negative premium is a real possibility. Hunt and Hoisington (2003) and Bernstein (1997b) likewise expect lower return prospects for stocks in the future. Mehra (2003) and Jones and Wilson (1999), however, argue that the equity risk premium is alive and well.

The view that the stock-bond correlation is stationary over time has been challenged by the likes of Bernstein (1997a) who suggests that covariances are "maddeningly unstable." Dopfel (2003) notes that while the correlation between the returns of stocks and bonds is typically assumed to be a constant value between +0.30 and +0.50, his examination of 36-month rolling correlations from 1929 through June 2003 shows four periods, including the most recent period since 2000, where the correlation was negative. In fact, the correlation between the returns from stocks and bonds was negative for a full 10-year period, averaging −0.23 from 1955 to 1964. Wainscott (1990) studies these correlations from 1925 through 1988 using rolling one, three, five, and 10-year periods. Not surprisingly, the one-year periods show that correlations are very unstable, while the results are smoothed out over longer time periods. Ziering and McIntosh (1997) note that the returns and correlations of stocks, bonds, REITs, and real estate vary considerably over subperiods from 1972 to 1995. Gulko (2002) observes the decoupling phenomenon whereby the correlation between stocks and bonds breaks down during market crashes.

Several studies relating to tactical asset allocation have attempted to improve on simply using historical inputs into mean-variance analysis. Ilmanen (1997) finds that factors such as the shape of the yield curve, the real rate of return, and changes in stock market wealth could be used to predict bond returns. Flavin and Wickens (2003) use a multivariate GARCH model to predict the conditional means and covariances of asset returns, arguing that macroeconomic factors affect these mean-variance inputs. Bharati and Gupta (1992) use several predictive variables in regression equations to estimate asset returns, and the regression residuals are used to estimate the covariance matrix. Giliberto, Hamelink, Hoesli, and MacGregor (1999) use a QTARCH model, partitioning input based on factors such as inflation and GDP growth, to find asset returns and correlations for different states of the world.

The preceding discussion shows that there is a considerable amount of uncertainty

Table 1
Summary statistics and correlation matrix (annual returns 1926 to 2002)

	Large company stocks	Long-term government bonds	U.S. Treasury bills
Arithmetic mean return	12.2%	5.8%	3.8%
Standard deviation	20.5%	9.4%	3.2%
Correlation coefficients			
Large company stocks	1.000		
Long-term government bonds	0.127	1.000	
U.S. Treasury bills	−0.016	0.229	1.000

Source: Ibbotson (2003)

associated with the stock-bond risk premium and the stock-bond correlation. In this article, we provide a framework within which to evaluate the pertinence of the issues being debated. We employ a mean-variance utility function to examine the impact on the portfolio asset mix (stocks, bonds, and bills) of changing estimates (or of being off in estimates) of these factors. We also report the marginal effects of changes in the expected return on stock and the stock-bond correlation on portfolio weights for investors with different levels of risk aversion. If differences in any of these key stock-bond forecasts do not significantly affect optimal portfolio decisions, then the discussion may be moot. If, on the other hand, changes in some forecasts have a significant impact on the portfolio mix, then they would merit closer scrutiny. At the very least, investors should be aware of the influence of these input factors on optimal asset allocation decisions.

2. Historical data

Over the 1926 to 2002 period, the 12.2% arithmetic mean annual return generated by large company stocks provided an average 6.4% premium over long-term government bonds, as shown in Table 1. The extra return generated by stocks, however, was accompanied by a volatility level more than double that of bonds. U.S. Treasury bills provided modest arithmetic mean returns of 3.8% per year with a relatively low standard deviation of 3.2%. Data for large company stocks, long-term government bonds, and U.S. Treasury bills is derived from annual total returns from the Ibbotson Associates SBBI 2003 Yearbook (Ibbotson Associates, 2003). These will subsequently be referred to as stocks, bonds, and bills to simplify the narrative. The historical stock-bond correlation coefficient was positive, but a relatively low 0.127. This figure was pulled down somewhat by recent experience that had the stock and bond markets moving in opposite directions. The stocks-bills correlation coefficient was approximately zero, while the bonds-bills correlation coefficient was 0.229. In the mean-variance analysis that follows, we assume all of the historical variables shown in Table 1 to be fixed inputs into the decision-making process except for the mean return on stocks and the stock-bond correlation.

Fig. 1 shows the risk premium on stocks versus bonds and the stock-bond correlation for the period of 1930 through 2002. Because annual data display extreme volatility, the stock-bond risk-premium is calculated using a five-year moving average, and the stock-bond

