Kaput's Multiple Linked Representations and Constructivism

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With the introduction of readily available technology, the subject of physical representations of mathematical concepts has become more of an issue for mathematics educators (Kaput, 1989). How technology is to be used for student learning may depend on the choice of representations inherent in the particular technological tools used. In discussing the idea of reflective abstraction and representations Steffe said:

> Our operationalization of reflective abstraction is compatible in its general outline with Kaput's (1988) view of how mathematical representational systems arebuilt in computer learning environments. (1989, p. 7)

The purpose of this paper is to explore one example of Kaput's model of multiple linked representation systems and how that example is compatible with constructivism.

Kaput's Representational Systems

James Kaput (1989) has stated that mathematical meaning-building can come from essentially four sources, grouped in two categories:

Referential Extension

Via translations between mathematical representation systems

Via translations between mathematical representation systems and non-mathematical systems...;

Consolidation

Via pattern and syntax learning through transformations within and operations on the notations of a particular representation system;

Via mental entity building through reification of actions, procedures, and concepts into phenomenological objects which can serve as the basis for new actions, procedures, and concepts at a higher level of organization...[i.e.] reflective abstraction (p. 168)

For the purposes of this paper, we will consider only the processes of translations between groups of mathematical representation systems. According to Kaput, the translations occur in two different areas (See figure 1), the mental world and the physical world. The mental world is the "...essentially private world of mental events and state changes..." 1989, p. 169) and the physical world is the "...public world of characters and lines in some physical medium..." (1989, p. 169). Both of these world views, as well as the correspondences between them, make up a representation system, and in this particular case, a mathematical representation system.

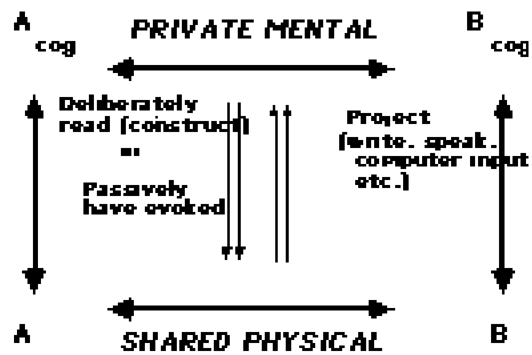

Figure 1: Correspondences between mental and physical representation systems (Kaput, 1989, p. 169).

Both the mental and physical representations may be used by the learner to make changes in a particular mathematical concept of the learner. The correspondences shown comprise a representation system. These systems act to help organize the thought processes so the learner can build an internal representation based on an external model (Kaput, 1988).

Referring to Kaput, Goldin defines a representation system as containing five components:

(a) a represented world;
(b) a representing world;
(c) aspects of the former which are represented;
(d) aspects of the latter which do the representing;
(e) the correspondence between them.
(Goldin, 1987, p. 126)

Based on the above definitions, Kaput (1989) illustrates three representation systems of algebra that he calls "core" systems. These three systems include the notational system of algebra, the graphical system associated with the notations, and the tabular system associated with both the graphical and notational systems. These three systems are linked in a way that essentially provides equal importance to each component. The formal, set theoretic system associated with the other systems is included as a central, unifying structure. Each node of the network is considered to be a notation system[1] and the notation systems are linked to form a broader, super-system in algebra (See figure 2). Kaput goes on to state that the balance of the importance of each of the individual notational systems depends on the way in which we define what it means to "know algebra".

Since working with the external representations of mathematical functions of one variable is a major emphasis of the teaching of algebra and of mathematics in general, the linked representation example in figure 2 will be the model used as the basis for the remainder of the paper. Figure 2 shows Kaput's model for functions of one variable. As an example of representation systems, Kaput's multiple linked representations model lends itself well to further analysis. The system E-2 represents single valued functions of real numbers written as an equation such as $y = x^2$ or $f(x) = 3x + 1$. Syntax rules in E-2 are the familiar rules for solving equations that are common in the algebra curriculum. The system of G-2 consists of the graphs in the Cartesian coordinate system of functions of one variable such as the familiar parabola graph of a quadratic function or the line graph of a linear function. The system T-2 contains the values of the independent and dependent variables that would be generated for an arbitrary number of substitutions in E-2 or an arbitrary number of discrete points of G-2. G-2 may be thought of as the familiar "T" tables of first year algebra. The three core representations are linked by virtue of the fact that all of these notation systems are used to represent the concept of function.

