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Sustainable retirement income for the socialite, the gardener, and the uninsured

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Abstract

This paper advances the literature on the sustainability of retirement income by making consumption a stochastic variable instead of a constant real value, as previous papers have done. The paper continues to make the rate of return and date of death stochastic variables, as Milevsky and Robinson (2000, 2005) do. The sustainability of retirement income depends on the nature of the lifestyle that the retiree chooses. The difference in shortfall probabilities or risk of ruin between the variable cases and the fixed consumption case is significant, and so the adviser needs to take this into account. The difference in shortfall probabilities between making consumption a nonconstant but deterministic amount, and making it also stochastic, is not as important, because it does not reduce risk enough to make more aggressive consumption rates secure. Finally, making consumption correlated with the rate of return, which implies the family adjusts consumption as its wealth changes, reduces shortfall probabilities to a moderate extent. In general, an initial consumption of more than 4% of initial wealth is not sustainable for any likely set of conditions. In the very best case, an initial consumption rate of 6% is sustainable, but we think that case will fit very few people. © 2010 Academy of Financial Services. All rights reserved.

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1. Introduction

One of the most important and difficult questions a financial adviser faces is how much a client can spend in retirement and not run out of money. As the baby boomer cohort starts

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to retire in large numbers, at the same time as traditional defined benefit pension plans are disappearing, this problem becomes even more significant.

The typical analysis makes assumptions about how long the person will live and the rate of return, along with a fixed spending amount. The answer is then a simple yes or no based on the present value of the retirement spending, compared with the expendable wealth the client has at his or her retirement date. The adviser then explains that this plan does involve some risk—living too long, earning a lower rate of return, a difference in the spending pattern.

Researchers have improved on this model by recognizing that the rate of return and the date of death are variables with statistical distributions. They have then modeled the variables to obtain the probability of running out of money before natural death (the "shortfall probability" or "probability of ruin"), given a fixed consumption rate, a lognormal return distribution, and either a range of dates of death or some approximation of mortality statistics.

Milevsky and Robinson (2000, 2005) model it analytically, with some assumptions, to incorporate rate of return and mortality into an analytic solution that yields a probability of shortfall. They conclude that a retiree age 65 is most likely not able to spend more than 4% of an initial endowment annually in real terms, without facing a significant probability of shortfall. Bengen (1994), Cooley et al. (1998), Guyton (2004), Ho et al., (1994), and Pye (2000) have run financial experiments or simulations using historical returns to quantify the sustainability of specified ad hoc spending policies and consumption rates for retirees. They generally find sustainable rates of 4% to 6%.

All these papers are assuming a fixed consumption throughout retirement in real terms, and that is also the basic assumption of most financial advisers' solutions. This assumption ignores the reality that retirees do not all follow the same pattern of spending throughout retirement, the spending may be uncertain for any pattern applied and retirees can and often do adjust their spending to some extent to respond to changes in their endowment because of higher or lower than expected investment returns.

In this article, we make the rate of consumption a variable as well to investigate more deeply the effect of different spending patterns. Milevsky and Robinson (2005) argue for the benefits of the analytic approach as opposed to simulations, and that is the approach we follow here.

We find that advisers and retirees need to consider the effect of spending patterns on sustainability of retirement consumption, just as you would expect. If their spending is likely to follow a declining pattern over time, which is what we most often expect to see, then they can consume more initially than a fixed consumption model recommends. The reverse is true for climbing spending. When spending is stochastic, it also affects the risk of the retirement plan, but not monotonically. Sometimes the risk rises, sometimes it falls. Finally, if the retiree is going to pay attention to the realized rate of return and wealth effects, and adjust spending somewhat, the risk of shortfall is also lessened.

The rest of the paper follows this plan. The second section explains what sort of personal spending patterns we want to model and why. The third section sketches the derivation of the formal model that we use to estimate shortfall probabilities when rate of return, date of death, and consumption rate are all stochastic variables. The fourth section uses the model to estimate shortfall probabilities for a number of different cases to provide us with insights that advisers and retirees can follow, and to demonstrate how our extension to the existing research improves planning capability. Finally, we conclude briefly and suggest some useful future research that we would undertake to improve the value of our model in practice.

2. What patterns matter? The socialite, the gardener, and the uninsured

The money consumed is a mixture of principal and income, and the objective is always to be able to spend the specified amount every year until death. Everything is expressed in real terms. Note that the initial level of annual spending may be less (usually) or more (rarely) than the person or family spent during the last years of working life. We are not modeling the initial retirement spending as a drop from preretirement spending. We take the initial value as something the retiree has determined and then we estimate the sustainability of that initial value, depending on the subsequent pattern of spending in retirement. For example, the financial planning industry's rule of thumb is to assume 70% of preretirement income is needed at the start of retirement. Our model starts with that 70% figure, or whatever initial starting point is selected.

The pattern of spending that most of the researchers have assumed is a constant real dollar value of consumption. We express that pattern very simply as a percentage of the initial wealth from which it is drawn. Who would spend in such a pattern? The usual mantra of a financial adviser is to enumerate all the spending that will decline in retirement, work clothing, dry cleaning, lunches eaten out, travel to and from work, total calorie consumption, and hence food costs. Nonetheless, some people do not experience a drop in consumption. Those who do not will include people who were earning very modest incomes that just covered living expenses, and their pensions, savings, and so forth, just match survival consumption in future. More interesting is the person who retires from work, but moves into a very social retirement life of parties, travel, theater, receptions, patron of the arts, and so forth. Because this person has more free time and energy when freed from work, he or she can spend quite a lot, and this spending need not decline for a very long time the way spending on vigorous sports might. We name this person, with a constant real spending pattern, "The Socialite."

A much more common pattern is declining real consumption every year. You can only spend so much on clothing, travel, parties, food, and so forth. As the years wear on, you need fewer new clothes, your food intake decreases, and you stay at home more often. Who likes this quieter life? We name the declining consumption pattern in retirement "The Gardener."

A less happy case is the person who was perhaps forced to retire a bit early, because of somewhat poorer health. This person faces an increasing pattern of consumption over time as he or she has to buy more medicine and pay more for services to help in the normal business of living. We are not talking of a catastrophic situation like a crippling stroke, but something gradual like arthritis. We call this person and pattern of consumption "The Uninsured." This pattern may become more prevalent as the baby boomers age and the enormous pressure on the health care system forces reductions in government and insurance supported medical care.