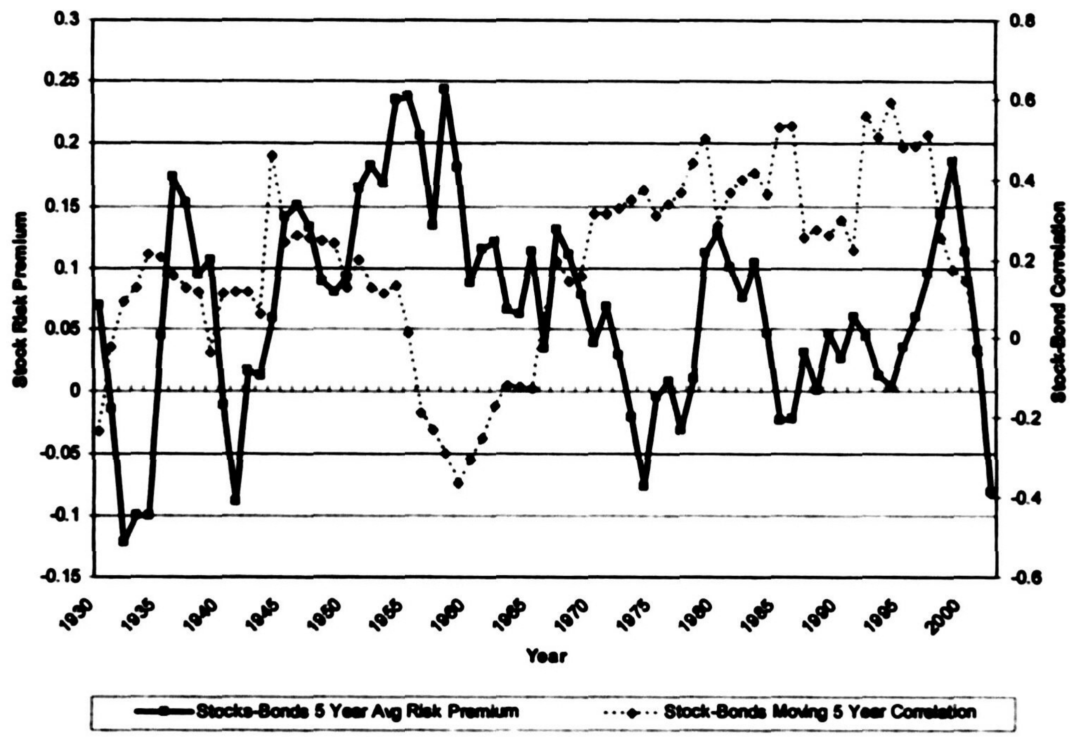

Fig. 1. Stock risk premium and stock-bond correlation, 1930 to 2002 with five-year overlapping periods.

correlation is computed with five-year rolling periods. (Correlation coefficients are based on monthly total returns for the 60-month periods.) Despite the smoothing effects of the moving average, the figure clearly shows that neither of these variables was very stable over time. While we will always have positive expectations of stock returns, the risk associated with stocks resulted in realized stock returns that were less than those of bonds in some periods. Likewise, we generally expect a positive stock-bond correlation, but there were also periods where the correlation was negative, including the most recent period ending in 2002. The most recent period of negative correlation occurred when stocks underperformed compared to bonds, but the previous periods of negative stock-bond correlation were associated with strong stock performance in late-1950s to mid-1960s. The last 15 years encompassed the extremes of the stock-bond correlation ranging from a high of 0.592 for the five years ending in 1994 to a low of −0.390 for the five years ending in 2002. As previously noted, Wainscott (1990) and Dopfel (2003), who included 12- and 36-month stock-bond correlation coefficients, found much more volatility when shorter periods were examined.

Showing the stock-bond risk premium and the stock-bond correlation in 10-year rolling periods, as in Fig. 2, smoothes out many of the bumps revealed in Fig. 1, but still raises some questions. Those who doubt the pessimistic forecasts of lower stock premiums in the future might reconsider their convictions after examining the realized returns from the most recent 30 years. The average risk premium for the most recent 30-year period was a mere 2.3%, which is a far cry from the 6.4% long-term average. Fig. 2 also reveals that the ten-year rolling average of risk-premiums has shrunk in recent periods in comparison to the pre-1970 period. So will the next few years bring a reversion to the long-term mean or more of what

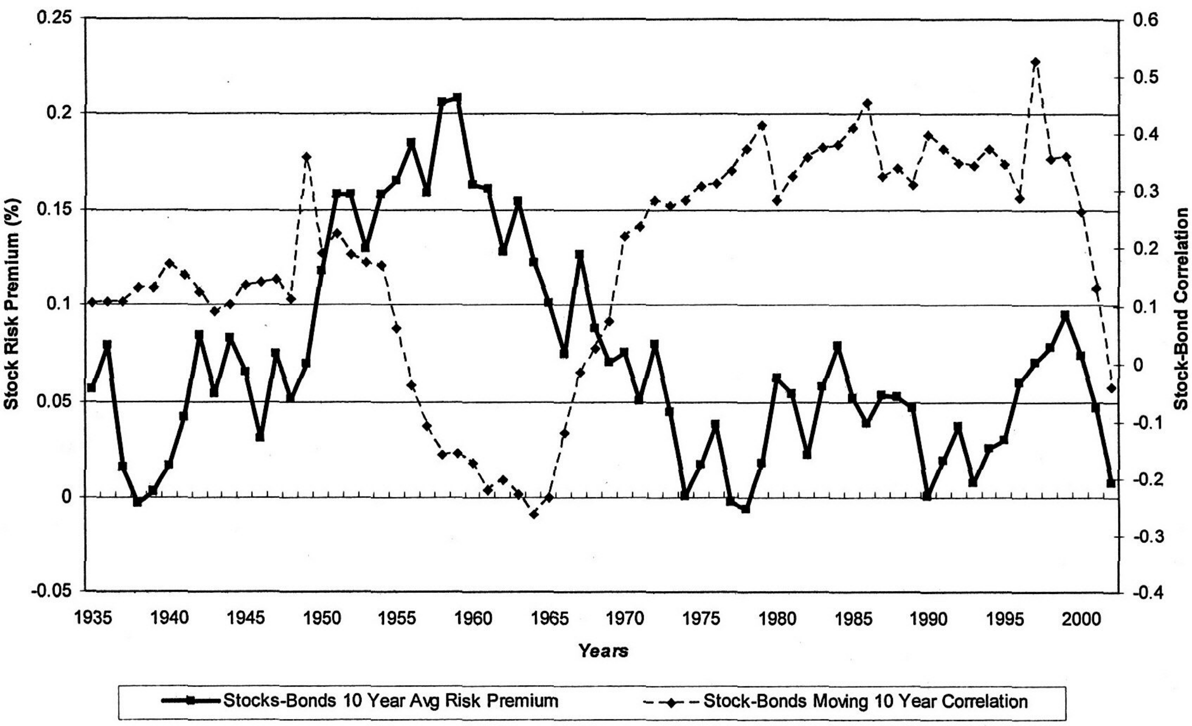

Fig. 2. Stock risk premium and stock-bond correlation, 1935 to 2002 with 10-year overlapping periods.

