The Historical Development of the Concept of Angle (2) [1]

José Matos

In the first part of this article published in the Summer 1990 issue of the *Mathematics Educator*, we discussed the use of directions by paleolithic and neolithic cultures. We saw how angles, as distinct geometric entities, had their origin in Greek culture, and we discussed several problems associated with them. We also saw how other cultures like the Chinese and the Hindus could solve geometric problems without angles. In this second part, we will proceed by discussing the notion of angle during the middle ages, the narrowing of the concept by seventeenth century mathematicians, and the enlargement of the notion of angle by nineteenth century mathematicians. Finally, we will briefly discuss contemporary approaches to angle, namely its disappearance in a branch of contemporary geometry.

Proclus' conception of angle

Proclus, who lived in Athens and died in 485, is our main source of information about the geometrical concepts of the high middle ages. He provided an extensive discussion on the concept of angle in his comment on Euclid's *Elements* (Morrow, 1970) and described a classification of angles that seems to have its origin in the work of Geminus (Heath, 1956). Angles for Proclus were formed by two intersecting surfaces or lines. The surfaces did not need to be flat, nor did the the lines need to be straight. Angles between two straight lines (rectilinear angles) fell into three cases: acute, right, and obtuse angles. The angles between a circle and its tangent included an angle in a semicircle and a horned angle. These angles were also called mixed angles. Figure 1 illustrates these different types of angles. In the special case of the angles between two circles, Proclus distinguished among biconcave, biconvex, and lunular angles (Figure 2). Figure 3 on the following page provides a summary of Proclus' classification of angles.

For Proclus, angles, as all mathematical entities, had strong philosophical, ethical, and ontological meanings. For example, he made an interesting distinction between a perpendicular drawn to a line and a line erected from another at right angles (Morrow, 1970).

> The line erected at right angles is an imitation of life lifting itself to the upper world from the hollows below, rising undefiled and remaining uninclined towards worse things, whereas the perpendicular is a likeness of life following the path downwards and holding itself free of the indeterminateness in the world of generation. (p. 226)

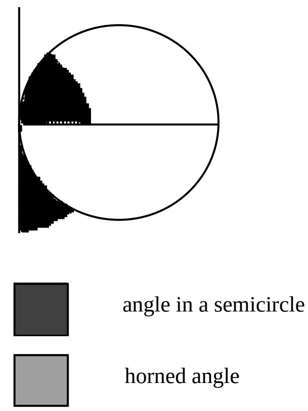

Figure 1: Proclus' description of angles between a circle and its tangent

The perspective that mathematical entities had strong connections to philosophical, ethical, and metaphysical problems was also present in two important issues about angles confronting the mathematicians of his time: the nature of angle, and the paradoxes produced by mixed angles. First, let us consider Proclus' discussion about the nature of angles, namely, whether an angle is a relation, as Euclid thought, a quality of a surface or of a solid, or a quantity.

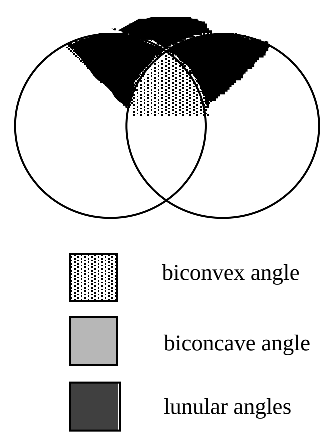

Figure 1: Proclus' description of angles between two circles

Proclus (Morrow, 1970) argued that an angle could be divided by a line or by a surface, like an ordinary