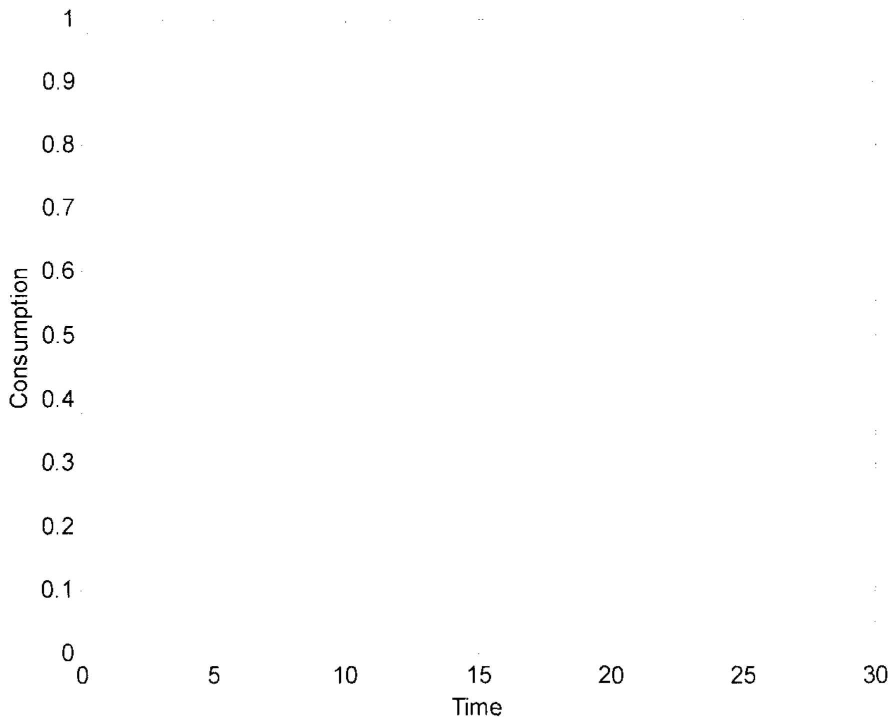


Fig. 1. Deterministic exponential consumption. Shows the case of a deterministic exponentially decreasing consumption pattern with $\alpha = 0.1$ and $\beta = 0$.

2.1. The model of shortfall probability and retirement consumption

Milevsky and Robinson (2000, 2005) present a model of retirement consumption from an initial endowment or nest egg in which the real return on investment is stochastic, the date of death is stochastic (based on mortality statistics), and consumption is a constant real amount, expressed as a percentage of the initial nest egg. We expand this model to incorporate the following additional factors:

1. The initial consumption amount can vary, either deterministically or stochastically. The model assumes a Geometric Brownian motion (GBM) for this process, with a drift of $-\alpha$ and a volatility of β.
2. When $\beta = 0$, the consumption process is deterministic, with an exponential decline shown by a positive value of α.
3. When $\beta > 0$, the consumption process is stochastic.
4. We also allow for correlation between the real return and real consumption values. This is analogous to a person reducing spending when returns are bad, and increasing it when they are good. Because a lot of consumption is fixed, the correlation coefficient, ρ, cannot take on a very high value.[1]

This setting allows for more flexibility and different consumption patterns can be modeled within the proposed framework. Figs. 1 and 2 show, respectively, the case of a deterministic exponentially decreasing consumption ($\alpha = 0.1$, $\beta = 0$), and the case of a stochastic GBM consumption ($\alpha = 0.04$, $\beta = 0.1$).

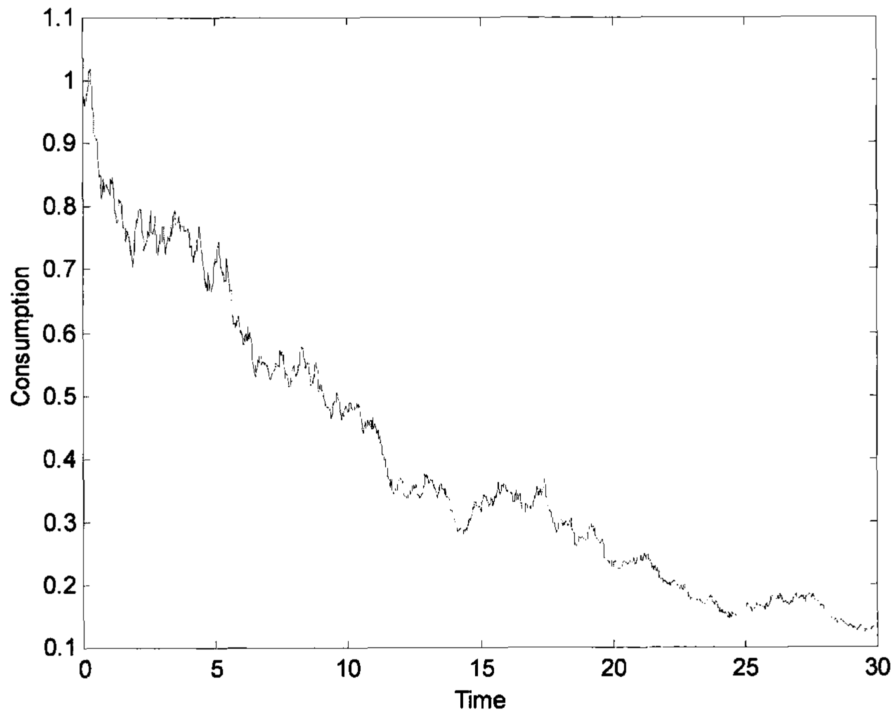


Fig. 2. Stochastic GBM consumption. Shows the case of a stochastic Geometric Brownian consumption pattern with $\alpha = 0.04$ and $\beta = 0.1$.