we have seen recently? The historical perspective regarding the stock-bond correlation in Fig. 2 also leaves room for differences of opinion. Which is a better forecast of the stock-bond correlation coefficient going forward—the roughly 0.3 correlation we saw from 1963 to 1992, the long-term average of about 0.1, the recent trend with a figure just below zero, or something entirely different? Finally, what difference do investor expectations regarding returns and correlations have on optimal portfolio allocations?

We use the historical levels of the returns, standard deviations, and correlations of the asset categories as the starting point of our analyses and discussions. This does not mean that these are our personal expectations or any consensus estimates of the future. In fact, we make no attempt whatsoever to predict the future in this paper. As we previously noted, there is a considerable divergence of opinion regarding the future, so picking any one set of input variables for mean-variance analysis is problematic. What is certain, however, is the historical relationships of stocks, bonds, and bills. There are also many institutions and individuals still providing recommendations that are based in large part on what has happened in the past. By using history as the basis of our analyses, we are able to determine the effects and relative importance of changes in mean-variance input variables on optimal portfolio allocations.

3. Mean-variance utility and portfolio selection

Mean variance analysis is based on the premise that investors prefer higher returns and seek to avoid risk or the variability of returns. Investors attempt to maximize expected portfolio return r_P for a given level of portfolio risk (variance) σ_P^2 or minimize risk for a given level of return. For our purposes, we assume that investors choose the portfolio weights that maximize utility U with the common utility function:

$$U = r_P - \frac{1}{2} A \sigma_P^2 \tag{1}$$

where A is the risk aversion parameter for an individual investor.[1]

3.1. Risk aversion parameter "A"

While all risk-averse investors seek to avoid risk, different investors have different levels of risk aversion. Investor attitudes toward risk are a key input into determining appropriate portfolio allocations. Therefore, individual investors and financial planners must address the issue of attitudes toward risk. Accordingly, financial advisors use various techniques, often in form of questionnaires, to capture investor sentiment and aid in portfolio allocation decisions. Vanguard has an "Investor Questionnaire" to help investors ". . . think about your reaction to the risks of investing." TIAA-CREF has an "Asset Allocation Evaluator," and there are many others.

Hallahan, Faff, and McKenzie (2004) look at how several demographic factors relate to financial risk tolerance. They find that males have more risk tolerance than females and that higher levels of income and wealth both relate to higher levels of risk tolerance. They also find a negative relationship between age and risk tolerance, as expected, but they show that the relationship is not linear. Hallahan et al. also note that investors tolerance of risk, as measured by psychometrically derived financial risk tolerance scores, are generally consistent with both ex ante investor expectations of risk tolerance and their desired portfolio selections.

The A in Eq. (1) is designed to model investor risk tolerance. As an illustration, Fig. 3 plots different levels of A on the efficient frontier based on the historical relationships among asset classes. Low values of A are consistent with a higher tolerance for risk. While individual investors will not know their individual values of A, Hallahan et al. (2004) note that investors have fairly good notions of where they lie on the risk-return spectrum. We find that investors with an A of 1 or 2 would be what TIAA-CREF would classify as "aggressive" investors. Based on the long-term historical risk-return and correlation relationships of asset classes, these investors should be holding portfolios of from 80–100% stock. Investors with an A of 3 are within the average risk range, with an optimal portfolio weight in stocks of about 60%. More conservative investors with A's of 5 to 10 would have optimal portfolio weights in stock of about 20–40%.[2] We examine levels of A ranging from 1 to 10, which covers the bulk of the reasonable range of investor choices on the efficient frontier, as shown in Fig. 3.

The utility function in Eq. (1) allows us to compare investors with varying levels of risk tolerance and to hold investor risk tolerance constant across different environmental scenarios. Looking at the optimal portfolio allocations based on historical relationships that are discussed above and shown in Table 2, individual investors and financial planners can identify roughly where they or their clients are on the risk-return continuum as reflected by

