The Historical Development of the Concept of Angle [1]

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For more than two thousand years Euclid's Fifth Postulate produced a discussion on the foundations of geometry, and it was so important that we tend to forget many other issues that also played a role in shaping the field of geometry. The concept of angle is one of them. For more than two thousand years a debate has been developing on the nature of the concept of angle, and the discussion is not over yet.

This article will present a historical account of the development of the concept of angle in an effort to understand the ways in which mathematicians conceived angle, the properties that they attributed to the concept, and the problems they felt were solved and unsolved by their research. It is not the purpose of this paper to search for the mathematically correct way to define angle, or to refute the concepts of the several mathematicians who expressed their opinions about the angle.

My attempt to understand the historical development of the notion of angle is rooted in the belief that the historical genesis of a notion may illuminate its psychological development. The several forms that the concept assumed, its differential development in several cultures, its plural relationship with other mathematical topics, and its immersion in several cultures may provide insights that help educators to speculate and produce models of children's conceptions of a mathematical topic. This should not be taken to imply that historical development mirrors psychological development of children's thinking. There are enormous objections to this later point of view, namely that: (a) it underlies the assumption that there is a unique, additive, progression of mathematical knowledge, (b) it forces a psychological identification between an Egyptian scribe and a modern child, (c) it assumes that mathematical knowledge is separable from the rest of the culture. Instead, I will assume that a historical investigation on the origins of a mathematical concept is fruitful as a guide for developing a pedagogical perspective.

Angles and directions in neolithic cultures

Although the concept of angle seems to have originated only in recent times, ancient cultures had means to solve problems that related to angles. One of them is the movement of the stars and planets that many cultures investigated as a way to predict seasonal changes of the weather, to predict the future, or to know when it was time to conduct their rituals.

Alignments of neolithic or paleolithic monuments with the summer solstice and other celestial events have been reported by archaeologists and astronomers. The megalithic culture that flourished in the British Isles produced stone circles fit for astronomical observations, the most famous being the one at Stonehenge (Hicks, 1984). Also archaeological medicine wheels left by the ancestors of the Plains Indians seemed to have been used to predict the time of the sundance ceremony (Eddy, 1980). The Bighorn medicine wheel, that can be found in the Bighorn Mountains of Wyoming, has a diameter of 27 meters. In the center, there is a cairn from which radiate 28 spokes. Along the periphery, there are six other cairns. Archaeologists assume that this monument had the following function:

> Aldebaran's brief flashing in the sky ... would warn you that the day of summer solstice had arrived. An hour later, the location of the sunrise itself would confirm this. So would the sunset that night. One month later [28 days, to be exact], Rigel would appear in the morning sky; and one month after that [again 28 days, to be exact], Sirius. These alter events may simply have marked off the time during which the mountain could be occupied. The rising of Sirius would have been a good sign to leave Medicine Mountain because winter was coming (Eddy, 1980, p. 14).

A rudimentary system for comparing different directions seems to have been used together with the idea of measuring the days that separate the risings of the successive stars by the 28 spokes. Other cultures also used celestial events to measure time. The Incas, for example, built cylindrical towers on the highest hill west of Cuzco, and from a central pyramid they observed the position of the sunset relative to those towers. This permitted them to determine the times for planting and for holding the corresponding religious ceremonies (Aveni, 1980; Broda, 1982).

Early cultures also used angles to solve problems related to the construction of buildings. The Mayas of Uaxactun constructed their buildings in such a way that their position, when observed from the top of a central pyramid, would permit them to determine the sunrise on the days of the summer and winter solstices. The equinoxes were also embodied in the construction arrangements (Aveni, 1980; Broda, 1982). Mayan processes for the observation of the sky and for the determination of directions that matched the position of the sun at precise astronomical events were not followed by their rigor in the determination of right angles. Aveni and Hartung (1982) show how the precision in the layout of parallel lines in the direction of special positions of the sun equates the accuracy achieved in modern buildings, but that directions perpendicular to these are not as exact.