2.2. Stochastic consumption and cumulative return

More specifically, and following Milevsky and Robinson (2005), we use the notion of the *stochastic present value* (SPV) of the stochastic consumption at initial time zero. The SPV is like the usual present value of a stream of cash flows, but it is a random variable that can be known only through its probability distribution. We define it by (technical details are presented in the Appendix):

$$SPV = \int_0^{\tilde{T}} C_t R_t^{-1} dt = \int_0^{+\infty} 1_{(\tilde{T}>t)} C_t R_t^{-1} dt \tag{1}$$

where $\tilde{T}$ denotes the random time of death (in years), (C_t) and (R_t) are respectively the consumption and the cumulative return processes, and $1_{(\tilde{T}>t)}$ is the indicator function that equals 1 if $\tilde{T}>t$ and 0 otherwise. We assume that the processes (C_t) and (R_t) follow two correlated Geometric Brownian motions given by:

$$\begin{cases} dR_t = \mu R_t dt + \sigma R_t dB_t \quad ; \quad R_0 = 1 \\ dC_t = -\alpha C_t dt + \beta C_t dZ_t \quad ; \quad C_0 = 1 \\ d < B,Z >_t = \rho \, dt \end{cases} \tag{2}$$

where the last equation states that the correlation between the consumption and the cumulative return is equal to ρ. Using standard stochastic calculus, the solutions to Eq. (2) are given by:

$$R_t = \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right) \tag{3}$$

and

$$\begin{aligned} C_t &= \exp\left(-(\alpha + \frac{1}{2}\beta^2\right)t + \beta Z_t) \\ &= \exp\left(-\left(\alpha + \frac{1}{2}\beta^2\right)t + \rho\beta B_t + \sqrt{1-\rho^2}\beta\bar{B}_t\right) \end{aligned} \tag{4}$$

where $(\bar{B}_t)$ is a Brownian motion uncorrelated with (B_t). The last equation in Eq. (4) shows how the correlation coefficient affects the consumption. To value the probability of ruin using the SPV, we need to use the *discounted consumption* $(C_t R_t^{-1})$. Combining Eqs. (3) and (4), we can easily show that:

$$C_t^{-1} R_t \underset{\mathcal{D}}{=} \exp\left(\left(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2\right)t + \bar{\sigma}\bar{Z}_t\right) \tag{5}$$

where $(\bar{Z}_t)$ is a Brownian motion and:

$$\begin{cases} \bar{\mu} = \mu + \alpha + \beta^2 - \rho\sigma\beta \\ \bar{\sigma} = \sqrt{\sigma^2 + \beta^2 - 2\rho\sigma\beta} \end{cases} \tag{6}$$

This yields that the inverse of the discounted consumption $(C_t R_t^{-1})$ has the same probability distribution (and hence the same moments) as the GBM with a drift of $\bar{\mu}$ and a volatility of $\bar{\sigma}$.

The point of the mathematics is simple, even though it looks very complex. We want to be able to determine how likely it is that a person will run out of money in retirement. This more complex SPV allows us to do the same sort of hypothesis test that we all learned in basic statistics: what is the probability that the variable we are interested in will be greater than a given value? In the retirement problem, the question is: what is the probability that the SPV of future consumption will exceed the wealth we have to support that consumption? To solve the problem, we need some more mathematics.

2.3. Exponential lifetime random variable

The remaining lifetime random variable $\tilde{T}$ is assumed to be exponentially distributed with mortality rate λ, that is the probability distribution for $\tilde{T}$ is given by:

$$P(\tilde{T} > t) = \exp(-\lambda t) \tag{7}$$

The mortality rate can be fitted to a mortality table using the median value given by $Median = \frac{\ln(2)}{\lambda}$. The theoretical average (expected) lifetime is given by $Mean = \frac{1}{\lambda}$. The case λ=0 corresponds to the endowment (infinite life) case.

Although the paper focuses only on the exponential formulation of remaining lifetime, the following result can be extended to more realistic mortality laws such as Gompertz and Gompertz-Makeham.

3. Main result: Exponential-Reciprocal Gamma approximation

Following Milevsky (1997) and Milevsky and Robinson (2005), we can show that the probability of ruin, where *ruin* means that the SPV is higher than the initial wealth w, can be approximated with a *Reciprocal Gamma* distribution. More precisely, we have that:

$$P(SPV > w) \approx GammaDist\left(\bar{\alpha},\bar{\beta};\frac{1}{w}\right) \tag{8}$$

where $GammaDist\ (\bar{\alpha},\bar{\beta};x)$ is the cumulative distribution function of the Gamma distribution with parameters:

$$\begin{cases} \bar{\alpha} = \dfrac{2\bar{\mu} + 4\lambda}{\bar{\sigma}^2 + \lambda} - 1 \\ \bar{\beta} = \dfrac{\bar{\sigma}^2 + \lambda}{2} \end{cases} \tag{9}$$

For the endowment case ($\tilde{T}=+\infty$), Eq. (8) gives the exact distribution with parameters $\bar{\alpha}_\gamma = \dfrac{2\bar{\mu}}{\bar{\sigma}^2} - 1$ and $\bar{\beta}_\infty = \dfrac{\bar{\sigma}^2}{2}$. For the mathematical details of the proof, refer to Milevsky (1997).

We know the properties of the Gamma distribution, and so we can calculate the probabilities of ruin for a retirement plan in which the initial wealth and starting real consumption are given, and the mortality rate, rate of return and pattern of consumption are all based on probabilities rather than certainties.

4. Empirical results

We assume initially a balanced portfolio earning 4% real per annum and with a standard deviation of 14%. This assumption is quite critical to the absolute value of the shortfall probability, but does not affect the relative results of the different tables. We are using a less favorable return distribution than Milevsky and Robinson (2005). They use a mean of 7% and standard deviation of 20%. We think our lower rate is more suitable, because the actual retiree must also pay money management costs in one form or another.

Tables 1 to 5 provide a series of example of applying different values of return and consumption parameters to different starting real consumption rates. From these tables we can draw some clear conclusions for retirees and their advisers.

1. The most significant effect on probability of shortfall of the innovations in this paper is the different patterns of consumption.

Table 1 Probability of shortfall for constant real consumption: the "socialite"

Retirement age	Median age at death	Initial consumption rate								
		2%	3%	4%	5%	6%	7%	8%	9%	10%
Endowment	Infinity	31.6%	57.1%	75.9%	87.5%	93.8%	97.1%	98.7%	99.4%	99.7%
50	78.1	6.0%	15.0%	26.4%	38.4%	50.0%	60.4%	69.2%	76.5%	82.3%
55	83	6.0%	14.9%	26.3%	38.3%	49.9%	60.3%	69.1%	76.4%	82.2%
60	83.4	4.7%	12.0%	21.7%	32.4%	43.2%	53.3%	62.3%	70.1%	76.5%
65	83.9	3.4%	9.0%	16.7%	25.7%	35.2%	44.6%	53.4%	61.3%	68.3%
70	84.6	2.2%	6.0%	11.6%	18.5%	26.1%	34.1%	42.1%	49.6%	56.7%
75	85.7	1.2%	3.4%	6.9%	11.5%	16.9%	22.8%	29.1%	35.5%	41.8%
80	87.4	0.5%	1.6%	3.4%	5.9%	9.0%	12.6%	16.7%	21.1%	25.7%

The entries on the table show the probability of running out of money before death for different consumption rates in each column. Initial consumption means the amount drawn from the investment balance (no distinction between income and principal) in the first year. For the "socialite" case, this amount stays constant. All rates of return and consumption amounts are real. For example, if you retire at age 60 and plan to consume a constant real dollar income from your wealth that is 5% of the starting balance, you have 32.4% probability of running out of money (in actuarial terms, this is the "probability of ruin"). The first row is the endowment case, where the money has to last in perpetuity. μ is the real investment rate of return and σ is the standard deviation. α and β are the drift and volatility of the consumption amount. In this table: $\mu = 4\%$, $\sigma = 14\%$, $\alpha = \beta = \rho = 0$.