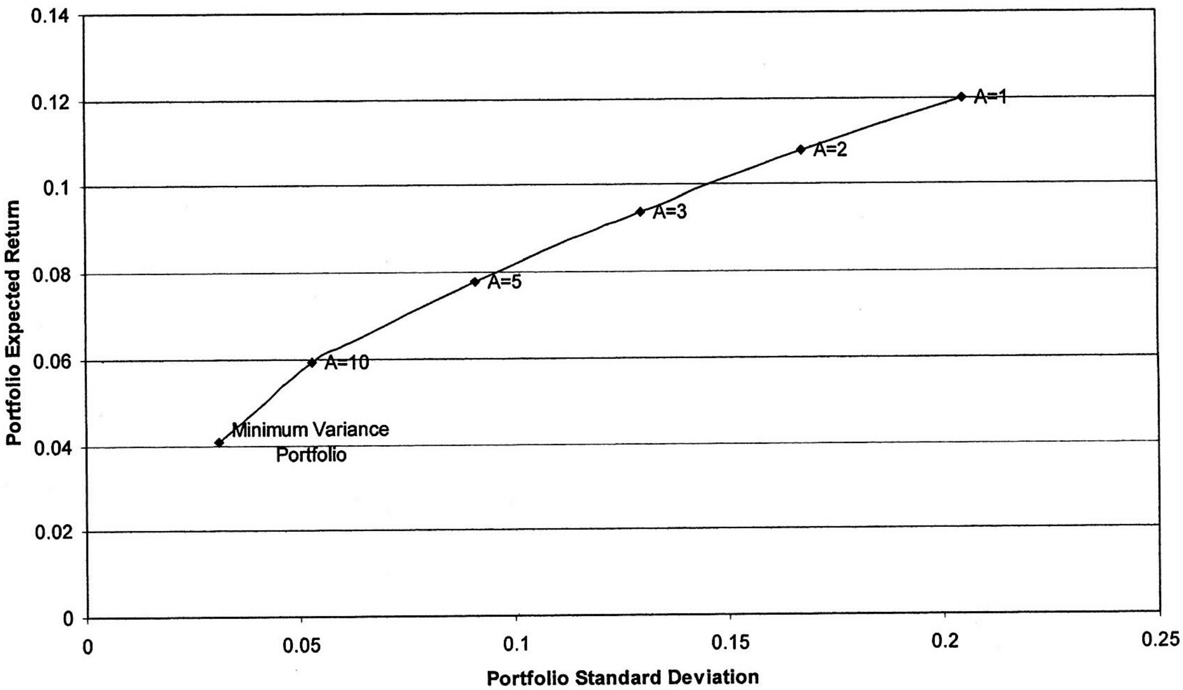

Fig. 3. Efficient frontier plotted assuming different levels of risk aversion A. Calculations are based on the long-term historical relations of the asset classes.

the risk aversion parameter A. The results that follow allow investors to see the effects of changes in expectations of stock returns and the stock-bond correlation on optimal portfolio holdings while holding this risk tolerance level constant.

3.2. Optimal portfolio weights and marginal effects equations

In the three-asset case with stocks, bonds, and bills, the portfolio return is calculated as

$$r_P = w_S r_S + w_B r_B + w_C r_C \quad (2)$$

where w_S, w_B, and w_C are the portfolio weights of stocks, bonds, and bills, respectively. These are the weights that investors adjust to maximize utility. The expected returns of stocks, bonds, and bills are given by r_S, r_B, and r_C, respectively. The portfolio variance is calculated as

$$\sigma_P^2 = w_S^2\sigma_S^2 + w_B^2\sigma_B^2 + w_C^2\sigma_C^2 + 2w_S w_B \rho_{SB}\sigma_S\sigma_B + 2w_S w_C \rho_{SC}\sigma_S\sigma_C + 2w_B w_C \rho_{BC}\sigma_B\sigma_C \quad (3)$$

where ρ_{SB} is the stocks-bonds correlation, ρ_{SC} is the stocks-bills correlation, and ρ_{BC} is the bonds-bills correlation. The standard deviations of stocks, bonds, and bills are given by σ_S, σ_B, and σ_C, respectively. Two constraints rule out short selling and assure portfolio completeness:

$$w_S, w_B, w_C \geq 0$$

$$w_S + w_B + w_C = 1$$

Table 2
Optimal portfolio allocations and marginal effects (stock-bond correlation equal to 0.1)

r_S		A				
		1	2	3	5	10
12%	w_S^*	1.000	0.803	0.584	0.382	0.203
	w_B^*	0.000	0.197	0.416	0.406	0.218
	w_C^*	0.000	0.000	0.000	0.212	0.579
	$\partial w_S^*/\partial r_S$	0.213(a)	0.106	0.071	0.047	0.024
	$\partial w_B^*/\partial r_S$	−0.213(a)	−0.106	−0.071	−0.013	−0.007
	$\partial w_S^*/\partial \rho_{SB}$	0.079(a)	0.025	0.007	−0.014	−0.007
	$\partial w_B^*/\partial \rho_{SB}$	−0.079(a)	−0.025	−0.007	−0.083	−0.044
10%	w_S^*	1.000	0.590	0.442	0.288	0.156
	w_B^*	0.000	0.410	0.558	0.432	0.231
	w_C^*	0.000	0.000	0.000	0.280	0.613
	$\partial w_S^*/\partial r_S$	0.213	0.106	0.071	0.047	0.024
	$\partial w_B^*/\partial r_S$	−0.213	−0.106	−0.071	−0.013	−0.007
	$\partial w_S^*/\partial \rho_{SB}$	0.044	0.007	−0.005	−0.016	−0.009
	$\partial w_B^*/\partial \rho_{SB}$	−0.044	−0.007	0.005	−0.061	−0.033
8%	w_S^*	0.606	0.377	0.300	0.194	0.109
	w_B^*	0.394	0.623	0.700	0.458	0.244
	w_C^*	0.000	0.000	0.000	0.348	0.647
	$\partial w_S^*/\partial r_S$	0.213	0.106	0.071	0.047	0.024
	$\partial w_B^*/\partial r_S$	−0.213	−0.106	−0.071	−0.013	−0.007
	$\partial w_S^*/\partial \rho_{SB}$	0.009	−0.010	−0.016	−0.018	−0.010
	$\partial w_B^*/\partial \rho_{SB}$	−0.009	0.010	0.016	−0.039	−0.022

All mean-variance input factors other than r_S and ρ_{SB} are at their historical levels, as shown in Table 1.
$\partial w_S^*/\partial r_S$ and $\partial w_B^*/\partial r_S$ are divided by 100 to show approximate change per a 1% change in return.
$\partial w_S^*/\partial \rho_{SB}$ and $\partial w_B^*/\partial \rho_{SB}$ are divided by 10 to show the approximate change per a 0.1 change in correlation.
(a) The marginal change is not sufficient to affect the portfolio weights. The optimal allocation will still be 100% stock.

