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Stochastic optimization of retirement portfolio asset allocations and withdrawals

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Abstract

Stochastic optimization identifies the asset allocation that minimizes the probability of exhausting the retirement portfolio, thereby minimizing risk, from unmanaged (constant) and optimally managed withdrawals over the retirement life span. Optimal equity compositions and minimized probabilities of prematurely exhausting the portfolio increase with higher withdrawal rates and earlier retirements with both managed and unmanaged withdrawals. However, optimal withdrawal management from optimally managed portfolios reduces the sensitivity of premature portfolio exhaustion to higher initial withdrawal rates or earlier retirements, thereby reducing the increase in the risk of exhausting the portfolio necessary to support the improved lifestyles from higher withdrawals, longer retirements, or both. © 2008 Academy of Financial Services. All rights reserved.

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1. Introduction

The success of a retirement portfolio is one of a set of problems in which the objective function cannot be evaluated precisely because it depends upon uncertain independent variables. In such problems, Monte Carlo simulation is widely accepted as a means of estimating the objective function by randomly generating values for uncertain outcomes from known or hypothesized distributions of input variables. The outputs from Monte Carlo simulation are statistical distributions of one or more output variables, and although based

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upon thousands of randomly generated values of input variables, the outputs retain their random characterization because they are a function of the randomness in the distribution sampling process.

Stochastic optimization refers to the optimization of a function in the presence of randomness in the input data or the optimization algorithm. Stimulated by the acceptance of Monte Carlo simulation, some simulation packages now include stochastic optimization procedures that recognize simulated outcomes as random inputs to an optimization algorithm. The technique is commonly applied in the physical and natural sciences, but it has not been applied in retirement portfolio and withdrawal management.

Some retirement withdrawal planning models or distribution planning models have employed Monte Carlo simulation to examine the sustainability of a constant stream of inflation-adjusted withdrawals from a retirement portfolio subject to uncertain portfolio returns and uncertain remaining retiree lifetimes, and more recently, Monte Carlo simulation has been used to examine the effects of withdrawal management decision rules in variable withdrawal models. Outputs from these simulations include the distribution of the probability of running out of money before death, the distribution of ending portfolio values, and, in variable withdrawal rate models, the distribution of withdrawal rates.

This paper applies stochastic optimization in a Monte Carlo simulation model to identify the optimal portfolio allocations for minimizing the probabilities of depleting the retirement portfolio before death, oftentimes termed portfolio ruin or shortfall risk, from constant inflation-adjusted withdrawals. Optimal portfolio allocations and minimum probabilities of portfolio ruin are regressed on the withdrawal rate and age at retirement to more clearly quantify the impacts of earlier retirement, higher withdrawal rates, or both, on optimum portfolio allocations and the minimum probability of ruin.

Monte Carlo stochastic optimization is also applied in a model of variable withdrawal rates to identify both the optimal portfolio allocations and the optimal withdrawal rate decision rule parameters for minimizing the probabilities of portfolio ruin. Optimal portfolio allocations and minimized probabilities of portfolio ruin are regressed on the initial withdrawal rate and age at retirement to quantify the impacts of earlier retirement, higher initial withdrawal rates, or both, on the minimum probability of ruin from optimally managed withdrawals from optimum portfolios.

The withdrawal management process, indicative of flexibility in withdrawals, improves the performance of optimum retirement portfolios by reducing the probability of ruin, increasing the lifetime average withdrawal rates, and reducing the excess portfolio accumulation. Furthermore, the withdrawal management process reduces the sensitivities of optimal equity allocations, and therefore probabilities of portfolio ruin, to higher initial withdrawal rates and earlier retirements.

This research makes several important contributions to the financial services literature. The first and perhaps the most important contribution is the application of stochastic optimization to a financial services problem typified by an uncertain objective function. More specifically, the contribution to the retirement and distribution planning literature is the discovery of a stochastically optimal asset allocation for minimizing the probability of exhausting the retirement portfolio over the uncertain retirement life span, which, in turn, enables the specification of the relationships between the optimal asset allocation, the minimum probability portfolio ruin, the withdrawal rate, and the age at retirement. Therefore, it adds quantitative, but stochastic, precision to prior literature that estimates the optimal retirement portfolio asset allocation by examining a limited number of combinations of portfolio asset allocations and key independent variables. In addition, this research specifically extends Stout and Mitchell (2006) and Milevsky, Ho and Robinson (1997) by identifying the optimal portfolio allocations for unmanaged and for optimally managed withdrawals and more clearly specifying the sensitivities of the probability of retirement portfolio failure and optimal portfolio allocations to the retirement age and withdrawal rate.