2. A person who retires in the normal age range of 60 to 65 cannot generally expect to sustain an initial consumption rate in retirement greater than 4% of initial wealth.
3. A person who plans to continue a constant rate of consumption, "the socialite," cannot sustain a rate of more than 3%.
4. Someone whose health or other needs entail an increasing rate of consumption in retirement cannot sustain an initial rate of more than 2%.

Table 2 Probability of shortfall for different patterns of deterministic real consumption

	Initial consumption rate								
Retirement age	2%	3%	4%	5%	6%	7%	8%	9%	10%
Constant real consumption - "The Socialite": $\alpha = 0$, $\beta = 0$, $\rho = 0$									
55	6.0%	14.9%	26.3%	38.3%	49.9%	60.3%	69.1%	76.4%	82.2%
65	3.4%	9.0%	16.7%	25.7%	35.2%	44.6%	53.4%	61.3%	68.3%
75	1.2%	3.4%	6.9%	11.5%	16.9%	22.8%	29.1%	35.5%	41.8%
Exponentially increasing deterministic real consumption - "The Uninsured" : $\alpha = -1\%$, $\beta = 0$, $\rho = 0$									
55	11.1%	23.5%	37.0%	49.8%	61.1%	70.5%	77.9%	83.8%	88.2%
65	6.0%	13.9%	23.6%	34.0%	44.1%	53.6%	62.0%	69.3%	75.5%
75	1.9%	5.1%	9.5%	15.1%	21.3%	27.9%	34.7%	41.3%	47.7%
Exponentially decreasing deterministic real consumption - "The Gardener" : $\alpha = +2\%$, $\beta = 0$, $\rho = 0$									
55	1.5%	5.2%	11.6%	20.1%	29.9%	40.1%	49.9%	59.0%	67.1%
65	1.0%	3.4%	7.7%	13.6%	20.8%	28.8%	37.1%	45.3%	53.1%
75	0.4%	1.5%	3.5%	6.4%	10.2%	14.7%	19.8%	25.3%	31.0%

The entries on the table show the probability of running out of money before death for different consumption rates in each column. Initial consumption means the amount drawn from the investment balance (no distinction between income and principal) in the first year. α and β are the drift and volatility of the consumption amount; in this table consumption is deterministic and so $\beta = \rho = 0$. The "uninsured" case has slightly increasing consumption every year and the "gardener" case has slightly faster decreasing consumption every year. As in Table 1, $\mu = 4\%$ and $\sigma = 14\%$.

Table 3 Probability of shortfall for different patterns of stochastic real consumption

Retirement age	Initial consumption rate								
	2%	3%	4%	5%	6%	7%	8%	9%	10%
Constant real consumption–"The Socialite": $\alpha = 0$, $\beta = 0$, $\rho = 0$ [for comparison]									
55	6.0%	14.9%	26.3%	38.3%	49.9%	60.3%	69.1%	76.4%	82.2%
65	3.4%	9.0%	16.7%	25.7%	35.2%	44.6%	53.4%	61.3%	68.3%
75	1.2%	3.4%	6.9%	11.5%	16.9%	22.8%	29.1%	35.5%	41.8%
Pure stochastic real consumption: $\alpha = 0$, $\beta = 10\%$, $\rho = 0$									
55	6.6%	15.0%	25.3%	36.1%	46.5%	56.1%	64.5%	71.6%	77.6%
65	3.8%	9.4%	16.7%	25.0%	33.7%	42.3%	50.4%	57.9%	64.6%
75	1.4%	3.8%	7.2%	11.7%	16.8%	22.5%	28.4%	34.4%	40.4%
Exponentially increasing stochastic real consumption–"The Uninsured": $\alpha = -1\%$, $\beta = 10\%$, $\rho = 0$									
55	11.3%	22.5%	34.6%	46.2%	56.6%	65.5%	73.0%	79.0%	83.9%
65	6.4%	13.9%	22.8%	32.3%	41.7%	50.4%	58.4%	65.4%	71.5%
75	2.2%	5.3%	9.7%	15.0%	21.0%	27.2%	33.6%	39.8%	45.9%
Exponentially decreasing stochastic real consumption–"The Gardener": $\alpha = +2\%$, $\beta = 10\%$, $\rho = 0$									
55	2.0%	6.0%	12.3%	20.2%	29.0%	38.1%	47.0%	55.4%	62.9%
65	1.3%	4.0%	8.3%	14.0%	20.7%	28.1%	35.7%	43.2%	50.4%
75	0.6%	1.8%	3.9%	6.8%	10.5%	14.9%	19.7%	24.9%	30.3%

The entries on the table show the probability of running out of money before death for different consumption rates in each column. Initial consumption means the amount drawn from the investment balance (no distinction between income and principal) in the first year. α and β are the drift and volatility of the consumption amount. As in Table 1, $\mu = 4\%$ and $\sigma = 14\%$.