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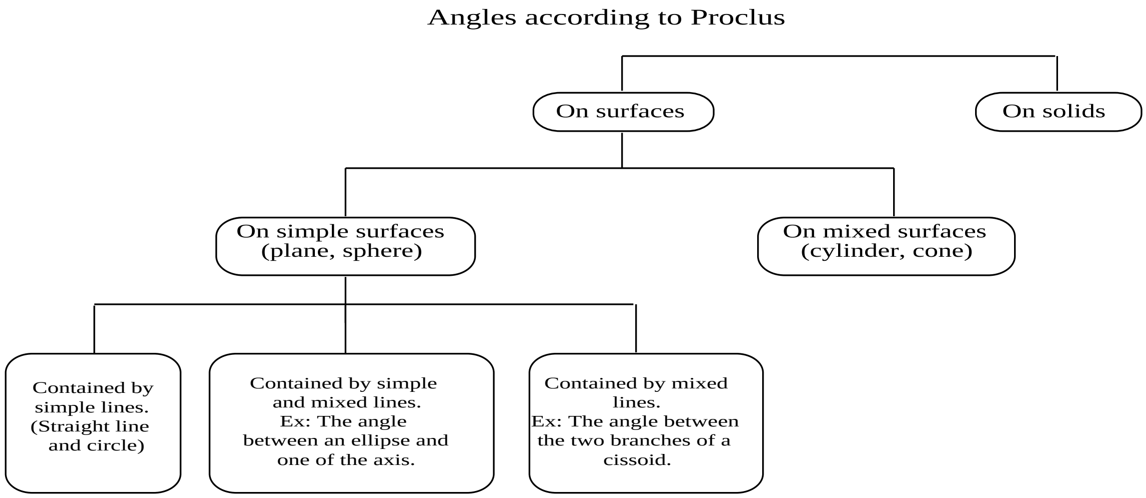

Figure 3: Proclus' classification of angles

quantity or magnitude. This meant that it was not a linear quantity, since linear quantites could be divided by a point; thus it should be either a surface or a solid. This fact seemed to deny an angle the attribute of being a quality because qualities can not be divided.

If we assume that an angle is a quantity, we run into problems from another area. Euclid's Proposition III.16, asserting that a horned angle (remaining angle, in Euclid's terminology) is smaller than any rectilinear angle, shows that there exist angles that no matter how much they increase, they will never become greater than a right angle. So there will never be a number that multiplied by any given remaining angle will produce an angle greater than a rectilinear angle. This contradicts Euclid's Axiom X.1 (currently called Archimedes' Axiom), and presents an immense obstacle to conceiving angles as a quantity.

A third possibility is to conceive angles as a relation between lines (or surfaces), as Euclid did. Proclus, however, pointed out that a relation between lines is not solely determined by the lines themselves. He gave an example of a cone cut by a triangle that passes through its apex. Although there is only one inclination between the sides of the triangle, two different angles are formed, namely, the angle in the plane of the triangle, and another on the surface of the cone. Both angles are contained by the two lines. Finally Proclus chose to follow the opinion of his teacher, Syrianus, and claimed that an angle was a combination of all of these categories. He equated an angle with a triangle that also presented these three attributes. He concluded that an angle was a quantity that provided it with the ability to be divided and compared, it was a quality by virtue of its shape, and it incorporated a relation because it needed the relation of the lines or surfaces that bound it (Morrow, 1970).

Proclus also provided some interesting developments on a second issue about angles confronting mathematicians of his time, the concept of non-rectilinear angles. In his discussion of Postulate IV of the *Elements*, which says that all right angles are equal, he quoted Pappus as saying that the converse is not true, namely, that there are non-rectilinear angles which equal one right angle and are not equal to a rectilinear right angle. As an example, Proclus drew the following figure where AB and BC were equal and formed a right angle, and AEB and BFC were semicircles.

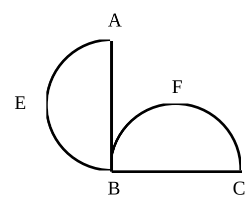

Figure 4: Curvilinear angles that equal right angles

Angle AEB is equal to angle BFC because the semicircles are equal. Let us add the angle ABF to each of those angles. Then <ABF + <FBC is equal to <ABC which is a right angle. But then <AEB + <ABF is also equal to a right angle. So we have an example of a curvilinear angle that is equal to a right angle. Proclus commented that the same held true if AB and BC met at angles other than right angles. Figure 5 illustrates two such cases (Morrow, 1970).

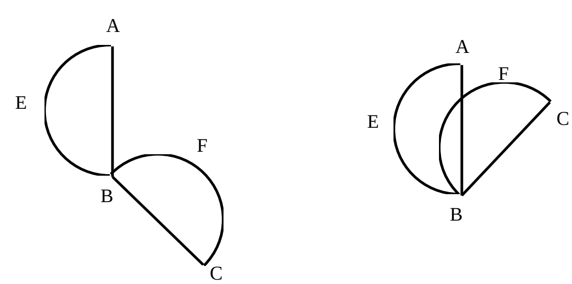

Figure 5: Curvilinear angles that are

For any given rectilinear angle, Proclus was able to construct an equal lunular angle (Morrow, 1970).