2. Literature review

Although many authors have explored the sustainability of inflation-adjusted withdrawals from a retirement portfolios for a fixed number of years (payout period), relatively fewer have examined the sustainability of constant inflation-adjusted withdrawals over the uncertain retirement life span. Lacking a methodology or avoiding the mathematical complexity for determining the optimal portfolio for a given withdrawal rate and planning horizon, these studies have generally reported combinations of inflation-adjusted withdrawals and portfolio allocations. Milevsky and Robinson (2005) assume an exponential distribution of mortality calibrated to the median life span from (Canadian) mortality tables. This assumption allows a closed-form (finite) solution for the probability of ruin with continuous compounding of lognormal returns. They argue that the solution can be stochastically approximated by the reciprocal gamma distribution, and they show the impact of the portfolio asset allocations for a 65-year-old retiree by examining the 35 combinations of seven withdrawal rates and five portfolio allocations.

Even fewer authors have studied the optimal asset allocation in retirement portfolios. Ho, Milevsky and Robinson (1994) present the closed-form mathematical solution for the allocation to equities that minimizes the probability of failing to earn the portfolio return necessary to support the desired inflation-adjusted consumption rate. Assuming that the returns are normally distributed with a mean and standard distribution, respectively, of 8% and 17% to equities, and 2% and 3.5% to T-Bills, they report that the optimal equity allocations to support a 6.25% real withdrawal rate is 71.0% for males and 89.3% for females. Milevsky et al. (1997) report the optimal two-asset retirement portfolio allocations that minimize the probability of living beyond the stochastic number of periods of portfolio survival for five retirement ages (from 55 to 75) and eight withdrawal rates (from 3.7% to 10%). They employ Monte Carlo simulation of lognormal (Canadian) asset returns and probabilities of survival from (Canadian) mortality tables to conclude that, except for the earliest retirement ages and highest withdrawal rates, the probability of ruin is relatively insensitive to the allocation to equities so long as the equity allocation is substantial (greater than 50%).

Some authors explicitly model withdrawal correction mechanisms for portfolios showing signs of impending failure. These models generally show that the flexibility to lower the withdrawal rate in response to market underperformance improves the sustainability of withdrawals over a fixed planning horizon, and Stout and Mitchell (2006) extend those results to an uncertain retirement life span. Their simulation results of a single portfolio of 65% equities and 35% intermediate-term U.S. government bonds show that withdrawal rate reductions, induced by either market underperformance or superannuation, can be combined with affordable withdrawal rate increases to improve the average lifetime withdrawal rate, reduce the probability of ruin, and reduce excessive terminal portfolio balances.

3. Research design and data

Monte Carlo simulation is employed to generate expected values for the uncertain probability of exhausting the retirement portfolio during the retirement life span, subsequently termed the probability of ruin. Each simulation uses Latin Hypercube[1] sampling and consists of 50,000 iterations (portfolio outcomes) of annual retirement withdrawals and market returns over the retirement life span. The output from a single simulation is the probability of ruin, across 50,000 portfolios, based upon the hypothesized distributions of market returns and retiree aging.

Simulated market returns are based on annual data from Ibbotson (2007), and asset-class return relatives, (1 + return), are log-normally distributed. The log-normal assumption is widely accepted, and Cooley, Hubbard and Walz (2003) show that simulated returns generate approximately the same results as actual historic returns for reasonable withdrawal rates (4–8%), portfolio allocations (50–75% equities), and payout periods (greater than 20 years). The arithmetic mean inflation-adjusted returns to large-cap stocks and intermediate-term U.S. government bonds over the years 1926 to 2006 are 9.1% and 2.4%, respectively, and the standard deviations of the inflation-adjusted returns are 20.2% and 6.8%, respectively. The historic correlation between the returns to large-cap stocks and intermediate-term bonds is 0.14, and that correlation is retained in simulated returns. Similarly, the historic one-year serial correlation in intermediate-term bond returns, 0.23, is maintained in the simulated intermediate-term bond returns. The randomly generated asset-class returns exhibit the same distributional properties and correlations as the historic returns, and the Kolmogorov-Smirnov goodness of fit test concludes that the distributions of simulated asset-class returns are *not* different from the distribution of actual historic asset-class returns. The simulated returns, descriptive of historic asset-class returns, are assumed to represent future asset-class returns, as well. The simulated returns are to, and withdrawals are from, tax-deferred accounts. Portfolios are assumed to be rebalanced annually and transactions costs are ignored.

Retiree aging is simulated from the year-by-year binomial conditional probabilities of living another year. Probabilities are from the mortality table for the total (unisex) population from the Centers for Disease Control, National Center for Health Statistics for the total U.S. population (The Centers for Disease Control, 2002).

The mean probability of portfolio ruin before death across 20 simulations of 50,000 portfolios is the input to the algorithm to stochastically optimize (minimize) the simulated mean probability of ruin.[2] The optimization algorithm employs the Box method of stochastic optimization.[3] The Box method begins by developing an initial input value or set of input values that impact the mean probability of ruin. The algorithm then iteratively searches the possible input values to reduce the simulated mean probability of ruin, replacing the least optimal input values with more optimal values, until the mean probability of ruin converges to a single minimum solution, if such a (global) minimum exists.