5. In the most wildly optimistic case, a "gardener" whose initial consumption declines a significant rate during retirement, who invests very aggressively and does very well,

Table 4 Probability of shortfall when real consumption and real return are positively correlated

Correlation between rate of return and consumption	Initial consumption rate								
	2%	3%	4%	5%	6%	7%	8%	9%	10%
All cases are age 65, comparing different correlations with no correlation, $\mu = 4\%$ and $\sigma = 14\%$									
Pure stochastic consumption:									
$\alpha = 0$, $\beta = 0$	3.4%	9.0%	16.7%	25.7%	35.2%	44.6%	53.4%	61.3%	68.3%
$\alpha = 0$, $\beta = 10\%$, $\rho = 0$	3.8%	9.4%	16.7%	25.0%	33.7%	42.3%	50.4%	57.9%	64.6%
$\alpha = 0$, $\beta = 10\%$, $\rho = 0.2$	3.1%	8.1%	15.1%	23.5%	32.3%	41.2%	49.7%	57.6%	64.6%
$\alpha = 0$, $\beta = 10\%$, $\rho = 0.3$	2.7%	7.5%	14.3%	22.6%	31.6%	40.7%	49.4%	57.4%	64.6%
Increasing consumption: "The Uninsured"									
$\alpha = -1\%$, $\beta = 10\%$, $\rho = 0$	6.4%	13.9%	22.8%	32.3%	41.7%	50.4%	58.4%	65.4%	71.5%
$\alpha = -1\%$, $\beta = 10\%$, $\rho = 0.2$	5.3%	12.4%	21.2%	30.9%	40.5%	49.7%	58.0%	65.4%	71.7%
$\alpha = -1\%$, $\beta = 10\%$, $\rho = 0.3$	4.8%	11.6%	20.4%	30.1%	39.9%	49.3%	57.8%	65.4%	71.9%
Decreasing consumption: "The Gardener"									
$\alpha = +2\%$, $\beta = 10\%$, $\rho = 0$	1.3%	4.0%	8.3%	14.0%	20.7%	28.1%	35.7%	43.2%	50.4%
$\alpha = +2\%$, $\beta = 10\%$, $\rho = 0.2$	0.9%	3.2%	7.1%	12.6%	19.2%	26.7%	34.5%	42.3%	49.8%
$\alpha = +2\%$, $\beta = 10\%$, $\rho = 0.3$	0.8%	2.8%	6.5%	11.8%	18.4%	25.9%	33.8%	41.8%	49.5%

The entries on the table show the probability of running out of money before death for different consumption rates in each column. α and β are the drift and volatility of the consumption amount and ρ is the correlation between real consumption and real rate of return. A positive correlation means the person adjusts consumption up or down as the rate of return on investments rises and falls, and thus the person offsets the wealth effects to some extent. As in Table 1, $\mu = 4\%$ and $\sigma = 14\%$.

Table 5 Probability of shortfall for the best possible situation: high real returns for a gardener whose consumption declines during retirement and adjusts for return volatility

		Initial consumption rate								
Retirement age	Median age at death	2%	3%	4%	5%	6%	7%	8%	9%	10%
50	78.1	0.3%	1.2%	3.1%	6.1%	10.4%	15.7%	21.8%	28.4%	35.3%
55	83	0.3%	1.2%	3.1%	6.1%	10.4%	15.7%	21.7%	28.3%	35.2%
60	83.4	0.2%	1.0%	2.7%	5.4%	9.2%	13.9%	19.5%	25.5%	31.9%
65	83.9	0.2%	0.9%	2.3%	4.6%	7.8%	11.9%	16.7%	22.1%	27.8%
70	84.6	0.2%	0.7%	1.8%	3.6%	6.2%	9.5%	13.4%	17.9%	22.7%
75	85.7	0.1%	0.5%	1.3%	2.6%	4.4%	6.8%	9.7%	13.1%	16.9%
80	87.4	0.1%	0.3%	0.8%	1.6%	2.8%	4.3%	6.1%	8.4%	10.9%

The entries on the table show the probability of running out of money before death for different consumption rates in each column. Initial consumption means the amount drawn from the investment balance (no distinction between income and principal) in the first year. All rates of return and consumption amounts are real. For example, if you retire at age 60 and plan to consume in the first year a real dollar income from your wealth that is 5% of the starting balance, you have 5.4% probability of running out of money (in actuarial terms, this is the "probability of ruin"). μ is the real investment rate of return and σ is the standard deviation. α and β are the drift and volatility of the consumption amount and r is the correlation between real return and real consumption. In this table: $\mu = 7\%$, $\sigma = 20\%$, $\alpha = +4\%$, $\beta = 10\%$ and $\rho = 0.3$.

and who also adjusts consumption partially as real investment return varies, may be able to sustain an initial rate of consumption of 6% of initial wealth starting at age 65. This situation, shown in Table 5, assumes $\mu = 7\%$, $\sigma = 20\%$, $\alpha = +4\%$, $\beta = 10\%$, and $\rho = 0.3$. This gardener is getting high real returns, is reducing consumption pretty quickly year over year, and is able to adjust quite a lot for changes in investment returns.

6. Making consumption stochastic has a relatively small impact on the probability of shortfall. It increases the risk at low rates of consumption and decreases it at higher rates.
7. Making stochastic consumption partially correlated with investment return reduces the risk of shortfall, but not by a great deal.
8. Changing the assumed rate of return always has a significant effect, and we do not spend much time showing that in this paper. The relative relationship between the patterns remains the same. However, assuming a 7% real return, which is higher than the long-run return on U.S. or Canadian equities, only increases the safe level of consumption for the gardener to an initial 6% of wealth.

What do we consider to be an acceptable risk of shortfall? That is a decision for every retiree or adviser to think about, but our choice is 10%. We think that many people would choose 5%, but we know of no formal evidence on this question.

Table 1 presents the Socialite, which is what most prior research assumed. In this case: $\alpha = \beta = \rho = 0$. We can see that retirement at age 65, when most pensions start, encounters a shortfall risk of over 10% for a draw greater than 3%, and the risk rises very rapidly. At the 5% or 6% that some advisers recommend, the shortfall risk is one-quarter to one-third. Early retirement is only safe with a draw of 2% per annum. Milevsky and Robinson (2005) assume a much higher mean return of 7% with a standard deviation of 20% and at age 65 the retiree is just marginally safe at a draw of 4%.