Astronomical activities however did not serve the sole purpose of predicting natural phenomena. Cosmos, society, and ritual were a whole in these cultures and so the search for understanding phenomena that ruled the cosmos was also a means to regulate life on earth (Broda, 1982). Anthropological data suggest that for some cultures they were also a process of understanding philosophical, moral, and metaphysical dimensions (Reichel-Dolmatoff, 1982). Andean time and space and their relation to such broad dimensions as urban organization, social structure, ritual, and subsistence, were ordered by a complex system of astronomical observations. Andean villages were organized along a series of lines that radiated from a ceremonial center where astronomical observations were made. These lines were oriented towards a locally important geographical feature and showed the position of the rising or the setting of an important celestial body. These arrangements were embodied in a calendar system that related closely to the regulation of Andean life. Among the Inca, these spatial divisions were the responsibility of specific clans at specific times of the year. It is interesting to note that the Quechua Indians have the same word for space and time (Fabian, 1982). This very blend of space and time can also be found in the Aztec culture, where the count of days in many Aztec calendars was linked to directions in space (Aveni, 1980).

Egyptian use of inclination

In Egypt, the construction of pyramids and other buildings with inclined walls did not seem to make use of angles (Robins & Shute, 1985), nor did they have a specific word for angle (there is none in Gillings, 1972). To have an idea of how problems involving inclination were solved, let us observe Problems 56-60 of the *Rhind Papyrus* (c. 1650 BC.) Problem 56 asks: “if a pyramid is 250 cubits high and the side of its base 360 cubits long, what is its seked?” (Chace, 1986, p. 51). “Seked” (or “seqet”) is an Egyptian word whose meaning is close to our notion of gradient (literally, “that which makes the nature,” Heath, 1931, p. 79). It is a measure equivalent to the cotangent of the angle of slope. The solution of the problem above involves dividing 360 by two, which is half the base, and then finding the quotient of 180/250. Since one cubit is 7 hand’s breadths, the result is multiplied by 7 so that the result can be presented in hand’s breadths. In contemporary terms, we would say that the scribe was using either similar triangles or an early version of trigonometric ratios (Boyer, 1968; Gillings, 1972). A similar process seems to have been used in two-dimensional drawings, except that a drop of six units was used instead of seven (Robins & Shute, 1985).

Egyptian astronomers invented what is called a diagonal star clock, which was in use by 2400 BC (Neugebauer, 1957; Krupp, 1977). This star clock was in tune with the Egyptian calendar that accounted for 365 days divided into twelve months, each divided into three ten-day weeks, plus five extra days. Each “week” (decade) was marked by the heliacal rising of a star, which provided a rough division of the celestial globe into 36 zones. This star clock is simply a grid, in which each square represents a date and a time. The user just has to identify the star that is rising in the horizon at that particular time and, by knowing the decade of the year, determines the hour of the night. The Egyptians essentially measured the night time by the position of stars or groups of stars.

This method was replaced by the Ramsside star clock about 1500 BC. This latest system required that the position of the star in the sky be compared relative to the direction North-South and to the body of one of the astronomers. The two astronomers sat facing each other on a north-south line, and the northernmost astronomer, the observer, watched the progress of stellar transits in the sky behind his partner to the south. The observer apparently obtained a reading by superimposing his plumb line upon a star (Krupp, 1977).

This system apparently made use of a coordinate system where the horizontal lines represented the hours and the vertical lines the positions of the stars, but the records of the positions of the stars remained essentially qualitative (Neugebauer, 1957). The Egyptians also made use of shadow clocks that would permit the readings of daytime hours by observing the length of the shadow formed by the sun (Krupp, 1977).

Babylonian invention of the zodiac circle measurement

Although the Babylonians, as the Egyptians, had no word for angle (Bruins, 1964), they developed very sophisticated techniques for recording and predicting the movement of celestial bodies. A Chaldean cuneiform text, for example, shows a table recording the motion of the planet Jupiter including the constellations where the planet is located and the position in degrees and minutes of arc (Aveni, 1980). The fact that the planets, as the sun and the moon, move inside a narrow band that stretches through the sky seems to facilitate the development of this technique.