The optimal input values are then used in a second set of Monte Carlo simulations to estimate the minimum mean probability of portfolio ruin. These simulations are performed 40 times with 50,000 iterations each.[4]

This research applies Monte Carlo stochastic optimization to two withdrawal planning models. The first model is characterized by constant (unmanaged) inflation-adjusted, beginning-of-the-year withdrawals from a balanced portfolio of equities (large-cap common stocks) and intermediate-term U.S. government bonds over the uncertain retirement life span. Stochastic optimization is employed in this model to identify the portfolio allocation to equities that minimizes the probability of portfolio ruin. The second model is characterized by variable, but managed, inflation-adjusted withdrawals. Withdrawal rate changes are managed by illustrative decision rules patterned after Stout and Mitchell (2006). The decision rules exemplify withdrawal rate flexibility and are designed to reduce the withdrawal rate if portfolio sustainability is threatened by market underperformance or superannuation and to increase the withdrawal rate if excess portfolio accumulation or longevity is assured by market overperformance. Stochastic optimization techniques are used in the second model to identify both the optimal portfolio allocation and the optimal withdrawal management decision rule parameters to minimize the probability of ruin over the uncertain remaining life span.

3.1. Constant inflation-adjusted withdrawals until death

The first model assumes that all retirees begin their retirement with an initial withdrawal from the retirement portfolio. The after-withdrawal remainder of the portfolio is invested in the mix of large-cap common stocks and intermediate-term U.S. bonds and earns the inflation-adjusted rate of return, appropriately weighted to the portfolio composition, until the next annual withdrawal. In a discrete time setting, the portfolio rate of return is:

$$R_{t,i} = (Wt_E)E_{t,i} + (1 - Wt_E)B_{t,i} \quad (1)$$

where $R_{t,i}$ is the weighted average portfolio return for simulation number i at time t, Wt_E is the portfolio fraction allocated to equities, $E_{t,i}$ is the inflation-adjusted rate of return to large-cap equities for iteration (portfolio) number i at time t, and $B_{t,i}$ is the inflation-adjusted rate of return to intermediate bonds for simulation number i at time t.

Subsequent withdrawals are always the same inflation-adjusted amount from the portfolio measured in inflation-adjusted dollars. Hence, the value of the portfolio is given by Eq. (2) below:

$$V_{t,i} = [V_{t\text{-}1,i} - (W)\ V_0]\ (1 + R_{t\text{-}1,i}), \quad (2)$$

where V designates the value of the portfolio and w is the constant withdrawal fraction.

The model contains two conditional variables, $ALIVE_t$ and $RUIN_t$, which are either true or false. After the first year, $ALIVE_t$ takes its value from a randomly drawn number from the

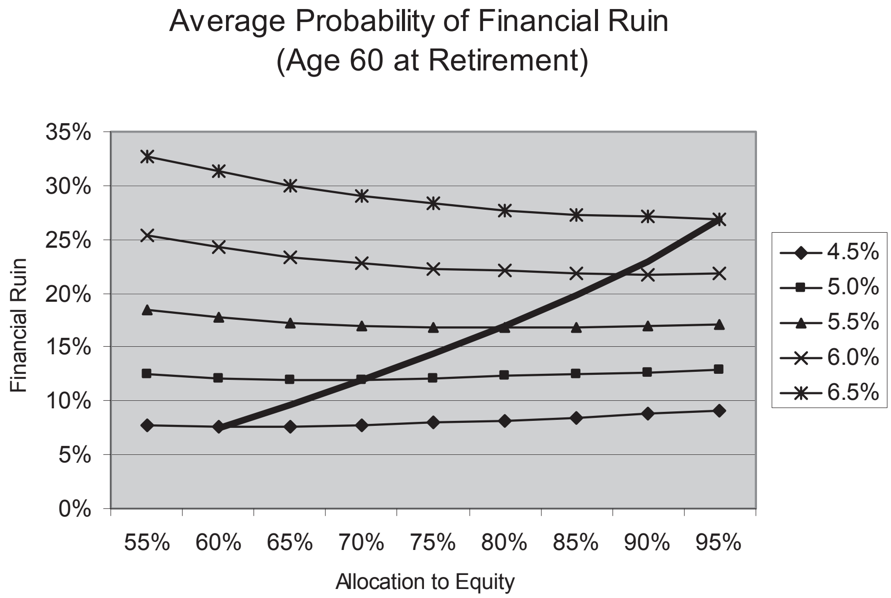

Fig. 1. The relationship between the average probability of ruin and the retirement portfolio allocation to equities with withdrawal rates of 4.5%, 5.0%, 5.5%, 6%, and 6.5% for a 60-year-old retiree.

binomial conditional probability of living to year t, having already lived to year (t-1). A withdrawal is attempted from the portfolio only if $ALIVE_t$ is true. $RUIN_t$ takes its value to indicate portfolio ruin (true) if $V_{t,i} - (W)\ V_0 < 0$. With the addition of these conditional variables, the first model is summarized in Eqs. (3) and (4) below:

$$V_{t,i} = [V_{t-1,i} - (W)\ V_0]\ (1 + R_{t-1,i}) \text{ if } ALIVE_t = \text{true and } RUIN_t = \text{false}, \tag{3}$$

otherwise

$$V_{t,i} = V_{t-1,i} \tag{4}$$

The t subscript increments annually to 101 minus the initial retirement age because the mortality table is truncated at age 101, causing $ALIVE_{101,i}$ to always be false. The i subscript increments from 1 to the 50,000 iterations (retirees or portfolios) of the simulation.