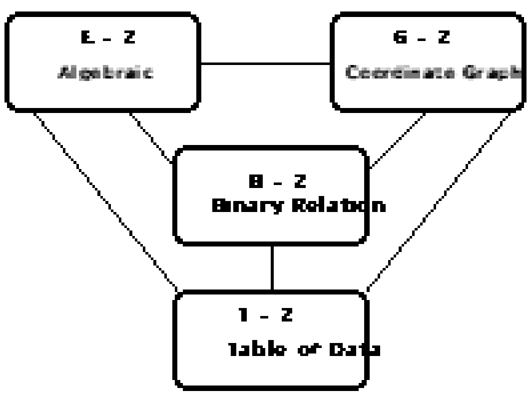

Figure 2: Interrelationships among various two-variable algebraic notational systems (Kaput, 1989).

B-2 is also a representation of function. It consists of the abstract definition of function given as a set of binary relations on real numbers. This system is "feature-bare" (Kaput, 1989, p. 170) but common in formal discussions of mathematics. "B-2 is usually thought of as occupying a privileged position in that the other systems are regarded as representing it" (Kaput, 1989, p. 170).

The system E-2 is an example of what Kaput (1987) describes as a symbol system, in that E-2 is a symbol scheme[2] along with a field of reference (the real number field) and a systematic rule of correspondence between them. E-2 also satisfies Goldin's definition of representation system. The represented world is the field of real numbers. The representing world is the system of symbols usually associated with algebra. The aspects of the represented world are the field properties of real numbers. The aspects of the representing world are the syntax rules that are associated with the manipulations involved in the process of solving equations. The correspondence between the worlds is the preservation of the field properties whenever the manipulations of algebra are performed. Thus E-2 is both a representation system and a symbol system. As a representation system, E-2 lacks the richness that we usually assume it contains. That is, it alone is not a strong enough representation system to represent the world of physical experience that we in education usually try to force upon it (Herscovics, 1989).

Likewise, the systems G-2 and T-2 are not complete representation systems in the sense that alone, they do not necessarily convey the information we expect. Information may be gained from Cartesian graphs, but the graphs themselves are not enough to give a complete picture of the function concept. Similarly, a table of data by itself is not complete. It can only convey a discrete picture of a concept that is, in most school situations, a continuous one. The least informative representation system is B-2 because the concept of binary relations and set-theoretic definitions are so general that the system fails to communicate much more than a mere sketch of the concept of function. But as a system, "...the cognitive linking of [these] representations creates a whole that is more than the sum of its parts ..." (Kaput,1989, p. 179).

A Constructivist View of the Example

The multiple linked representation system of Kaput's model generally refers to an external system of presentations that the learner may use to organize concepts (von Glasersfeld, in press). von Glasersfeld views the "didactic" use of what he calls graphic and schematic representation as being based on "selective isomorphisms." (von Glasersfeld, 1990 p.10) That is, graphic and schematics representations "help to focus the naive perceiver's attention on particular operations that are deemed desirable." (1990, p.10) This appears to be a basic requirement for building concepts. With the basic mental representations of notation, graph and table in place, and without answering the questions involved with construction of those basic structures, the linked physical representations of Kaput are consistent with the constructivist view. But, as Kaput states, "meanings are developed within or relative to particular representations or ensembles of such" (Kaput, 1989, p. 168).