In the results that follow, the portfolios of more risk-averse investors include all three of the asset classes. With the constraint against short selling, more aggressive investors (with lower values for A) shift more of their assets toward stocks and bonds and eliminate their holdings in bills. The most aggressive investment position of 100% stock eliminates holdings of both bills and bonds. So the optimal portfolio solutions may include one, two, or three assets.

We develop solutions for two and three-asset portfolio combinations below. The results for the three-asset portfolio do not generalize to the two-asset case because our optimal portfolio weight and marginal-effect equations cannot constrain short selling. To overcome this shortcoming, we first find the portfolio weights using an optimization program with security weights constrained to non-negative values. This optimization output reveals whether the optimal portfolio combinations include one, two or three assets. The number of positively weighted assets in the optimal portfolio determines which of the marginal effects equations are used.

The optimal portfolio weight in stock w_S^* can be found by setting $\partial U/\partial w_S = 0$ and solving for w_S. First the portfolio return and standard deviation must be put into more manageable formats before substituting into Eq. (1). In the two-asset case, $(1\text{-}w_S)$ is substituted in for w_B in Eqs. (2) and (3). In the three-asset case, substituting $(1\text{-}w_S\text{-}w_C)$ in for w_B into Eqs. (2) and (3) puts the problem in terms of just two unknowns. The resulting two-asset (three-asset) portfolio expected return and standard deviation are shown by Eqs. (4), (4a), 5, and (5a), respectively:

$$r_P = w_S r_s + (1 - w_S) r_B \tag{4}$$

$$r_P = w_S r_S + (1 - w_S - w_C) r_B + w_C r_C \tag{4a}$$

$$\sigma_P^2 = w_S^2 \sigma_S^2 + (1 - w_S)^2 \sigma_B^2 + 2 w_S (1 - w_S) \sigma_{SB} \tag{5}$$

$$\sigma_P^2 = w_S^2 \sigma_S^2 + (1 - w_S - w_C)^2 \sigma_B^2 + w_C^2 \sigma_C^2 + 2 w_S (1 - w_S - w_C) \sigma_{SB} + 2 w_S w_C \sigma_{SC} + 2(1 - w_S - w_C) w_C \sigma_{BC}. \tag{5a}$$

The covariances σ_{IJ} of the asset pairs are determined by the individual variability of the assets and their correlations with each other:

$$\sigma_{IJ} = \rho_{IJ} \sigma_I \sigma_J.$$

The optimal portfolio weight in stock for the two-asset scenario is then calculated as

$$w_S^* = \frac{r_S - r_B + A(\sigma_B^2 - \rho_{SB}\sigma_S\sigma_B)}{A(\sigma_S^2 + \sigma_B^2 - 2\rho_{SB}\sigma_S\sigma_B)} \tag{6}$$

and the optimal portfolio weight in bonds w_B^* is naturally $1 - w_S^*$. For the three-asset portfolio the optimal weight in stock is

$$w_S^* = \frac{[r_S - r_B + A(\sigma_B^2 - \sigma_{SB})]Y - [r_C - r_B + A(\sigma_B^2 - \sigma_{BC})]Z}{A(XY - Z)^2} \tag{6a}$$

where

$$X = (\sigma_S^2 + \sigma_B^2 - 2\sigma_{SB})$$

$$Y = (\sigma_B^2 + \sigma_C^2 - 2\sigma_{BC})$$

$$Z = (\sigma_B^2 - \sigma_{SB} + \sigma_{SC} - \sigma_{BC}).$$

We next find what impact a change in the return on stocks r_S has on the optimal portfolio weight in stocks w_S^*, which is given by $\partial w_S^*/\partial r_S$. This marginal effect shows how much of a change in the optimal weight in stock will occur as a result of a one unit increase in the expected return of stock. We derive solutions for the two-asset case in Eq. (7) and the three-asset case in Eq. (7a):

$$\frac{\partial w_S^*}{\partial r_S} = \frac{1}{A(\sigma_S^2 + \sigma_B^2 - 2\rho_{SB}\sigma_S\sigma_B)} > 0 \tag{7}$$

$$\frac{\partial w_S^*}{\partial r_S} = \frac{Y}{A(XY - Z^2)} > 0. \tag{7a}$$

Higher values for r_S lead to increased optimal weights for stock in the portfolio. The degree of the increase in portfolio weight is dependent upon the variances and covariances of security returns and the risk aversion of the individual investor.

To find the marginal impact of changes in ρ_{SB} on the optimal portfolio weight in stock, we find $\partial w_S^*/\partial \rho_{SB}$, which gives us two and three-asset results in Eqs. (8) and (8a), respectively:

$$\frac{\partial w_S^*}{\partial \rho_{SB}} = \frac{\sigma_S\sigma_B}{\sigma_S^2 + \sigma_B^2 - 2\rho_{SB}\sigma_S\sigma_B}\left[\frac{2[r_S - r_B + A(\sigma_B^2 - \rho_{SB}\sigma_S\sigma_B)]}{A(\sigma_S^2 + \sigma_B^2 - 2\rho_{SB}\sigma_S\sigma_B)} - 1\right] \tag{8}$$

$$\frac{\partial w_S^*}{\partial \rho_{SB}} = \frac{2\sigma_S\sigma_B Y(Y - Z)[r_S - r_B + A(\sigma_B^2 - \sigma_{SB})] - A\sigma_S\sigma_B Y(XY - Z^2)}{A(XY - Z^2)^2} + \frac{\sigma_S\sigma_B[(XY\text{-}Z^2) + 2\sigma_S\sigma_B Z(Z - Y)][r_C - r_B + A(\sigma_B^2 - \sigma_{BC})]}{A(XY - Z^2)^2} \tag{8a}$$

This shows us the changes in the optimal portfolio weight because of a change in the stock-bond correlation. The marginal effect is determined by the variances and covariances of security returns, the risk aversion parameter of the individual investor, the stock risk premium ($r_S - r_B$), and the current stock-bond correlation.