A simulation of 50,000 iterations generates a single probability of ruin (the number of ruined portfolios/50,000) that is the outcome from the portfolio allocation, the withdrawal rate, the age at retirement, the stochastic inflation-adjusted portfolio return, and the stochastic occurrence of death. Across a number of simulations, the probability of ruin is a stochastic variable, possessing the common statistics that describe its distribution. For a person retiring at age 60, Fig. 1 shows the average probability of portfolio ruin as a function of the withdrawal rate and the portfolio asset allocation to equities. The average probability of ruin is computed across 20 simulations of 50,000 iterations each. A visual approximation of relationship between the minimum average probability of ruin, the withdrawal rate, and the allocation to equities is added.

Monte Carlo stochastic optimization is employed to identify the portfolio allocation to equities (Wt_E) that minimizes the probability of ruin across 50,000 iterations (portfolios).

Table 1 Constant inflation-adjusted withdrawals until death

Retirement age	Inflation-adjusted withdrawal rate (%)	Optimal Wt_E (%)	Average probability of ruin (%)	Standard deviation average probability ruin (%)	Average end portfolio (multiple)	Standard deviation average end portfolio (multiple)
55	4.25	66.81	8.56	0.15	3.02	0.023
	4.50	69.73	10.91	0.18	3.04	0.026
	4.75	73.19	13.38	0.15	3.17	0.036
	5.00	76.52	16.01	0.15	3.20	0.030
	5.25	79.00	18.75	0.16	3.21	0.033
	5.50	82.56	21.45	0.24	3.32	0.034
	5.75	86.55	24.13	0.17	3.49	0.055
	6.00	89.59	26.97	0.26	3.57	0.065
	6.25	91.91	29.67	0.20	3.58	0.064
	6.50	94.58	32.37	0.26	3.67	0.074
60	4.25	55.38	5.68	0.20	1.84	0.013
	4.50	59.80	7.52	0.11	2.00	0.010
	4.75	64.18	9.64	0.10	2.00	0.009
	5.00	68.95	11.92	0.10	2.10	0.013
	5.25	75.00	14.37	0.09	2.22	0.013
	5.50	79.41	16.75	0.13	2.31	0.020
	5.75	83.45	19.30	0.20	2.37	0.026
	6.00	87.00	21.90	0.20	2.54	0.036
	6.25	91.78	24.32	0.20	2.76	0.044
	6.50	95.29	26.72	0.26	2.76	0.039
65	4.25	50.25	3.25	0.15	1.43	0.011
	4.50	53.82	4.55	0.15	1.44	0.007
	4.75	58.62	6.09	0.16	1.43	0.012
	5.00	61.10	7.78	0.12	1.44	0.009
	5.25	64.58	9.73	0.11	1.45	0.011
	5.50	67.16	11.88	0.16	1.43	0.015
	5.75	70.51	13.95	0.17	1.44	0.013
	6.00	74.78	16.20	0.18	1.47	0.017
	6.25	79.81	18.58	0.22	1.53	0.017
	6.50	83.20	20.77	0.17	1.54	0.013
Retirement age	**Inflation-adjusted withdrawal rate (%)**	**Nonoptimal Wt_E (%)**	**Average probability of ruin (%)**	**Standard deviation average probability ruin (%)**	**Average end portfolio (multiple)**	**Standard deviation average end portfolio (multiple)**
55	5.00	65.00	16.41	0.22	2.28	0.022
55	5.00	85.00	16.11	0.15	4.07	0.045
55	6.00	79.00	27.85	0.14	2.55	0.035
55	6.00	99.00	27.69	0.22	4.73	0.088
65	5.00	50.00	8.04	0.14	1.15	0.006
65	5.00	70.00	7.89	0.12	4.72	0.015

Optimal portfolio allocations to large-cap stocks and simulation results of the distributions of the probability of ruin and ending portfolio value. Several nonoptimal allocations are shown for comparison.

The stochastically optimal portfolio allocation is then applied in 40 simulations of 50,000 iterations each. Simulated results of optimal portfolio performance are presented in Table 1.

Table 2 Constant inflation-adjusted withdrawals until death

Dependent variable	Independent variable coefficients			Statistics	
	Constant	Retirement age	Inflation-adjusted withdrawal rate (%)	R^2	F
Optimal Wt_E (%)	81.824 (6.571)	−1.466 (0.097)***	14.997 (0.554)***	0.973	480.0***
Average Probability of Ruin (%)	19.013 (2.781)	−0.849 (0.041)***	9.378 (0.234)***	0.987	1035.5***

***Indicates statistical significance at the 0.001 confidence level; standard error in parentheses.
Linear regression results based upon Table 1 data.