The overriding concern of the constructivist epistemology is the issue of reality (von Glasersfeld, 1984). In this example, the question centers on the idea of what is the "real" definition of function. In the case of algebra, the formal, mathematical definition, as represented by B-2 in figure 2, is not the definition that the learner eventually constructs. The goal of the activities with the representations should be to lead the learner to a definition of function that is more than the sum of the parts of the representation. The representations should serve as a language with which the student organizes and reorganizes experiences about function (von Glasersfeld, 1987). Instead of viewing the representations of G-2, T-2, and E-2 as stepping stones to the formal definition of function, the constructivist would say that experiences with G-2, T-2, and E-2, as well as B-2, could lead to a concept of function that is always changing with new experiences and never complete. Thus the physical representations are all part of the same thing (are compatible or isomorphic), and may act as ways and means to communicate about the experiences. This compatibility and overlapping is needed for satisfactory communication (von Glasersfeld, 1987). Thus, for the constructivist, "there is no absolute meaning for the mathematical word function, but rather a whole web of meanings woven out of the many physical and mental representations of functions and correspondences among representations" (Kaput, 1989, p. 168).

The links in Kaput's model refer to the regularities that one finds in moving from one system of representation to another. The similarities of one system to another make it possible for a learner to use one representation to build a stronger concept in another system. Further, the learner must find "a way to fit available conceptual elements into a pattern that is circumscribed by specific constraints" (von Glasersfeld, 1987, p. 9). These "conceptual elements" are the assimilated constraints made in the language of the subsystem of the representation system. Thus, the act of moving from one system to another while comparing and reorganizing the concepts of function, promotes the reflection and self-monitoring necessary for further assimilation and abstractions. Multiple representations act to enrich the activities from which experiences are gained, thus perhaps leading to learning in both representations like the learning referred to by von Glasersfeld as "[drawing] conclusions from experiences and to act accordingly" (von Glasersfeld, 1987, p. 8).

In the constructivist view, the model for multiple linked representations would probably look similar to the model in figure 2 that Kaput has formulated. The main difference would be that the system B-2 would not have the prominence in the figure that Kaput has implied. The model would probably look more like figure 3, with B-2 lowered to the same cognitive level or a slightly lower level than the other nodes. The constructivist would view the formal definition of function as an important part of the definition, but not more important than the examples and experiences that go into making the concept of function for the learner (Steffe, 1989b). Thus B-2 could be effectively eliminated in the early stages of learning. Indeed, B-2 would be a minor part of the concept until the corresponding concepts of E-2, G-2, and T-2 are firmly in place. For school algebra, this would imply that the formal definitions be postponed until a rich background of concepts has been developed by the student.

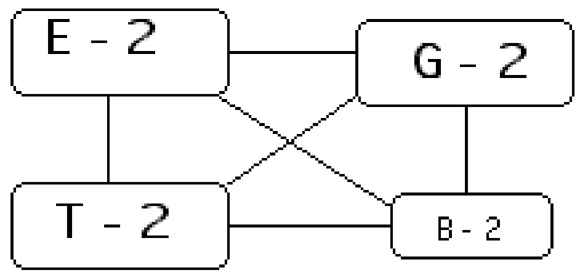

Figure 3: The constructivist view of Kaput's model.

The links of Kaput's model would still be in place in the constructivist view. Each of the nodes are connected to all the other nodes directly, with a bidirectional path of length one from each node to the others. A learner would move from one representation to another in much the same way as the marchers make "progress" in M. C. Escher's *Ascending and Descending* (a simplified version is shown in figure 4), always going upward by building on those concepts already formed, and then returning to previously formed concepts on a richer conceptual level, completing a "strange loop" (Hofstadter, 1979), that is a