The portfolio impacts on w_B^* are shown by $\partial w_B^*/\partial r_S$ and $\partial w_B^*/\partial \rho_{SB}$. In the 2-asset case, $\partial w_B^*/\partial r_S$ is equal to $-\partial w_S^*/\partial r_S$, and $\partial w_B^*/\partial \rho_{SB}$ is equal to $-\partial w_S^*/\partial \rho_{SB}$. When the expected returns on stock change, so do the optimal weights on stocks and bonds. In the 2-asset case, the increase (decrease) in the bond weight should be identical to the decrease (increase) in the stock weight. With three assets in the portfolio, a change in the weight of stock can be offset by changes in both bonds and bills.

4. Optimal portfolio mix and marginal effects

Table 2 shows optimal portfolio weights for stocks, bonds, and bills for different levels of investor risk aversion. It also reveals the marginal effects of changes in estimates of r_S and ρ_{SB} on w_S^* and w_B^*. The results are presented for r_S of 8, 10, and 12% with ρ_{SB} held at 0.1 (approximately its historical average). All other input items are fixed at the historical levels shown in Table 1. To aid in interpretation, $\partial w_S^*/\partial r_S$ and $\partial w_B^*/\partial r_S$ are each divided by 100 to reveal the approximate changes in the respective portfolio weights of stocks and bonds per 1% change in r_S. $\partial w_S^*/\partial \rho_{SB}$ and $\partial w_B^*/\partial \rho_{SB}$ are scaled by a factor of 10 to show the approximate changes in w_S^* and w_B^* per 0.1 change in ρ_{SB}. Each panel in Table 2 shows the portfolio combinations that form the efficient frontier based on the given input assumptions. The panel with r_S = 12%, near its historical average, serves as a frame of reference. Optimal portfolio weights for highly risk-averse investors ($A = 10$) are 20.3% stocks, 21.8% bonds, and 57.9% bills. As we would expect, the lower the degree of risk aversion, the higher the optimal allocation to stocks. The optimal portfolio combination for investors with a moderate level of risk aversion ($A = 3$) is 58.4% stocks and 41.6% bonds. This is near the classic 60-40, stocks-bonds mix. Aggressive investors ($A = 1$) would forego bonds and bills entirely and hold 100% stocks.

4.1. Marginal effects of the expected return on the optimal mix

Examination of Table 2 shows that forecast changes for r_S significantly affect the optimal asset mix, but the degree of the adjustment depends on the investor's level of risk aversion. The more risk-averse the investor, the smaller the change in the portfolio mix for a given change in r_S. A 1% increase in r_S leads to a 21.3% increase in w_S^* for investors with $A = 1$ but only a 2.4% increase in stock for investors with $A = 10$. The more conservative investors are beginning from much lower initial allocations to stock. The portfolio effects because of changes in r_S can be quite dramatic. With $r_S = 12\%$, $\rho_{SB} = 0.1$, and $A = 3$, a decrease in the forecast regarding r_S from 12 to 8% (a 2.2% risk premium over bonds) would mean cutting the allocation to stock nearly in half from 58.4–30.0%. This sizeable reduction in r_S likewise results in an approximately 50% cut in w_S^* for most other values of A, as well. Even conservative investors with $A = 10$ should reduce their stockholdings from 20.3–10.9% in response to the large drop in expected stock returns. So there are considerable differences in the optimal portfolio allocations for both market optimists and pessimists regardless of their degree of risk aversion.

Eqs. (7) and (7a) reveals that $\partial w_S^*/\partial r_S$ is not dependent on expected returns. This observation is reflected in Table 2, where holding A and ρ_{SB} at fixed levels results in constant values for $\partial w_S^*/\partial r_S$. For $A = 3$, for example, a 1% increase in r_S leads to a 7.1% increase in w_S^*, regardless of whether r_S is 8% or 12%.

Increases in r_S would naturally be expected to reduce w_B^*, and this is what we observe. $\partial w_B^*/\partial r_S$ is a constant negative amount for each level of risk aversion, regardless of the value of r_S. For less risk-averse investors with $A \leq 3$ where only stocks and bonds are held, $\partial w_B^*/\partial r_S$ is equal to $-\partial w_S^*/\partial r_S$. With only two assets, any increase in stockholdings must be exactly offset by a decrease in the bond position. For more risk-averse investors with $A \geq 5$ who are holding stocks, bonds, and bills, the decrease in bonds is not proportional to the increase in stock. In fact, most of the offsetting decreases in portfolio weights associated with the higher stock weight occur in bills rather than bonds. For $A = 5$, for example, $\partial w_S^*/\partial r_S$ equals 0.047, and $\partial w_B^*/\partial r_S$ equals -0.013.