The optimal equity allocations in Table 1 provide greater precision than prior literature but generally support the proposition that portfolio sustainability over relatively long retirement planning horizons requires aggressive portfolios (50–75% equities). It clearly reveals that greater withdrawal rates require more aggressive portfolio allocations, and the more aggressive allocations subject the portfolio to greater market risk and probabilities of failure, even as ending portfolio values are increasing. The table also provides more precision to the risk of earlier retirement. Earlier retirement also requires more aggressive portfolios to provide withdrawals to death, and the more aggressive portfolios combine with longer and more uncertain life spans to increase the probability of portfolio ruin.

Several nonoptimal portfolio allocation results are shown at the bottom of Table 1 for comparison purposes. Portfolios which are under (over) weighted to equities generate lower (higher) ending portfolio values. Nonoptimal allocations also increase the probability of ruin, but in support of the Milevsky et al. (1997) simulated optimization results, Fig. 1 and Table 1 show that the increases in the probability of ruin caused by deviations from the optimal allocations are small (but statistically significant) for deviations of ±10% from the optimal Wt_E. Greater increases in the probability of ruin are associated with greater combinations of the risks from excessive withdrawal rates and excessive longevity (higher constant withdrawal rates and earlier retirements), so portfolio allocation optimality is most beneficial under these conditions.

The interrelationships among the variables of the model are clarified by the linear regression results presented in Table 2.[5] The ruin-minimizing optimal portfolio allocation to large-cap equities increases about 14.997%/4 = 3.75% for each quarter-point increase in the withdrawal rate and 1.466% per year of earlier retirement. The minimized probability of ruin from the optimal portfolio increases about 9.378%/4 = 2.34% per quarter-point increase in the withdrawal rate and 0.849% per year of earlier retirement.

3.2. Variable, managed, inflation-adjusted withdrawals until death

Model 2 simulates portfolio returns from the same data as model 1, but it incorporates withdrawal rate changes “triggered” by decision rules patterned after Stout and Mitchell (2006).

The foundation of the decision rules is the present value interest factor for an annuity due for the retiree's expected remaining lifetime, L, at the average inflation-adjusted rate of return, Avg r, over historic overlapping time periods of the expected remaining life and portfolio allocation,[6] that is,

$$PVIFADue_{L,Avg\ r} = [1 + Avg\ r - (1 + Avg\ r)^{-(L-1)}] / Avg\ r. \quad (5)$$

This factor has theoretical support for use in personal decision-making, reflects financial history, integrates expected longevity, and provides a locus over time to a desired ending portfolio balance.

The simulation model assumes that the retiree, perhaps under the advice of a professional, selects the initial withdrawal rate based upon the risk-return tradeoff between higher projected lifetime probabilities of portfolio ruin and higher withdrawal rates. The maximum permissible initial withdrawal rate would likely be V_0 / $PVIFADue_{L,Avg\ r}$, using expected returns and expected longevity as though they were known with certainty.

Subsequently, the retiree computes, year-by-year, the portfolio value, V'_t, necessary to sustain the prior year's withdrawal amount, $(W_{t-1})(V_0)$, over the expected remaining lifetime if the portfolio earns the average inflation-adjusted rate of return, Avg r, over historic overlapping time periods of the expected remaining life, L, and portfolio allocation. The required portfolio amount is

$$V'_t = (W_{t-1})(V_0)\ (PVIFADue_{L,Avg\ r}). \quad (6)$$

The sufficiency of the retirement portfolio is monitored based on $PVIFADue_{L,Avg\ r}$, and corrective withdrawal rate changes are initiated at signals of portfolio inadequacy or excess accumulation. Withdrawal rate reductions are managed by a set of decision rules, termed *risk reduction rules*, designed to reduce or delay portfolio ruin. The risk of higher, earlier, or both, withdrawals is a higher probability of portfolio ruin, and the return is the improvement in the retirement lifestyle. Following Stout and Mitchell (2006), a second set of decision rules, termed *improved lifestyle rules*, manage withdrawal rate increases. The improved lifestyle rules are designed to cautiously allow affordable withdrawal rate increases whereas avoiding overreactions that might threaten future portfolio sustainability.

4. Risk reduction rules

If portfolio underperformance causes the value of the retirement portfolio at year t, V_t, to be less than V'_t, the withdrawal rate is reduced to the withdrawal rate that *is sustainable* over the remaining lifetime at the average inflation-adjusted rate of return over historic overlapping time periods of the expected remaining lifetime, $(V_t / PVIFADue_{L,Avg\ r}) / V_0$. Without an absolute minimum withdrawal rate, however, prolonged market underperformance may cause the portfolio-amortizing withdrawal rate to drop below a minimally acceptable level. Given information regarding the choice, the retiree must ultimately determine the maximum standard of living sacrifice he is willing to endure to improve portfolio sustainability. Accordingly, the model assumes that the retiree sets the minimally acceptable withdrawal rate, W_{Min}, as an alternative to portfolio ruin, and the simulation uses this personal minimum withdrawal rate as an absolute withdrawal rate floor. The risk reduction rules are summarized in Eqs. (7) through (9), below:

$$\textit{Risk reduction 1 if } V_t < V'_t, \text{ then } W_t = [V_t / PVIFADue_{L,Avg\ r}] / V_0, \tag{7}$$

$$\textit{Risk reduction 2 if } W_t < W_{Min}, W_t = W_{Min}, \text{ otherwise} \tag{8}$$

$$\textit{Risk reduction 3 if } W_t = W_{t\text{-}1} \tag{9}$$

For illustrative purposes, the minimum acceptable withdrawal rate (W_{Min}) for risk reduction rule 3 is set at two-thirds of the initial withdrawal rate, which limits the lifestyle sacrifice to one-third of the initial withdrawal rate.

