With apologies to the Chinese proverb,

I Teach and I Understand

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In mathematics we frequently hear it said "I didn't *really* understand that until I taught it." Consider how strange this is. Many very able students, who later have become teachers themselves, recognize that there is a level of understanding that is the "learning" level of understanding and another, deeper, level of understanding acquired while teaching. Why should teaching lead to a deeper level of understanding? What implications does that idea have for the ways in which we help our students learn?

What is it that teachers do? To drastically over simplify, teachers explain mathematics by showing and telling. In addition to what teachers themselves do, they also prescribe what students are to do. They may have students participate in the process by using some materials, asking or answering questions, duplicating the teacher's drawings, or imitating the process being explained. After the explanation students are asked to do something that reviews the explanation or uses the mathematics that was explained.

The word "explain" needs to be explained. Articles by Henderson (1955) and Skemp (1978) provide a basis for explaining explain. As both articles emphasize, mathematics is more than a collection of statements, useful as these statements may be. Mathematics is a process of discovering and verifying these statements. The statements are not simply true, they were discovered and verified in mathematical ways and participation in the discovery and verification is just as important as learning the statements. Certainly for a teacher learning the hows and whys is absolutely essential!

One may explain something using a physical model that embodies the mathematical idea to be explained. As in Figure 1, the corners are torn off a paper triangle. When they are placed in an adjacent position they form a straight angle.

One may show this to students or have students tear their own triangles and tell the students what it shows: That whatever triangle you have the angles add up to a straight angle. Students also might use protractors to measure the angles and add the measures. These explanations are *inductive* as well as physical since the conclusion is drawn from many instances. In either case, showing that this works for only one triangle would not be very convincing. For one triangle the result might be a coincidence.

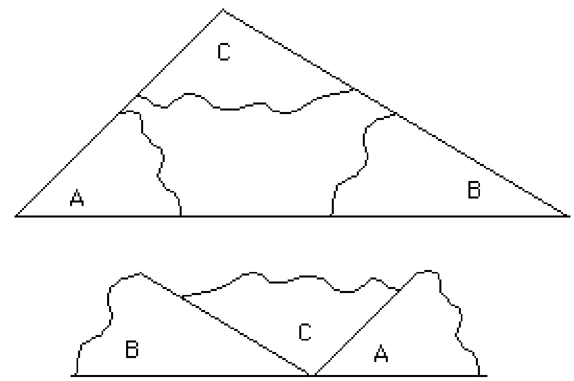

Figure 1

One may explain something by deduction. It is not necessary to have the entire structure of Euclidean geometry to have an informal deduction. If a student knows that when parallel lines are crossed by another line (a "transversal," you will remember) many pairs of equal angles are formed, the student can see that the angles of a triangle must add up to a straight angle as shown in Figure 2.

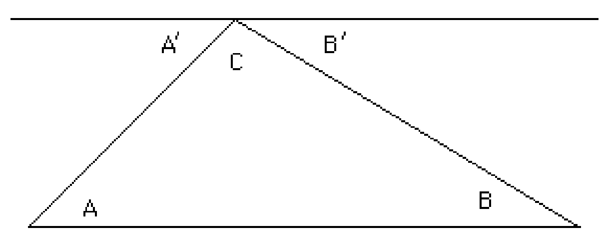

Figure 2

An item of knowledge can be explained by disclosing the source of the knowledge and having students retrace the path of that discovery. For example, rather than tell students that $\pi = C/d \approx 3.14 \approx 22/7$ you may have students measure circumference and diameter of a number of circles, make a table, and look for the relationship in the table. Looking at this relationship is probably much like the way the value of π was discovered originally.

One may explain something by "showing how." This is what Skemp would call instrumental understanding. There are many examples in mathematics in which the teacher shows how something is done. The process of long division, the procedure for bisecting an angle, and the way to plot points on a Cartesian coordinate system are examples of processes. Usually, behind a process there is a rationale that can be, and sometimes is, explained. Students are expected to learn and remember "how" and to do the process when it is appropriate.

When a teacher is explaining mathematics, the explanation is most meaningful to and best understood by the teacher. The degree of understanding among the students is variable but in all cases the students understand the material less well than the teacher. (I can think of exceptions to that rule, but only in unusual cases.) Some teachers believe that as they are explaining the students are receiving the explanation with the same degree of understanding as the teacher. This is never so. The explanation belongs to the teacher; the teacher has gone through the steps in teaching, deciding what to show and what to tell. The teacher has chosen words that communicate meaning to himself or herself. In going through these steps the teacher has personalized the knowledge, making it his or her own!

We want students to personalize their knowledge and make it their own. That is why we believe so strongly in the "I Do and I Understand" proverb. We made our knowledge more deeply our own as we tried to find ways to communicate it to our students. I suggest that it is the same with our students; to understand more deeply the mathematics we are teaching, students must struggle to find ways to communicate the mathematics to themselves or to others. One might say that they should plan what they would say and do, what they would show and tell, if they were to explain the mathematics to someone else. This might be accomplished in several ways: class presentations, peer tutoring, and writing assignments are currently popular. Another effective learning procedure, one that does not require group work, is to follow the teaching plan, mentally and on paper teaching the topic to yourself. What should students do to effectively plan the development of a mathematical idea? Let us look at the statement about the sum of the angles of a triangle as derived by tearing the angles as in Figure 1. There are two parts to this explanation, what is done physically to the angles and what is said to form the conclusion. The student should do both parts, making a triangle, tearing off the angles and forming them into a straight angle, and explaining what was done and what it means. If students work in pairs the students may explain it to each other in turn. If writing is used as a mathematics learning tool, the student may write the explanation accompanied by a diagram. In a similar way students should perform and rehearse the explanation of the deductive development.

One way to stimulate students to rehearse explanations of a mathematical idea is to call for students to explain mathematics. "How do you know" questions call for explanations: "How do you know that the diagonals of a rhombus are perpendicular?", "How do you know the formula for the area of a trapezoid $A = {}^1/_2\ h(b_1 + b_2)$?", "Why do you invert the divisor when you divide fractions?" Asking students to explain to the class will also produce explanations: "Show us why the product of two negative numbers is positive." In practice it is difficult (impossible) to have every student respond to such questions in class discussion. Some teachers use small group work with assignments such as this: "Discuss and agree on ways you could show that the diagonals of a rectangle are equal. Write the explanation and use diagrams."

Of course, it's not the point of this note to give answers to those questions. Our students should be able to give those answers. Why else do we bother to "explain" at all? Let me rephrase that: In our teaching and in our textbooks we spend a great deal of time and effort in presenting "explanations." It seems we do this because we want students to understand. How do we know whether or not a student does understand? One way to find out whether they understand is to ask them to explain the mathematical idea themselves, to another student, to you, or in writing. To return to the original point of this note, we would like to bring our students to the deeper "I Teach" level of understanding and to do this we need to put them in the position of selecting and organizing the words and pictures and materials that communicate mathematical ideas.

References

Skemp, R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher*, *26*(3), 9-15.

Henderson, K. B. (1955). On explanation. *Mathematics Teacher, 48*(5), 310-313.

Quotations

One must learn by doing the thing; for though you think you know it, you have no certainty until you try.

Sophocles

What we know is not much.
What we do not know is immense.

Pierre-Simon de Laplace