Note that that it could also be proven that the angle between two equal (congruent) curves was equal to the angle between the two tangent lines.

Proclus carried this notion of angle a step further. In a comment about the exterior angle of a triangle (Euclid's Proposition I.32) and the sum of the three angles of a triangle, Proclus added that the converse of this theorem may not be true for figures with curved sides as in Figure 6.

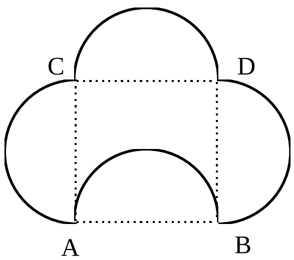

Figure 6: An example of a figure with curved sides

Figure 6 has four curved sides (ACDB is a square) and angle A and angle B are equal to right angles, as shown previously. Proclus concluded that the sum of the interior angles of this figure was two right angles, yet the figure was not a triangle (Morrow, 1970). It is not clear why Proclus did not consider angles C and D. It may be conjectured that he did not recognize a mixed angle at C in the interior of the figure. This was confirmed in at least another instance. Proclus argued that not all triangles had three sides and gave as an example Figure 7 which he said was a four-sided triangle (Morrow, 1970).

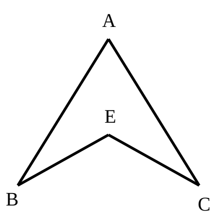

Figure 7: Proclus' four sided triangle

> This also is one of the paradoxical problems in geometry, to find a four-sided triangle, such as BAC. Though bounded by four sides BA, AC, CE, and EB, it has three angles, one at B, another at A, and a third at C. Consequently the figure here presented is a four-sided triangle (p. 257).

The angle BEC is said to be outside the figure. For Proclus, this confirmed the idea, as for Euclid, that angles greater than two right angles were not accepted (Morrow, 1970; Heath, 1956). Proclus referred to Zenodorus, a geometer of the second or first century B.C., who called Figure 7 hollow-angled (Morrow, 1970). It is worth noting that defining an angle as the inclination between two lines, did not allow one to recognize the internal angle BEC of the figure as an angle at all. In fact, it did not permit the recognition of angles beyond two right angles.

Along the same lines, Proclus rejected zero and straight angles. He pointed out that Euclid carefully worded the definition of perpendicular lines to make sure that the definition would only hold if the straight lines would make an angle at all, excluding intersecting straight lines with the same direction. He made the following comment referring to the possibility that one line was placed at the extremity of the other in the same direction.

> Suppose it [a straight line] stands at the extremity of the [other] straight line and makes one angle with it. Would it be possible for this to be equal to two right angles? Obviously not, for every rectilinear angle is less than two right angles, just as every solid angle is less than four right angles. Even if you take the angle that seems the most obtuse, you can only give it such a magnitude as will still fall short of the measure of two right angles (p. 228).

Proclus' concept of angle is quite different from our contemporary perspective. For him, an angle was essentially a quality, or a relation between two lines (or more than two lines in the case of the solid angles) not necessarily straight. Only in some cases are we able to quantify this relation and measure an angle.

After Proclus' time, the debate continued about the nature of angle. Simplicius (beginning of the sixth century A.D.), extending Apollonius' opinion that an angle was the contraction of a surface or a solid to one point, said that a surface angle (an angle determined by the intersection of any two arcs) was a new entity intermediate between a line and a surface. Moreover a solid angle was midway between a surface and a solid. Aganis, a contemporary of Simplicius, said that an angle is a quantity having two or three dimensions whose extremities come together at a point. Philoponus (5th and early 6th centuries A.D.) extensively discussed the nature of mixed angles (Bello, 1983; Knorr, 1986).

Middle ages' perspectives

The debate about both the nature of angle and mixed angles continued during the 11th through the 15th century. The focus, however, was more on the philosophical assumptions behind the concept, and no important mathematical developments were produced.