5. Improved lifestyle rules

In the event that portfolio underperformance signals that the existing withdrawal rate is not sustainable ($V_t < V'_t$), risk reduction rule 1 forces an immediate and complete recognition of the portfolio deficiency through the withdrawal rate reduction. Similarly, portfolio overperformance may cause the current portfolio value to exceed the value deemed reasonably necessary to sustain the existing withdrawal rate, signaling the potential for a withdrawal rate increase. An immediate and complete increase in the withdrawal rate, however, may threaten the future sustainability of the portfolio as the portfolio lacks any reserve or buffer against future market reversals. To protect against an overreaction that is too quick, the improved lifestyle rules require that unanticipated increases in the retirement portfolio value remain in the portfolio until an adequate reserve, RES, is accumulated to protect the portfolio from subsequent ruin. The required reserve is a multiple of the required portfolio amount necessary to sustain the portfolio, V'_t. Serving as a precondition for any withdrawal rate increase, the reserve requirement limits withdrawal rate increases in response to transitory increases in the portfolio but permits responses to long-term growth trends. For a given initial withdrawal rate, greater portfolio reserves are expected to reduce the probability of ruin or provide greater protection from earlier retirements.

The improved lifestyle rules also protect the survivability of the portfolio from increases in the withdrawal rate that are too complete by permitting only partial adjustment of the withdrawal rate. Upon a signal that portfolio growth has met the reserve precondition for a withdrawal rate increase, the maximum withdrawal rate that the portfolio would sustain is $(V_t/PVIFADue_{L,Avg\ r}) / V_0$. Complete adjustment to the higher withdrawal rate, however, could cause drastic increases in the withdrawal rate. Therefore, to “smooth” the transition to higher withdrawal rate, the simulation model increases the withdrawal rate only a fraction, PART, towards the maximum rate. Higher adjustment fractions would be associated with higher probabilities of ruin as retirees adjust the withdrawal rate more completely in response to abnormally high market returns or shortened longevity, and therefore, more completely reduce the portfolio buffer against subsequent market declines or unexpected longevity. The size of the portfolio reserve and the rate of partial adjustment interact to protect the

Table 3 Variable, managed, inflation-adjusted withdrawals until death

Retirement age	Initial inflation-adjusted withdrawal rate (%)	Optimal Wt_E (%)	Optimal portfolio reserve (RES multiple)	Optimal withdrawal adjustment (PART fraction)
55	4.25	61.83	2.132	0.203
	4.50	63.97	2.176	0.262
	4.75	65.98	2.174	0.263
	5.00	68.28	2.213	0.259
	5.25	70.35	2.352	0.218
	5.50	72.34	2.368	0.278
	5.75	74.54	2.405	0.277
	6.00	77.23	2.519	0.249
	6.25	79.84	2.473	0.295
	6.50	82.52	2.529	0.264
60	4.25	56.75	2.251	0.252
	4.50	60.25	2.481	0.298
	4.75	62.58	2.507	0.259
	5.00	65.38	2.734	0.312
	5.25	68.23	2.800	0.305
	5.50	71.54	2.867	0.248
	5.75	73.35	2.905	0.283
	6.00	76.58	3.045	0.292
	6.25	78.17	3.086	0.288
	6.50	81.01	3.185	0.254
65	4.25	56.69	1.725	0.253
	4.50	61.37	1.676	0.243
	4.75	61.38	1.670	0.300
	5.00	62.77	1.549	0.307
	5.25	65.29	1.540	0.307
	5.50	68.45	1.525	0.225
	5.75	73.24	1.508	0.238
	6.00	77.32	1.494	0.242
	6.25	80.71	1.468	0.245
	6.50	85.62	1.453	0.224

Optimal (ruin minimizing) portfolio allocations, reserve multiples, and partial adjustment fractions if the minimum withdrawal rate is two-thirds of the initial withdrawal rate.

sustainability of the portfolio with lower adjustment fractions providing protection for lower portfolio reserves.