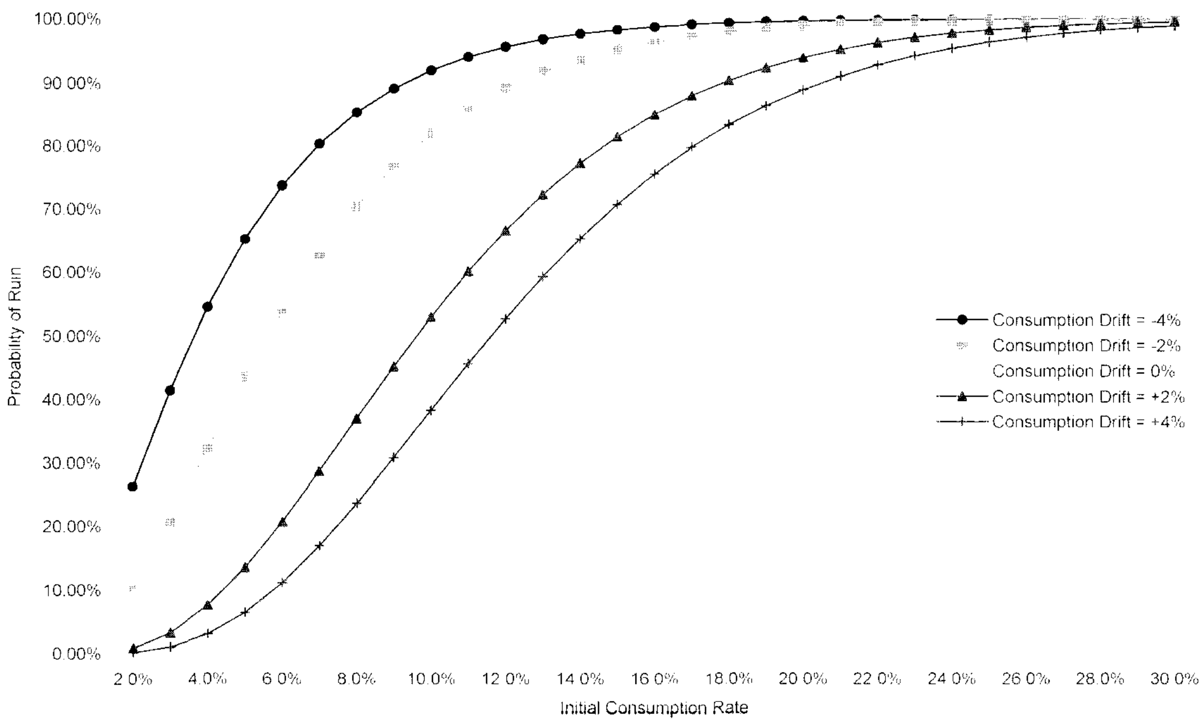
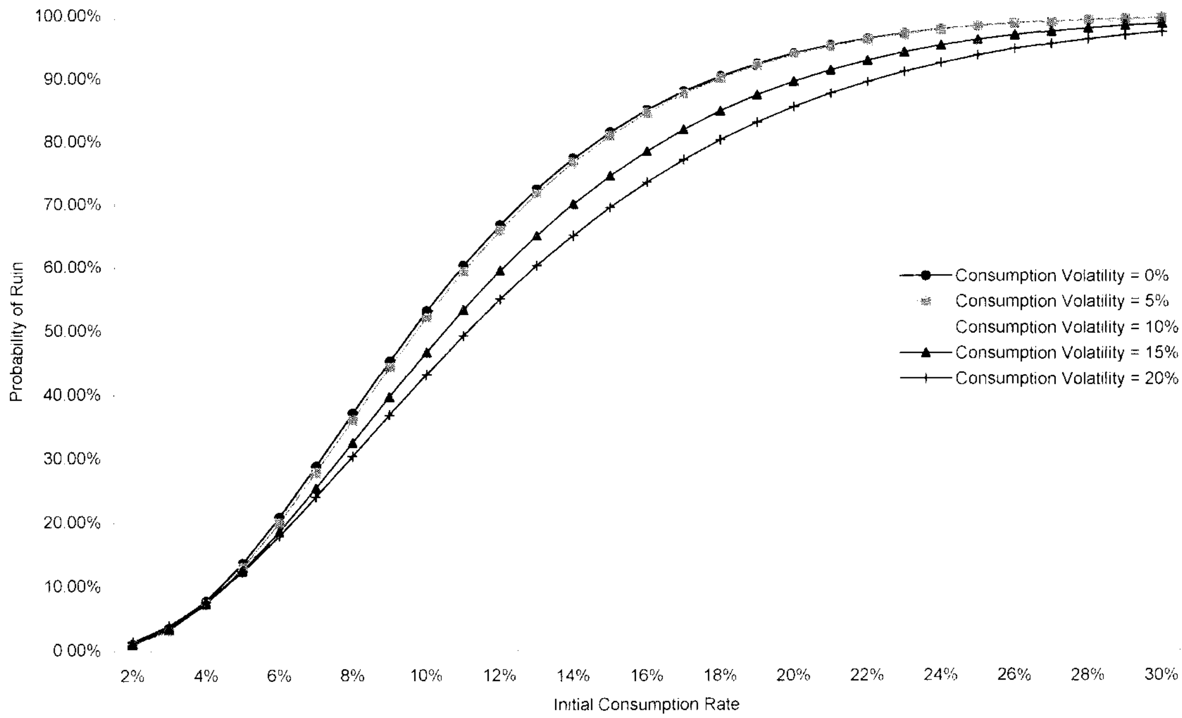


Fig. 3. Probability of shortfall under stochastic real consumption for different consumption drifts (α). Shows the probability of shortfall for different values of the consumption drift (α). The other parameters are: Retirement age = 65, μ = 4%, σ = 14%, β = 0%, and ρ = 0.

Table 2 shows the results for ages 55, 65, and 75 for the new patterns we have proposed: gardener, uninsured, and the socialite for comparison, and all deterministic, so that $\beta = \rho = 0$. The 65 year old gardener has a better prospect than the socialite, and a sustainable draw of 4% real, with higher draws increasingly risky, but less so than for the socialite. The 65 year old uninsured is naturally worse off, and should only start at a 2% draw. Fig. 3 shows the sensitivity of the probability of shortfall to the consumption drift (α) for a large range of draw rates with the following parameters: Retirement age = 65, μ = 4%, σ = 14%, β = 0%, and ρ = 0.

Table 3 now builds in stochastic consumption, with β = 10%. We do not know what the 'right' rate would be, but it certainly must be fairly low, because people do not change their spending habits randomly, and a significant number of living expenses are primarily fixed in real terms, rent, many utilities, municipal taxes, and so forth. The effect is interesting, and we did not expect it, though we can explain it. Stochastic spending increases the shortfall probability slightly for the lower consumption rates and lowers it for the higher spending rates. For example, the "pure stochastic consumption"[2] shortfall risk increases for 2% and 3% draw, and is identical between stochastic and nonstochastic cases at a 4% draw. For all higher draws, the shortfall probability is lower for stochastic consumption. The same pattern appears for the uninsured and the gardener. The changes in the shortfall are not so large that they change our conclusions, however. The higher draws still create unacceptably high shortfall risks. Figs. 4a and 4b show clearly this effect for a large range of draw rates with the following parameters: Retirement age = 65, μ = 4%, σ = 14%, α = +2%, and ρ = 0.2.