The arrangement of stars into constellations that formed the zodiac signs seems to be their first device devoted to recording the motion of the planets. This first approach was followed by a numerical procedure to record the position of celestial bodies. This procedure combined the notions of space and time differently than Andean cultures. It seems that a Sumerian (3000 BC - 2500 BC) unit of calculating distances was the “mile”, corresponding roughly to 10 km. This unit quite naturally became an unit of time, one “mile” being the amount of time it took to walk one “mile”. Gradually the time associated with these “time-miles” became quite independent of their “length-miles” roots. A parallel process took place in astronomy. In the first part of the first millennium BC, Babylonian astronomers transferred the “mile” to the measurement of celestial events, by stipulating that the number of miles contained in one day is simply equivalent to one revolution of the skies. Each day contained 12 “miles” (danna) and so the circumference of the sky contained also 12 “miles”. Each of these miles was then divided into 30 subdivisions, called “lengths” (us), which produced 360 parts for the whole sky. This seems to be the origin of the astronomical division of time into 360° and, consequently, the division of the circle into 360°. These “degrees” were the fundamental units for the measurement not only of arcs, but also for the measurement of time. This way of measuring time using a distance unit carried its influence to the Greeks and Romans (Neugebauer, 1983).

By 700 BC, the Babylonians had organized the fixed stars in three “roads”, the middle one being an equatorial belt of 30° width. By this time there were systematical observations of celestial events, and by 400 BC the Babylonians created the zodiacal circle. The constellations which lent their names to the zodiacal signs are, of course, much older. But it was only for mathematical reasons that a definite great circle which measured the progress of the sun and the planets with respect to exactly 30°-long sections was introduced (Neugebauer, 1957).

From this time on, the Babylonians produced accurate tables of astronomical events which would, for example, give the days and the positions measured in degrees within each zodiacal sign. Their theoretical model involved the use of zig-zag functions in order to account for the variation of the speed of the sun across the sky in different epochs of the year. Its predictions were almost accurate, and it seems that the Babylonians were not very concerned with a precise determination of the position of celestial bodies. In fact, they even seem to lack accurate instruments to do so. The Babylonians produced detailed tables for the astronomical events of the moon and several planets, which seemed to be used essentially to determine favorable and unfavorable conjugations (Seidenberg, 1975). Their calculations involved complex concepts such as the angular velocity of the celestial bodies, but the complexity of their astronomical methods did not transfer to other geometrical problems.

Greek pre-Euclidean conceptions of angles and directions

Early Greek astronomy was essentially qualitative. A quotation from Eudoxus (408-355 BC) gives us a perspective of how early Greeks described the position of the stars :

> There are on this the middle section of the Crab and the longitudinal part of the body of the Lion, a small part of the upper section of the Virgin, the neck of the gripped Serpent, the right hand of the Kneeler (i.e. Hercules), the head of Ophiuchus, the neck of the Bird (i.e. Cygnus, the Swan) and its left wing, the feet of the Horse (i.e. Pegasus) (quoted in Dicks, 1970, p. 158).

This shows the kind of verbal description of the position of stars common in those times. Although Eudoxus conceived the stars as evolving on concentric spheres that can be plotted on a celestial sphere and dividing it into twelve parts, his descriptions are essentially qualitative.

Although their astronomical methods are rudimentary when compared to the Babylonians, Greek geometers seem to be the first to have a word for angle (Gray, 1979) and, as reported by Plato, to distinguish between acute, right, and obtuse angles (Heath, 1956). They were also the first to develop geometrical proofs. Pythagoreans, for example, proved that the angle sum of a triangle equals two right angles (Gray, 1979).

Aristotle is one of the early Greek philosophers whose mathematical observations are well documented. He made explicit references to angles (as he does to lines, surfaces, and other mathematical objects). One very interesting reference can be found in his book *Analytics Posterior*. He discussed the rule that the two premises of every syllogism must have between them an affirmative and a universal proposition (Maziarz & Greenwood, 1968). From his argument a valid proof of the equality of the base angles of an isosceles triangle can easily be constructed, provided we accept his underlying conception of angle. The proof goes like this (Aristotle used different notations and terminology). Let AOB be an isosceles triangle inscribed in a circle of center O. Consider now the angle A, which is the mixed angle between the diameter that contains OA and the circle (angle $\alpha+\gamma$ of the Figure 1). Identically, consider the

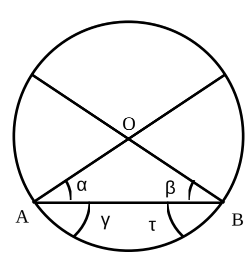

Figure 1

mixed angle B. We know that angles A and B are equal because “angles of semicircles are equal”. We also know that angles γ and τ are equal because “the two angles of all segments are equal” (segments here refer to chords). Consequently, $\alpha = A - \gamma = B - \tau = \beta$ (Heath, 1956; Heath, 1949).