The improved lifestyle rules are presented in Eqs. (10) and (11) below:

$$\textit{Improved lifestyle } 1 \textit{ if } V_t > V'_t\,(1 + \mathrm{RES}), \text{ then } W_t = W_{t-1} + (\mathrm{PART})\,\{[(V_t / \mathrm{PVIFADue}_{L,\mathrm{Avg}\ r}) / V_0] - W_{t-1}\}, \text{ otherwise} \quad (10)$$

$$\textit{Improved lifestyle } 2 \textit{ if } W_t = W_{t-1} \quad (11)$$

Monte Carlo stochastic optimization is employed in the model to identify the portfolio allocation to equities and the two control parameters, RES and PART, which minimize the probability of ruin across 50,000 iterations. The optimal portfolio allocations and control parameters are then employed in 40 simulations of 50,000 iterations each. Optimal param-

Table 4 Variable, managed, inflation-adjusted withdrawals until death

Retirement age	Initial withdrawal rate (%)	Average probability of ruin (%)	Standard deviation average probability ruin (%)	Average end portfolio (multiple)	Standard deviation average end portfolio (multiple)	Average lifetime withdrawal rate (%)	Standard deviation average lifetime withdrawal rate (%)
55	4.25	4.09	0.09	1.43	0.010	6.13	0.017
	4.50	5.24	0.12	1.37	0.008	6.35	0.017
	4.75	6.52	0.10	1.36	0.087	6.49	0.015
	5.00	7.79	0.11	1.38	0.012	6.63	0.021
	5.25	9.11	0.11	1.47	0.010	6.65	0.012
	5.50	10.60	0.08	1.40	0.012	6.86	0.017
	5.75	12.15	0.01	1.43	0.011	7.00	0.015
	6.00	13.66	0.13	1.52	0.018	7.10	0.021
	6.25	15.37	0.13	1.48	0.018	7.36	0.015
	6.50	16.86	0.19	1.57	0.012	7.46	0.014
60	4.25	2.94	0.08	1.15	0.006	5.73	0.015
	4.50	3.75	0.13	1.17	0.008	5.94	0.019
	4.75	4.76	0.16	1.20	0.008	6.04	0.016
	5.00	5.88	0.13	1.19	0.008	6.25	0.017
	5.25	7.08	0.13	1.21	0.007	6.41	0.014
	5.50	8.33	0.14	1.28	0.008	6.51	0.010
	5.75	9.64	0.18	1.25	0.007	6.69	0.013
	6.00	9.71	0.15	1.34	0.012	6.81	0.019
	6.25	12.45	0.18	1.29	0.017	6.97	0.018
	6.50	13.92	0.24	1.34	0.020	7.08	0.020
65	4.25	2.14	0.09	0.95	0.005	5.89	0.021
	4.50	2.85	0.10	0.96	0.004	6.16	0.013
	4.75	3.69	0.11	0.90	0.005	6.29	0.020
	5.00	4.65	0.12	0.86	0.005	6.48	0.014
	5.25	5.67	0.14	0.85	0.006	6.62	0.022
	5.50	6.50	0.18	0.89	0.006	6.75	0.022
	5.75	7.72	0.20	0.90	0.007	7.03	0.019
	6.00	8.90	0.24	0.91	0.007	7.28	0.023
	6.25	10.23	0.24	0.91	0.006	7.50	0.023
	6.50	11.66	0.16	0.95	0.010	7.74	0.028

Simulation results of the distributions of the probability of ruin, ending portfolio value, and the lifetime average withdrawal rate from the optimal simulation parameters in Table 3 (the minimum withdrawal rate is two-thirds of the initial withdrawal rate).

eters are presented in Table 3, and simulated results of portfolio performance from the optimal parameters are presented in Table 4.

A comparison of Tables 1 and 3 exposes the impact of withdrawal management on optimal portfolio allocations. The threat to portfolio ruin results from the interactions of the withdrawal rate, the retirement age, and the portfolio allocation. The greatest risk to age 55 retirees is outliving the portfolio, and optimal management reduces the optimal portfolio allocations to equity from unmanaged withdrawals (Table 1 to Table 3) at all withdrawal rates presented, with greater reductions at higher withdrawal rates. As the retirement age increases, the longevity risk decreases relative to the risk of excessive withdrawal rates, and the optimal allocations to equity begin to increase as the risk of excess withdrawals surpasses

Table 5 Variable, managed, inflation-adjusted withdrawals until death

Dependent Variable	Independent variable coefficients			Statistics	
	Constant	Retirement age	Inflation-adjusted withdrawal rate (%)	R^2	F
Optimal Wt_E (%)	26.985 (4.903)	−0.240 (0.073)***	10.709 (0.413)***	0.962	341.3***
Average probability of ruin (%)	4.204 (1.915)	−0.374 (0.028)***	4.902 (0.161)***	0.976	548.0***

***Indicates statistical significance at the 0.001 confidence level; standard error in parentheses.
Linear regression results based upon Table 4 data.

the longevity risk. By age 65, the risk of excessive withdrawal rates dominates the risk of excessive longevity, and portfolio allocations to equity increase at all withdrawal rates shown, understandably by less at higher initial withdrawal rates.

Table 3 shows that the optimal reserve multiple (RES) increases to protect the sustainability of the portfolio from greater allocations to equities necessary to minimize ruin from earlier retirements, higher initial withdrawal rates, or both. The optimal rate of adjustment to higher withdrawal rates (PART) seems to be between 0.2 and 0.3, supporting caution, although no correlation to the other variables is obvious. In general, Table 4 shows that greater portfolio protection from ruin causes greater ending portfolio values, particularly if the withdrawals threat is compounded by the longevity threat (higher initial withdrawals and earlier retirement ages).