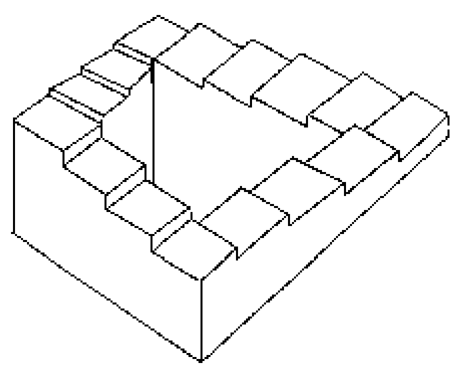

Figure 4: Simplified version of *Ascending and Descending*. (Jacobs, 1974, p. 538).

circular journey that ends at the beginning but on a different level. One can imagine that the steps are drawn in such a way that the learner could move directly upward to any of the other three nodes without going through any intervening nodes. Likewise, the learner could move backwards to a comfortable level and then take another path upward. Each step would involve self-monitoring and possible reconceptualization.

One suggestion that Kaput (1989) has proposed is that the representations model be implemented in a computer environment in such a way that the links from one representation to another be actual translations from one setting to the other on the computer screen. In this form, a learner could have windows containing the three basic representations of a function. The student could change parameters in one representational window and the modifications would automatically be made in the other windows. For example, suppose $f(x) = x^2 + 1$ is shown in the equation window, the graph of $y = f(x)$ is shown in the graph window, and a table of the data points is shown in the table window. The learner could change the function to $f(x) = x^2 - 1$, for instance. The resulting changes in the graph would be made in the graph window and the table of data would be changed accordingly. Likewise, the learner could modify the shape of the graph in the graph window, perhaps by using a mouse as in FUNCTION PROBE by Confrey (1989), and the resulting change would be made to the equation displayed in the equation window. The appropriate correction would also be made automatically in the table display. Or an alteration to the data would result in a changed graph and modified equation. Thus the work-space would provide a dynamic setting for activities in each of the representation windows.

This dynamic window setting would fit the constructivist view of the way the learner builds knowledge. Switching from one window to another and watching the changes made, could lead to reflection and reorganization of concepts. Operating in one window by using the references of the other windows could help the learner build new operations from the old ones. The teacher's responsibility, in such an environment, would be to help with the decisions about which representation to turn to next and to make suggestions about possible misconceptions (Steffe, 1989b).

So, the example of the representation system of Kaput, with some modifications, would be consistent with the constructivist view. The differences between constructivism and other world views is in the way reality is defined. For the constructivist, the reality of function is the learner's own definition of function. Making a physical representation for the learner to use is not a sufficient condition for the construction of the function concept, but activities with those representations may help the learner to develop the concept.

Notes:

1. "When A and its syntax are considered apart from a field of reference, they are called a *notation system"* (Kaput, 1989, p. 169).

2. "A *symbol scheme* is a concretely realizable collection of characters together with more or less explicit rules for identifying and combining them" (Kaput, 1987, p. 162).

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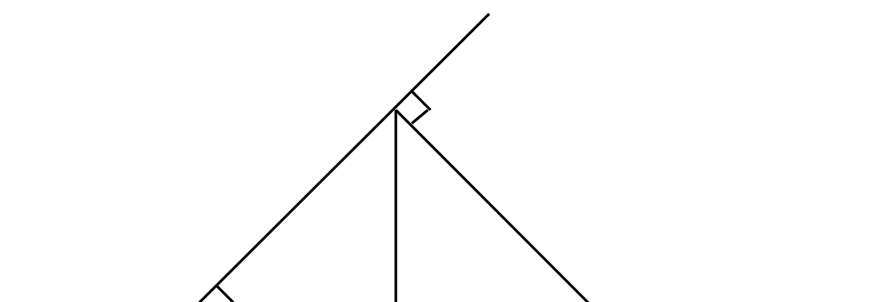
From Dr. Wilson's Notebook...

Spider Web

A spider has woven a web beginning with segment CA and then zigzagging between lines BA and BC as shown in the diagram below. Suppose that BC = 1 and that angle ABC has measure 45 degrees. Calculate the length of the zigzag path from C to B.



Possible Extensions/Variation: Vary the measure of angle ABC.