If we accept the terms angle of a semicircle and angle of a segment, this proof is logically coherent. His works do not discuss mathematics *per se* because although Aristotle had a strong background in mathematics (after all it was one of the topics extensively taught in Plato’s school), he was primarily a philosopher and not a mathematician. In spite of the fact that this proof does not play a central role in the point he is making (Maziarz & Greenwood, 1968), it does provide a perspective of his concept of angle.

A central issue in Aristotle’s discussion of geometrical objects was the determination of their nature. Length, for example, was a quantity; parallelism, was a relation, and a triangle was a quality. Angle, straight, and circular were kinds of figures for him. Aristotle himself is believed to have had the opinion that an angle was a quality. In some writings, Aristotle seems to have conceived of angle as a deflection or breaking of a line: “a line, if it be bent, but yet continuous, is called one” (Heath, 1956, p. 159). Eudemus, a disciple of Aristotle who wrote books about the history of arithmetic, astronomy, and geometry and is said to have written a book about angle, is quoted by Proclus as saying that an angle is a quality. Moreover Eudemus had the opinion that an angle is a fracture or a deflection of a line (Heath, 1956; Morrow, 1970).

Another issue is the way the diverse types of angles relate to each other. Aristotle discussed the priority of the right angle in comparison with the acute. On the one hand, the right angle is defined first and so should have priority. On the other hand, the right angle is composed of acute angles, which seems to give priority to acute angles. Aristotle also hints at what could be a nice definition of right angles. A right angle is reached when a thing falls on the ground and rebounds, making similar angles on both points of impact. The right angle is seen here as the limit of the opposite angles (Heath, 1956).

These early Greek geometric efforts were also concerned with the issue of tangency which, as we shall see, will relate closely to the concept of angle. The concept of the tangent to a circle in a point led to some controversy about the nature of this contact, i.e., on how many points does the tangent touch the circle. Protagoras, the Sophist, appealing to common sense arguments, claimed that the tangent actually must have more than one point common to the circle or, in other words, that it touches the circle in some length. A related problem arose when discussing the nature of the tangency between a sphere and a plane. Democritus seemed to have answered this critique by a circle and a tangent (Heath, 1949; Maziarz & Greenwood, 1968).

Euclid’s conception of angle

“A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line... when the lines containing the angle are straight, the angle is called rectilinear” (Heath, 1956, p. 176), is the definition that Euclid presented in *Book I* of the *Elements*. On the one hand Euclid seems to define angle as a set of two lines with specific characteristics, but on the other hand, he suggests that angle is a kind of area contained by two lines. This definition of angle is included in the beginning of *Book I*, together with the definition of points, lines, surfaces, figures, and parallel lines.

According to Heath (1956) the concept that angle is an inclination is a new departure. The prevailing tendency at that time, as we saw in Aristotle, was that an angle resulted when a straight line would break or deflect, that is, the two sides of the angle originated from the same line. Nevertheless, as we saw above, Euclid also used the terminology that an angle is contained, much in the same way that he talks about figures being contained by lines (Heath, 1926). It seems that Euclid thought of an angle as the space “in between” the two lines. Heath (1956) argues that this shows that Euclid also conceived of angle as a quantity, as can also be seen, for example, by his quantification of angles in Propositions I.9 and I.13.

Euclid’s definition of rectilinear angles excludes the zero angle and angles greater or equal to a straight angle. This view was taken consistently across the *Elements*. Whenever straight angles are involved, Euclid instead talked about two right angles. Even on two propositions (Propositions III.20 and VI.33) that could be applied to angles greater than two right angles, Euclid did not even hint at that possibility, and if such angles were presented to him, he would not recognize them as angles at all or would consider them a composition of angles (Heath, 1926). That is exactly what Heron (third century AD) did in his discussion of Proposition III.20 of the *Elements*, where he considered “angles” (plural) whenever the angle BEC in figure 2 below exceeds two right angles. Only the Middle Ages’ commentators (Tartaglia, Peletier, and Clavius) considered angles greater than two right angles to be a whole angle