4.2. Marginal effects of the stock-bond correlation on the optimal mix

With a base assumption of $\rho_{SB} = 0.1$, the marginal effect of a change in ρ_{SB} on w_S^* is relatively small in most cases across the entire range of values for r_S, especially compared to the marginal effects associated with changes in r_S. The largest relevant value for $\partial w_S^*/\partial \rho_{SB}$ that is observed in Table 2 is 4.4% when $A = 1$ and $r_S = 10\%$. For $A = 1$ and $r_S = 12\%$, neither a 0.1 change in ρ_{SB} nor a 1% change in r_S is significant enough to change the optimal allocation from 100% stock. Unconstrained investors with these expectations would have large short positions in bonds or bills and stock allocations in excess of 100%. The effect of changes in ρ_{SB} on w_B^* is, however, quite significant when the optimal portfolio combinations include stocks, bonds, and bills. With $A = 5$ and $r_S = 12\%$, for example, a 0.1 increase in ρ_{SB} leads to a roughly 8% reduction of the portfolio weight in bonds. $\partial w_B^*/\partial\rho_{SB}$ generally declines both with increases in r_S and with decreases in A. Higher expected stock risk premiums make stock more attractive in comparison to bonds, and increases in the correlation make bonds even less appealing. Lower levels of risk aversion likewise imply a greater emphasis on return rather than lowering risk through diversification, thereby making bonds less important in the portfolio.

We find the sign of $\partial w_S^*/\partial\rho_{SB}$ to be either positive or negative depending on r_S and A. For $A = 3$ and $r_S = 8\%$, for instance, a 0.1 increase in the ρ_{SB} forecast would lead to a 1.6% reduction in w_S^*. But for $A = 3$ and $r_S = 12\%$, a 0.1 increase in ρ_{SB} means a 0.7% increase in w_S^*. Looking across the rows in Table 2 shows that for a given level of r_S, investors with low risk aversion will hold just stocks or a mix of stocks and bonds. As risk aversion increases, bills eventually enter into the optimal portfolio mix. As A increases from that point on, w_S^* decreases, approaching the weight of stock in the minimum variance portfolio, and $\partial w_S^*/\partial\rho_{SB}$ approaches 0 from below. In other words, higher levels of risk aversion mean smaller optimal allocations to stock, but only to a certain point. With the smaller stock weights held by more conservative investors, an increase in correlation would still decrease w_S^*, but the change would be negligible.

The stock risk premium (r_S-r_B) is conspicuous in the solution to $\partial w_S^*/\partial\rho_{SB}$. Holding A constant by looking down the columns in Table 2 reveals that $\partial w_S^*/\partial\rho_{SB}$ declines monotonically with reductions in r_S. For less risk averse investors holding only stocks and bonds, an increase in ρ_{SB} means a choice must be made between the two assets. In most of the scenarios, increasing the portfolio weight of stock is seen as the more attractive alternative. Of course this is dependent upon the stock risk premium being sufficiently high. For optimal portfolio combinations that include stocks, bonds, and bills, an increase in ρ_{SB} negatively impacts the diversification benefits of the stock-bond combination in general. This typically leads to reduced optimal portfolio weights for both stocks and bonds, with bills picking up the slack. As noted previously, the bulk of the reductions in portfolio weights are registered in the bond, rather than stock, position.

Table 3, which holds r_S fixed at 12%, examines the effect of the stock-bond correlation level on the optimal portfolio composition and the marginal effects thereof because of changes in r_S and ρ_{SB}. With r_S near its historical average, changes in expected ρ_{SB} do not generally have a large effect on w_S^*, regardless of the assumed initial level for ρ_{SB}. For most investors, however, $\partial w_B^*/\partial\rho_{SB}$ is heavily dependent upon the stock-bond correlation assumption. At higher expected values of ρ_{SB}, the riskiness of the stock-bond combination increases, and bills become a preferred alternative, primarily replacing bonds. For instance with $A = 5$ and $\rho_{SB} = -0.1$, the optimal portfolio combination is 42.2% stocks and 57.8% bonds. With this expectation of negative correlation, the stock-bond combination has a comparatively low level of risk. $\partial w_S^*/\partial\rho_{SB}$ is equal to -0.005, and $\partial w_B^*/\partial\rho_{SB}$ is 0.005. If ρ_{SB} increases from -0.1 to 0.3, the optimal portfolio mix shifts to 36.9% stock, 24.2% bonds, and 38.9% bills. While the weight in stocks does decrease somewhat, the most significant portfolio change is

Table 3
Optimal portfolio allocations and marginal effects (expected return on stock equal to 12%)

ρ_{SB}		A				
		1	2	3	5	10
0.3	w_S^*	1.000	0.862	0.600	0.369	0.196
	w_B^*	0.000	0.138	0.390	0.242	0.131
	w_C^*	0.000	0.000	0.011	0.389	0.673
	$\partial w_S^*/\partial r_S$	0.255[a]	0.127	0.086	0.052	0.026
	$\partial w_B^*/\partial r_S$	−0.255[a]	−0.127	−0.063	−0.038	−0.019
	$\partial w_S^*/\partial \rho_{SB}$	0.112[a]	0.036	0.002	0.001	0.001
	$\partial w_B^*/\partial \rho_{SB}$	−0.112[a]	−0.036	−0.138	−0.085	−0.045
0.1	w_S^*	1.000	0.803	0.584	0.382	0.203
	w_B^*	0.000	0.197	0.416	0.406	0.218
	w_C^*	0.000	0.000	0.000	0.212	0.579
	$\partial w_S^*/\partial r_S$	0.213[a]	0.106	0.071	0.047	0.024
	$\partial w_B^*/\partial r_S$	−0.213[a]	−0.106	−0.071	−0.013	−0.007
	$\partial w_S^*/\partial \rho_{SB}$	0.079[a]	0.025	0.007	−0.014	−0.007
	$\partial w_B^*/\partial \rho_{SB}$	−0.079[a]	−0.025	−0.007	−0.083	−0.044
−0.1	w_S^*	1.000	0.760	0.572	0.422	0.226
	w_B^*	0.000	0.240	0.428	0.578	0.315
	w_C^*	0.000	0.000	0.000	0.000	0.460
	$\partial w_S^*/\partial r_S$	0.183[a]	0.091	0.061	0.037	0.023
	$\partial w_B^*/\partial r_S$	−0.183[a]	−0.091	−0.061	−0.037	0.004
	$\partial w_S^*/\partial \rho_{SB}$	0.058[a]	0.018	0.005	−0.005	−0.016
	$\partial w_B^*/\partial \rho_{SB}$	−0.058[a]	−0.018	−0.005	0.005	−0.054