A comparison of Tables 1 and 4 shows that the management process has reduced the average probability of ruin by about 50% whereas reducing the excess accumulation in the ending portfolio by about 50%, and furthermore, the withdrawal management process has generated lifetime average withdrawal rates in excess of the constant withdrawal rates of Table 1 and the initial withdrawal rates of Table 4.

The interrelationships among the variables of the managed withdrawals model are clarified by the linear regression results presented in Table 5. The optimal portfolio allocation to large-cap equities increases about 10.709%/4 = 2.68% for each quarter-point increase in the withdrawal rate and 0.24% per year of earlier retirement; and the minimum probability of ruin increases about 4.902%/4 = 1.23% per quarter-point increase in the withdrawal rate and 0.37% per year of earlier retirement.

The effects of withdrawal rate management are revealed by comparing the results in Tables 2 and 5. When compared to unmanaged withdrawals from optimally allocated portfolios, optimal withdrawal management from optimally allocated portfolios (Table 5) reduces the risk of prematurely exhausting the retirement portfolio by reducing the equity allocation increase necessary to support higher initial withdrawals or earlier retirements, thereby limiting market risk exposure. Therefore, the withdrawal management process has caused the optimal portfolio allocation to equities and the minimum probability of ruin to be less sensitive to the withdrawal rate and the age at retirement. The optimal equities allocation increases only 2.68% per quarter-point increase in the withdrawal rate instead of 3.75%, and about 0.25% per year of earlier retirement instead of near 1.5%, and the minimum probability of ruin increases only 1.23% per quarter-point increase in the withdrawal rate instead of 2.34%, and 0.37% per year of earlier retirement instead of near 0.85%.

6. Conclusions

Retirees and perspective retirees face a risk-return tradeoff in selecting higher, earlier, or both, withdrawal rates from the retirement portfolio. The risk of running out of money during the remaining retirement lifetime increases as the return of improved retirement lifestyles increases. Informed of the alternatives, the decision is ultimately that of the retiree. This research is part of the growing body of literature that contributes to the requisite risk-return information.

A first step in improving the risk-return alternatives is selecting the optimal portfolio allocation for the desired withdrawal rate and expected retirement life span, and Monte Carlo stochastic optimization is appropriate to determining the portfolio allocation that minimizes the probability of portfolio ruin. The increased-probability-of-ruin cost for the improved-lifestyle benefit from a higher withdrawal rate is thereby minimized. The optimal allocation to equities in a two-asset portfolio increases about 3.75% for each quarter-point increase in the withdrawal rate and 1.466% per year of earlier retirement, and the minimum probability of portfolio ruin from the optimal portfolio increases about 2.34% per quarter-point increase in the withdrawal rate and 0.849% per year of earlier retirement. The ruin-minimizing portfolio allocation to equities and the minimized probability of ruin are decreasing functions of the retirement age and increasing functions of the withdrawal rate, so the reward to portfolio optimality is greatest at higher withdrawal rates and earlier retirement ages.

A second step in improving the risk-return alternatives is withdrawal flexibility, possibly a form of withdrawal management. A set of decision rules to meet the objectives of the retiree is adopted, and Monte Carlo stochastic optimization is used to select the decision rules parameters and portfolio allocation to minimize the probability of portfolio ruin. Optimal parameter values for an illustrative portfolio performance-based withdrawal rate management scheme generate optimally managed withdrawals from optimal portfolios. The illustrative optimal withdrawal management from optimal portfolios reduces the probability of ruin and the excess portfolio accumulation by about 50%. In the withdrawal-managed model, the ruin-minimizing portfolio allocation and the minimized probability of ruin remain decreasing functions of the retirement age and increasing functions of the initial withdrawal rate. However, optimal managed portfolios contain fewer equities, and optimal management of optimally allocated portfolios reduces the sensitivity of the probability of ruin to higher initial withdrawal rates (from 2.34% to 1.23% per quarter-point increase in the withdrawal rate) and earlier retirements (from 0.85% to 0.37% per year). Therefore, the flexibility in retirement withdrawals from the illustrative withdrawal management parameters reduces the probability-of-ruin cost of the improved-lifestyle benefits by about one-half. This cost reduction will prove particularly beneficial at higher initial withdrawal rates and earlier retirement ages.

Notes

1. Latin Hypercube sampling insures that sampling from a probability distribution uniformly spans the range of possible values.
2. The 50,000 iterations stabilizes optimized variables. The coefficients of variation for minimized probabilities of ruin, for example, are of the order of magnitude 0.01.
3. The Box Method, also termed the "Box complex" method is one of a class of algorithms which search the feasible solutions to find the optimal solution. The method is described in detail by Box (1965).
4. A reported minimum average probability of ruin, for example, is the average, across 40 simulations, of the probability of ruin from 50,000 iterations within a single simulation.
5. Nonlinear regression produced very similar results with a slightly lower coefficient of determination.
6. The expected remaining life is computed from the same mortality table and is rounded up to the next whole number of years.

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