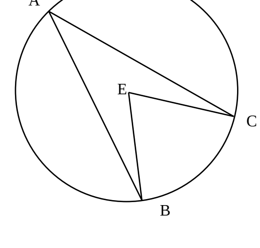

Figure 2

(Heath, 1926). Another proof of the absence of straight angles in Euclid’s geometry can be found in his definition of perpendicular lines. When two straight lines intersect at equal angles, Euclid called these right angles, and the lines were said to be perpendicular to each other (Definition 10). If Euclid had conceived of zero or straight angles, this definition would not hold because two collinear straight lines would also intersect at equal angles, and would consequently be perpendicular to each other. Euclid’s word for perpendicular means literally “let fall” which shows its connection with the plumb line (Heath, 1956). Proclus informed us that in ancient times the perpendicular was called “gnomon-wise” because the gnomon was set up at right angles to the horizon (Morrow, 1970).

The contemporary requirement that angles have straight sides was not followed by Euclid. In his Proposition III.20, when talking about angles at the circumference of a circle (inscribed angles, in our terminology, Euclid referred to a portion of the circle as the base of an angle (Figure 3). But, more important than that, Euclid’s definition of angles holds for both angles with curved and straight sides. These different types of angles are further elaborated in *Book III* (Heath, 1926). In this book, as illustrated by Figure 4, Euclid defined the segment of a circle as “the figure contained by a straight line and a circumference of a circle”, an angle of a segment as “contained by a straight line and a circumference of a circle”, and an angle in a segment as “the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the base of the segment, is contained by the straight lines so joined” (p. 1). Later in *Book III*, Euclid used the “remaining angle”, that will be known as a “cornicular angle” or as a “horned angle” in Proclus’ terminology (Morrow, 1970). Euclid also used this type of angle in his book *Optics* (Knorr, 1986). Although Heath (1926) considers that this type of angle plays no significant role in the *Elements*, its use in *Book III* is not easy to dismiss as an archaism. It seems that Euclid’s concept of angle was similar to a corner between two lines that could be straight or curved. In several propositions he stresses the fact that he is referring to rectilinear angles (Propositions 9, 23, 42, 44, 45 from *Book I*, for example.)

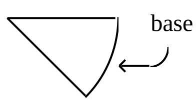

Figure 3

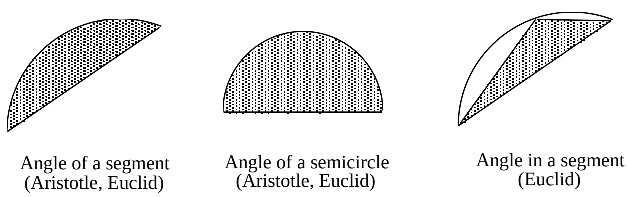

Figure 4

As we shall see, this type of angle raised much controversy which lasted until at least the 17th century (Heath, 1956). In Euclid’s time, the discussion was about ways to compare the angle of a semicircle with rectilinear angles, and the *Elements* provide a crucial proof that establishes this relationship. Proposition III.16 of the *Elements* states that the angle of a semicircle is greater than any acute rectilinear angle, and that the remaining angle is less than any acute rectilinear angle. Euclid proved it in three steps. Let AB be the diameter of the circle ACB as shown in Figure 5, and let AE make a right angle with AB. First he proved that any straight line drawn from A making a right angle with AB lies outside the circle. Then he proved that no straight line can fall in the space between AE and the circumference. To conclude he showed that “the angle of the semicircle contained by the straight line BA and the circumference… is greater than any acute rectilinear angle, and the remaining angle contained by the circumference… and the straight line AE is less than any acute rectilinear angle” (Heath, 1926, p. 38). This was certainly a paradoxical result because it showed that there were some quantities (namely remaining angles) that no matter how much they were increased would never become greater than other quantities (acute rectilinear angles, in this case.)