A



B

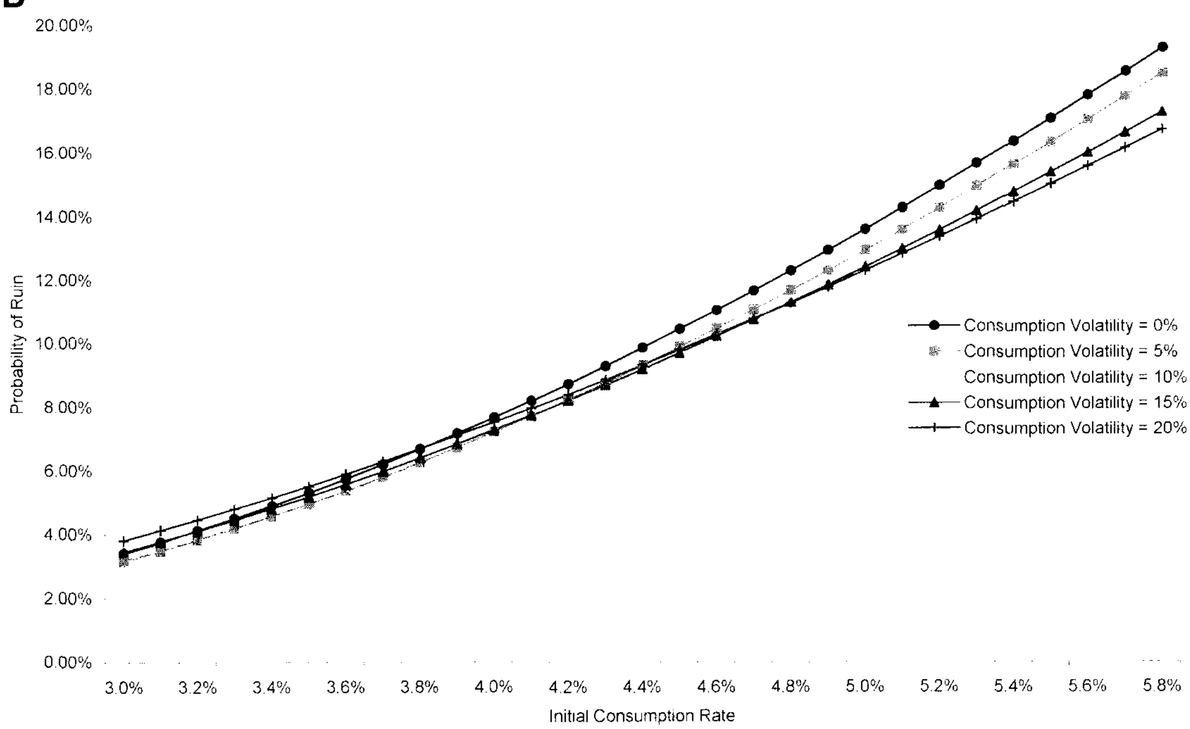


Fig. 4. Probability of shortfall under stochastic real consumption for different consumption volatilities (β). (A and B) Show the probability of shortfall for different values of the consumption volatility (β). The other parameters are: Retirement age = 65, μ = 4%, σ = 14%, α = +2%, and ρ = 0.2.

The shortfall risk decreases with the consumption volatility (β) for higher draw rates and increases for lower draw rates. Fig. 4b zooms in on lower draw rates and we can see that the inflection point is around a draw rate of 4.5%.

The explanation for this odd-seeming result lies in the nature of shortfall risk when a retiree is at high risk. If a sequence of consumption leads to ruin, as it may do, then increasing expenses because of chance does not affect that result. However, if the consumption is stochastic, the part of the distribution that reduces spending enough to avoid ruin does matter. Just like an option, the upside may cause a benefit and the downside cannot hurt any more. If the retiree is pretty secure, however, the tail of the distribution that increases the shortfall probability is more likely to occur with stochastic consumption, than the tail of the distribution that will decrease the already low probability of shortfall.

Table 4 brings in correlation of spending with return, at rates of 20% and 30%, and showing now the comparison of different correlations for the socialite, uninsured, and gardener, for age 65. We show only positive correlations, and in behavior terms this means that the retiree reduces spending somewhat when realized returns are low and increases it when realized returns are high. This should 'diversify' away some of the risk of investment returns, and it shows in the shortfall probabilities, which all fall with higher correlation. The effect is not enough to reduce shortfall probabilities enough to make draws of a full percentage point move to less than 10% shortfall risk when they were over it with no correlation. We might ask what the reasonable correlation coefficient is. It cannot be very high, because some spending is fixed, and most of it has at least a fixed component. You may choose to buy hamburger instead of steak, but you do have to eat. Fig. 5 shows the sensitivity of the probability of shortfall to the correlation (ρ) for a large range of draw rates with the following parameters: Retirement age = 65, μ = 4%, σ = 14%, α = +2%, and β = 10%. Unlike the sensitivity to the consumption volatility, the shortfall risk increases with the correlation (ρ) for higher draw rates and decreases for lower draw rates, with an inflection point around a draw rate of 12%.

Table 5 brings together the most desirable effects of all the previous tables and increases the rate of return as well to an aggressive equity portfolio. The gardener is the person, and a gardener who is reducing consumption every year and also adjusting spending when returns change. This lucky and frugal gardener can retire with reasonable security on an initial draw of 6% of wealth at any age from 50 onwards. We do not think many people would fit this profile, but that is an empirical question, particularly the α value of +4%, which is essential to this result.

5. Conclusion and directions for future research

We have demonstrated a model that incorporates three stochastic variables in determining the probability that retirement spending is sustainable; rate of return, consumption in retirement and date of death. This model advances all previous work that has kept consumption as a constant amount. This model is easily programmed into an Excel spreadsheet, all the numerical results are in fact taken from such a spreadsheet.