Avicenna, an Arabic mathematician at the end of the eleventh century, said that we could use the word "angle" for the quantity itself, or for the quality of being "angular" (although he did not use this last word). The same applies to the words "square" and "squaring." As we shall see, this opinion was pursued by Albertus Magnus. Averroes, another Arabic mathema tician of the twelfth century, claimed that angle might be considered a fourth genus, the others being point, line, and surface (Tummers, 1984).

Albertus Magnus (1193-1280) was Thomas Aquinas' teacher and one of the first medieval scholars to write a commentary on the works of Aristotle. In a commentary on Euclid's *Elements*, he continued the debate, started by Aristotle, about whether an angle is a quantity, a quality, or a relation. From that discussion, we were able to make a composite picture of his concept of angle. He considered two dimensions in an angle: "breadth" and "length" (or "latitude" and "longitude").

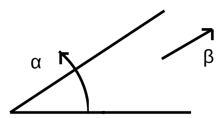

Figure 8: Dimensions of an angle

In Figure 8, α indicates a direction of increasing breadth and β a direction of increasing length. Solid angles also have "depth" which apparently would constitute a third dimension. Most of Albertus Magnus' arguments were almost certainly taken from Anaritius (end of the ninth century A.D.), who proposed that:

1) As an angle has breadth it is not a line. Nor is it a body because it might not have depth. Neither is it a surface, because it cannot be divided breadthwise, only lengthwise. An angle is the indivisible contact of two lines.
2) It does not seem to be a quantity because when a particular angle, the right angle, is doubled it is no longer the same kind of continuous quantity.
3) An angle is a property of a surface or a body so it is not a quantity.
4) An angle has an ability to divide a figure, and this ability is a kind of quality.
5) But angles can be increased and decreased so they seem to be a quality.
6) Acuteness and obtuseness are conditions of quantity.
7) An angle has breadth and length and so is a quantity (Bello, 1983; Tummers, 1980, 1984).

In his comments on the *Elements*, Albertus Magnus concluded that an angle tells us a quality about a certain quantity. An angle (angulus) was a quantity, but to be angular (angulatio) was a quality. However in his comments of Aristotle' *Metaphysics*, he presented a different conclusion asserting that an angle was a relation because the angle was a "medium" between a line and a surface (Tummers, 1984).

The problem of the nature of mixed angles continued to be discussed in the Middle Ages. Johannes Campanus who edited the *Elements* in the 13th century, inferred that Euclid's Proposition III.16, discussed earlier, posed a problem. Imagine that the diameter AC is rotated about C (Figure 9). As long as it cuts the circle, it will make consecutive acute angles with its initial position. This angle is less than the angle of the semicircle ADC. But in the moment that it ceases to cut the circle, it makes a right angle greater than that angle of the semicircle. During this process, the rectilinear angle is never equal to the angle of the semicircle. "The transition from the less to the greater, or vice versa, takes place through all intermediate quantities and therefore through the equal" (Heath, 1926, p. 41). Campanus concluded that these angles were not of the same kind. The same was observed by Cardano in the 16th century. He used the term angle of contact.

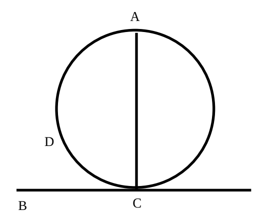

Figure 9: A diagram to illustrate Campanus' objection with Euclid's Proposition III.16

The problem of the corruption of the soul by an infinite number of sins produced very interesting arguments of a mathematical nature in the twelfth and thirteenth century. William of Auxerre, who died in 1231 or 1237 and was a teacher of theology in Paris, used contingency angles (his terminology for the horned angle) to claim that the soul contained an infinite number of parts, if not more, as a right angle contained an infinite number of angles of contingency (Tummers, 1980). He also argued that daily sins are infinitely smaller than mortal sins, as the relationship of a point with a line, or a line with a plane, or a contingency angle with a right angle.

Developments after the sixteenth century

The end of the 16th century saw a continuing discussion between the French geometer Peletier and Clavius about the nature of the angle of contact, the angle between a circle and one of its tangents.

> Peletier held that the "angle of contact" was not an angle at all, that the "contact of two circles"… is not a quantity , and that the "contact of a straight line with a circle" is not a quantity either; that angles contained by a diameter and a circumference whether inside or outside the circle are right angles and equal to rectilinear right angles, and that angles contained by a diameter and the circumference in all circles are equal (Heath, 1926, p. 41).