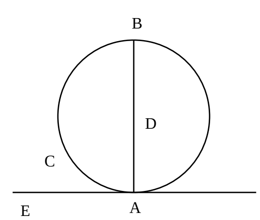

Figure 5

As we have seen, the issue of lines tangent to a circle was a much debated topic in Greek geometry. Euclid contributed to this debate by showing that a straight line is tangent to a circle at a point if any other straight line that contains the point of tangency also contains another point of the circle. This concept of tangent was also used by Archimedes (Knorr, 1986). Although Euclid does not use the word “tangent”, he drew the conclusion that the straight line AE (Figure 5), drawn at right angles to the extremity of the diameter of a circle evidently touches the circle. This notion that straight lines touch circles appears in many other places of the *Elements* to refer to tangents. Euclid’s insertion of this proposition seems to be his contribution to the controversy that was current in his time about the nature of tangency.

Post-Euclidean conceptions of angle

The discussion about the nature of angle in post-Euclidean geometry centered around three related problems: what is a good definition of angle; where do angles fit the Aristotelian categories of quantity, quality, and relation; what is the nature of curvilinear angles.

Apollonius of Perga (262-190 BC) is quoted by Proclus as having provided a new general definition of angle. For him an angle was “the contracting of a surface at a point under a broken line, or of a solid under a broken surface” (Morrow, 1970, p. 99). Note the movement implied by this definition. Again a reference to a broken line or surface shows up. He conceived a cone as an angle formed by lines that came to an end at the apex (Bello, 1983). Geminus (first half of the first century BC) is quoted also by Proclus as saying that an angle is formed when a line is broken (Morrow, 1970).

Heron (third century AD) said that an angle is a quantity which a simpler related quantity encloses when it comes to a point. Albertus Magnus gave the following explanation for this definition (Bello, 1983):

> For a surface angle is enclosed by lines, since it is midway between a quantity that has one dimension and another that has two, while itself it has two. A solid angle, though, terminates in surfaces, being midway between a surface, which has two dimensions, and a solid, which has three. (pp. 9-10)

The surface of a cone, for example, was thought to enclose the solid angle that terminates at the apex of the cone. Heron was the representative of a Babylonian tradition in geometry, and was more concerned with the actual calculation of solutions that opposed the new proof-oriented approach (Bruins, 1964).

A different perspective was taken by Plutarch of Athens who was Proclus’ teacher. Plutarch contended that an angle is “the first interval under the point, [because] there must be some first interval under the inclination of the containing lines or planes” (Morrow, 1970, p. 101). Plutarch seems to have been thinking about a rate of divergency of the lines under consideration (Heath, 1956). Similarly Carpus of Antioch (a Pythagorean) conceived angle as a quantity, namely a distance between the lines or surfaces containing it. But paradoxically, he did not think of this distance as a line (Morrow, 1970).

Mixed angles or horned-shaped angles continued to exist in the literature. Heron used them in his book *Catroptrics*, as did Theon (Knorr, 1986). Diocles, author of a book about burning mirrors written around 190-180 BC, made occasional use of them. In his second proposition he considered the line AB (Figure 6) which is reflected on B (reflected by a mirror) to produce the line BC such that angle α is equal to angle β (Toomer, 1976).

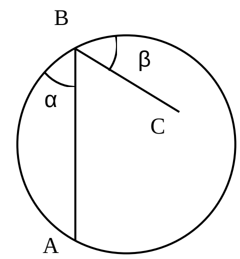

Figure 6

Another interesting use of mixed angles can be found in a mathematical fragment of disputed origin, called the *Bobbio Mathematical Fragment* which seems to have been influenced by the work of Diocles. The manuscript may be dated from the late antiquity before the seventh century AD, and presents a proof about the focal property of the parabola. Figure 7 illustrates an important part of this proof. A parabola is drawn with A and B referring to the marked rectilinear angles. All the Greek letters refer to mixed angles either between the parabola and its tangent (α and β) or between the parabola and other straight lines (γ and θ). In the proof the author made

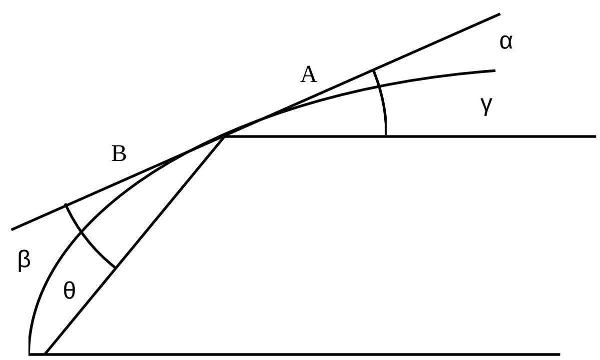

Figure 7

the following point: as <β = <α then <θ is equal to <γ, probably because <A is equal to <B, although the author did not add this last condition (Toomer, 1976). The remarkable feature of this is that the author was equating two mixed angles that are in fact not visually equal. The manuscript is just a fragment and does not allow us to go any further. Mixed angles were also briefly used by Pappus with no demonstrative purpose (Jones, 1983).