We conclude that the spending pattern itself, without a stochastic factor, causes the most

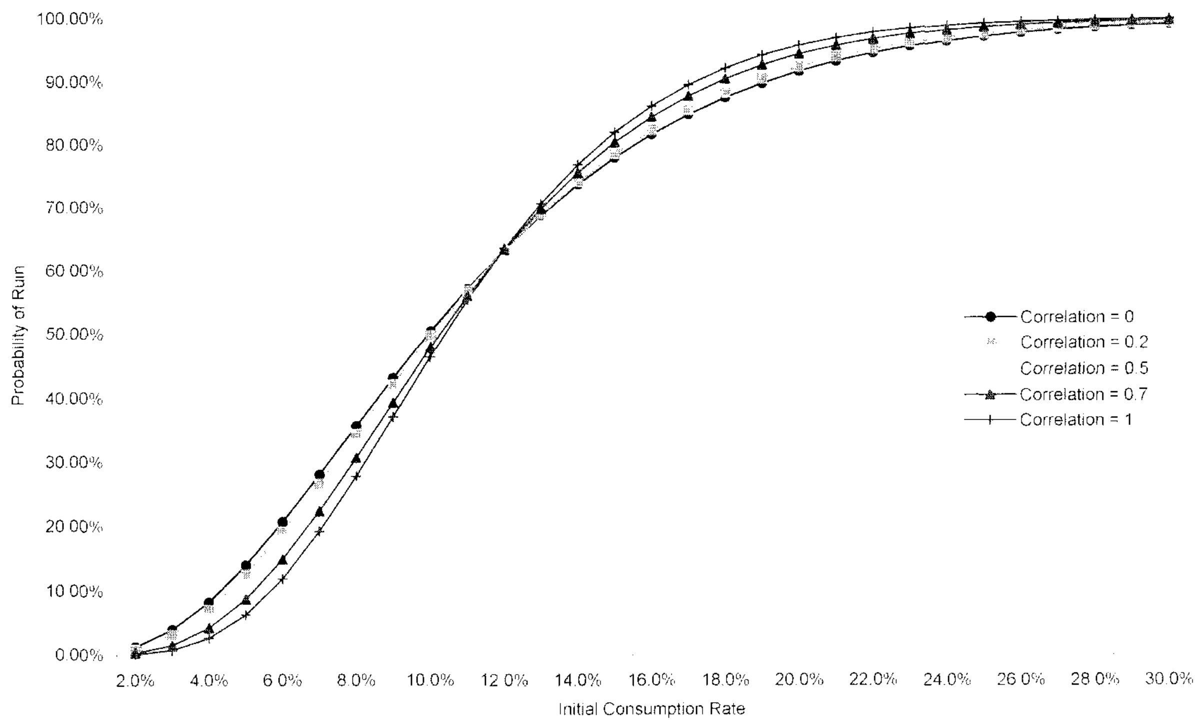


Fig. 5. Probability of shortfall under stochastic real consumption for different correlations (ρ). Shows the probability of shortfall for different values of the correlation (ρ) between the consumption rate and the rate of return. The other parameters are: Retirement age = 65, $\mu = 4\%$, $\sigma = 14\%$, $\alpha = +2\%$, and $\beta = 10\%$.

significant effects in changing the probability of shortfall. Nonetheless, the results still suggest that an initial spending rate of more than 4% of starting wealth at age 65 is going to be riskier than most people are likely to want. In the best case, this may improve to 6%.

We would also like to characterize two other patterns of consumption, but we cannot incorporate them into the model presented in this paper, and will be left for future research. One pattern is the person who has saved a lot all his or her life and not spent much on enjoyment. Retirement comes as an unexpected release and this person spends increasing amounts on travel as she or he expands horizons. You can only physically handle a lot of travel for so long, and eventually the spending declines again and fits one of the previous patterns. We call this person "The Explorer" after the new worlds that are opened up in retirement.

In the other pattern, the true disaster situation starts with any of the previous patterns, and a disaster (illness or accident) that does not kill, but disables so seriously that a very great spending increases happens at once. We have not named this person yet because all the appropriate names are so unhappy. This sort of jump process is more like an insurance problem and we do not model it in this article. The problem for the adviser is how to allow for a disaster in retirement when no insurance exists or is affordable. Most people will not suffer it, and so simply saving enough for any eventuality is not a feasible solution.

Finally, one particularly useful piece of empirical research remains undone to support our conclusions. There is lots of information on long run investment returns and so we can support our choices of those parameter values. However, our choices of α, β, and ρ are instinctive, not evidence-based. They are not simple statistics that are immediately observable from information that is widely known. What we need to find is longitudinal data on spending patterns that provides evidence of how people change their total spending in their retirement years. We have not been able to find such data. It would be very valuable for government policy and retirement planning in general, not just for our model.

Notes

1 We could conceive of negative correlation in a perverse way. If the stock market treats you badly, you go shopping to compensate. In this paper, we only show results for positive correlation.

2 By "pure stochastic consumption," we mean that the parameter $\alpha = 0\%$.

Acknowledgment

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Appendix

In this Appendix, we show that the probability of ruin can be defined by the probability that the SPV is below the initial wealth w.

Following the model given in Eq. (2), let us define the investment value (R_t) and the consumption (C_t) as two correlated GBMs given by:

$$\begin{cases} dR_t = \mu R_t\, dt + \sigma R_t\, dB_t & ; \quad R_0 = 1 \\ dC_t = -\alpha C_t\, dt + \beta C_t\, dZ_t & : \quad C_0 = 1 \\ d < B,Z >_t = \rho\, dt & \end{cases} \tag{A.1}$$

where the correlation between the two Brownian motions is ρ. Assuming that we can only invest in R and consume C per time unit, the wealth W_t at time t can be given as:

$$dW_t = (\mu W_t - C_t)dt + \sigma W_t\, dB_t\, ; \; W_0 = w \tag{A.2}$$

Using Itô's lemma, we can easily show that:

$$W_t = R_t \times \left[w - \int_0^t C_S R_S^{-1} ds \right] \tag{A.3}$$

solves the SDE (A.2). It is obvious that:

$$W_t \leq 0 \Leftrightarrow w \leq \int_0^T C_S R_S^{-1} ds \tag{A.4}$$

for any future time T. This shows that the probability of ruin before some time T is equal to:

$$P(\inf\{W_S; 0 \leq s \leq T\} \leq 0) = P(W_T \leq 0)$$
$$= P\left(w \leq \int_0^T C_S R_S^{-1} ds \right) \tag{A.5}$$

where the first equality comes from the fact that the integral $\int_0^T C_S R_S^{-1} ds$ is nondecreasing with respect to T.

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