All mean-variance input factors other than r_S and ρ_{SB} are at their historical levels, as shown in Table 1.
$\partial w_S^*/\partial r_S$ and $\partial w_B^*/\partial r_S$ are divided by 100 to show approximate change per a 1% change in return.
$\partial w_S^*/\partial \rho_{SB}$ and $\partial w_B^*/\partial \rho_{SB}$ are divided by 10 to show the approximate change per a 0.1 change in correlation.
(a) The marginal change is not sufficient to affect the portfolio weights. The optimal allocation will still be 100% stock.

between bonds and bills. At higher levels of ρ_{SB}, the stocks-bills combination, with an assumed correlation of −0.016, provides risk and return benefits superior to that of stocks and bonds. With ρ_{SB} at 0.3, $\partial w_S^*/\partial \rho_{SB}$ is equal to 0.001, and $\partial w_B^*/\partial \rho_{SB}$ is a sizeable −0.085. Only for aggressive investors with $A = 1$ does the expected correlation not matter—they will hold 100% stock regardless. For the case with $A = 1$, $\partial w_S^*/\partial \rho_{SB}$ and $\partial w_B^*/\partial \rho_{SB}$ are actually heavily influenced by the level of ρ_{SB}, but the short sale constraint results in optimal portfolio weights of 100% stock.

Table 3 also reveals that $\partial w_S^*/\partial r_S$ increases for all investors with increases in ρ_{SB}, but more so for less risk-averse investors. For instance, with $A = 3$ and $\rho_{SB} = -0.1$, $\partial w_S^*/\partial r_S$ is 0.061 compared to 0.086 with $\rho_{SB} = 0.3$. With $A = 5$ and $\rho_{SB} = -0.1$, $\partial w_S^*/\partial r_S$ is 0.037 versus 0.052 with $\rho_{SB} = 0.3$. Likewise, $\partial w_B^*/\partial r_S$ decreases as ρ_{SB} increases. With higher stock-bond correlations, the 12% historical mean return of stock (a 6.2% premium over bonds) makes it a more attractive choice than bonds.

5. Conclusion

Mean-variance analysis allows investors to select optimal portfolio combinations specific to their risk tolerance levels. Unfortunately, the resulting asset allocation decisions are heavily dependent upon the input assumptions. The expected risk premium for stocks versus bonds and the stock-bond correlation are two crucial inputs that have drawn widespread scrutiny lately. By examining a mixed portfolio of stocks, bonds, and bills, we have shown that changes in these two factors can significantly impact the asset allocation decision. The extent of the portfolio consequences depends considerably on the degree of risk aversion of the individual investor, so a firm grasp on the investor attitude toward risk is essential.

Increases in the expected return on stock will always increase the optimal portfolio allocation to stock, but the impact is much more pronounced for less risk-averse investors. While aggressive investors might make large portfolio adjustments if stock return forecasts change, more conservative investors should be less inclined to do so. Likewise as the risk premium on stock increases, bonds become less appealing to investors and their portfolio weights drop. For investors with less risk tolerance, increases in stock returns actually result in optimal portfolio weights for bills dropping more than those of bonds. For a given correlation between stock and bond rates of return, the marginal effect of a change in the expected stock return on optimal portfolio weights remains constant as the expected return changes.

For the most part, changes in the stock-bond correlation have a relatively small impact on portfolio allocations to stock. For investors with lower levels of risk tolerance, however, changes in the stock-bond correlation do result in a significant shift between bonds and bills. As the stock-bond correlation increases, these investors should move funds out of bonds and into bills. The marginal effects associated with changes in the stock-bond correlation are not fixed as the stock-bond correlation changes. Increases in the stock-bond correlation increase the magnitude of the impact (positively or negatively) of changes in the stock-bond correlation on the portfolio weights of both stocks and bonds.

An average investor who had been holding a traditional portfolio mix of 60% stocks and 40% bonds based on historical expectations should certainly make portfolio adjustments because of changes in the forecasts of stock returns. For every 1% reduction in the forecast of the stock-bond risk premium relative to the historical average (about 6%), the optimal allocation to stocks for the average investor falls by about 7%. On the other hand, an increase in the expected stock-bond correlation by 0.1 would have only minimal effects on the optimal portfolio allocation. Investors starting from different optimal portfolio mixes based on their level of risk aversion would rebalance their portfolios differently in response to changes in stock returns and the stock-bond correlation.

Investors using mean-variance analysis need to understand the implications of differing forecasts of input items on the asset allocation decisions they make. While there is considerable room for debate and disagreement regarding the expected returns on stock and the stock-bond correlation (and all of the other input factors), there is science behind the process. Once certain assumptions are made, asset allocation decisions are a natural byproduct. A better appreciation of how all of these mean-variance factors interrelate should improve the decision-making process.

Notes

1. Discussion of this utility function can be found in investments textbooks such as Bodie, Kane, and Marcus (2005).
2. The portfolio weights come from Table 2, which is discussed later.

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