The development of trigonometry

One of the developments of post-Euclidean geometry was the systematical development of trigonometry. Aristarchus (300 BC) seems to be the first to have developed a sketch of trigonometry in Greek mathematics. In his book, *On the Sizes and Distances of the Sun and the Moon*, he set out to prove several propositions about distances and apparent diameters of the sun, the moon, and the earth. To do this, he used properties of what we know now as trigonometric ratios. These properties allowed him to calculate intervals for the values of sines and cosines of 3° and 1°. The apparent diameters of the earth, the sun, and the moon, were given in fractions of the zodiac circle. The sun's apparent diameter, for example, was estimated as 1/15 of a sign (i.e., 2° in our units), which is a gross overestimation (Heath, 1931).

Hypsicles (second half of the first century BC) provided the first evidence of the division of the ecliptic into 360° in Greece. It is interesting to take a look at his method:

> The circumference of the zodiac circle having been divided into 360 equal arcs, let each of the arcs be called a degree in space, and similarly, if the time in which the zodiac circle returns to any position it has left be divided into 360 equal times, let each of the times be called a degree in time (Thomas, 1968, pp. 395-397).

At about the same time trigonometric methods for astronomy were developed by Hipparchus (second century BC), who wrote a treatise on straight lines in a circle (i.e. chords) developing a table which related arcs of a circle to the chords subtending them (Dicks, 1970; Neugebauer, 1983). Hipparchus apparently was also the first to state the position of more than 850 stars in terms of latitude and longitude in relation to the ecliptic (Heath, 1931). Ptolemy (second century AD) completed Hipparchus' developments. He worked with a circle whose diameter was divided into 120 parts. Using both elementary geometry and approximation procedures he computed a table of chords at 1/2° intervals (Callinger, 1982).

Angles in India and China

Other cultures did not seem to have shared the Greek's interest in angles and other geometric topics of Euclidean geometry. The *Elements* of Euclid and a book on arithmetic were the first Western works translated into Chinese at the end of the sixteenth century, and they required the creation of new words for the concepts of point, line, straight line, curve, parallel lines, angle, right, acute, and obtuse angle, triangle, and quadrilateral (Yan & Shirán, 1987).

Chinese mathematics included the solution of concrete geometrical problems (determining the height and distance of a mountain, finding the width of a stream, measuring square and circular towns) using ratios. A Chinese mathematics book, the *Zhõubi suànjing*, dating from the end of the second century BC (some historians place it much earlier), uses proportions between similar right triangles to compute several distances (Yan & Shirán, 1987). Chinese mathematicians made use of the equivalent of our trigonometric methods based on an extensive use of similar right triangles and the Pythagorean Theorem, but the concept of a ratio of segments as a function of an angle was completely absent (Libbrecht, 1973).

Ratios were also used by the Hindus. In a book from the late fourth or early fifth century AD, *Siddhãntas*, a study of the relation between the half chord and the half angle subtended at the center was developed, thus producing our contemporary trigonometric functions (Boyer, 1968). Hindu theoretical astronomy measured radial distances using the same units as the length of the circumference. As they were also using the Babylonian division of the circle into 360°, they were compelled to use a radius of 57° 18' to obtain a circumference of 360° for the circle ($2\pi \times 57° 18' = 360°$) (Neugebauer, 1983).

Note

1. This is the first of two parts. Part 2 will be appear in the next issue.

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It hath been an old remark, that Geometry is an excellent Logic. And it must be owned that when the definitions are clear; when the postulata cannot be refused, nor the axioms denied; when from the distinct contemplation and comparison of figures, their properties are derived, by a perpetual well-connected chain of consequences, the objects being still kept in view, and the attention ever fixed upon them; there is acquired a habit of reasoning, close and exact and methodical, which habit strengthens and sharpens the mind, and being transferred to other subjects is of general use in the inquiry after truth.

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