The last part was proved in the following way. Let A and B be two angles of semicircles drawn on the same line with one point in common as in Figure 10 on the following page. Then A cannot obviously be smaller than B. But also A cannot be greater than B, because if it were so we could draw larger and larger angles of semicircles and eventually we would have an angle of a semicircle larger than a right angle, which is absurd. So, he concluded, all angles of a semicircle were equal, and their difference was nothing, which implied, he said, that they were non-angles (Heath, 1926). Still in the end of the 16th century and the beginning of the 17th century, Vieta supported Peletier and Galileo seems to have done the same thing. Clavius argued that the angles of two semicircles of different sizes cannot be equal since they do not coincide. Moreover, in order for the Archimedian Axiom to hold, it was only necessary to assume that these angles were a quantity of a different nature than rectilinear angles (Heath, 1926).

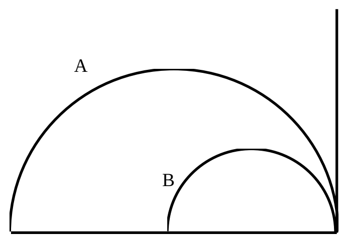

Figure 10: Angles contained by the diameter and circumference of circles

The controversy continued well into the 17th century, with the discussion between Wallis and Leotaud. Wallis, who wrote an essay of 100 pages on the various opinions about the nature of horned angles (Kasner,1945), argued that for two lines to make an angle it was necessary for an inclination to exist. But that was not the case between the circle and its tangent. In the point of contact, the straight line was not inclined to the circle but lied on it, was coincident with it. In the course of this argument Wallis expanded some interesting conceptions about angles. Let us quote Heath (1926):

> As a point is not a line but a beginning of a line, and a line is not a surface, but a beginning of a surface, so an angle is not the distance between two lines, but their initial tendency towards separation… How far lines, which at their point of meeting do not form an angle, separate from one another as they pass on depends on the degree of curvature… and it is the latter which has to be compared in the case of two lines so meeting (p. 42).

The use of the notion of degree of curvature provided a clarifying context to discuss angles of contact. Both Newton and Leibniz applied analytical methods to determine the curvature of a curve (Boyer, 1968; Kline, 1972), and Newton clearly distinguished in his writings the rectilinear angle from the contact angle (Struik, 1986) but none of them provided a way to actually measure these angles.

It is interesting to note that there is evidence that this whole discussion was not limited to the mathematical circles. For example, two of Voltaire's novels contain extensive references to the horned angle (Kasner, 1945).

The nineteenth century shift

The discussion of the nature of angles shifted significantly during the nineteenth century. A different concept of angle was developed by the works of Lobatchevsky, Gauss, and Bolyai on non-Euclidean geometries. Lobatchevsky extended the notion of angle by calling the angle made by two limit parallel lines the angle of parallelism. Geometry had ceased to be limited to the study of physical space, which meant that there was no longer a need for a visual interpretation of geometric entities. Another extension of the notion of angle was developed in the beginning of the nineteenth century when angles were used to express the time interval between two periodic events. The crucial idea came from the development of functions in trigonometric series by Fourier (Kline, 1972). With regards to the issue of horned angles, Cantor showed how these were a concrete illustration of geometries that did not obey the Archimedian Axiom and resolved the related paradoxes that had puzzled mathematicians for centuries (Kasner, 1945).

Research in Euclidean geometry continued to try to clarify the notion of rectilinear angles, and new perspectives emerged. Veronese argued that an angle was an entity in one dimension with respect to the ray, and an entity in two dimensions with respect to the points of the plane. His idea was to define angle as "the aggregate of the rays issuing from the vertex and comprised in the angular sector" (Heath, 1956, p. 180). For him, an angle was the set of all the rays that are "between" two given rays (Enriques, 1911). An angle was "a part of a cluster of rays, bounded by two rays (as the segment is a part of a straight line bounded by two points)" (Heath, 1956, p. 180). A different approach was the one undertaken by Bertrand. An angle of two straight lines was the portion of the plane that is common to the two semi-planes limited by the two lines, or it was the interference of these two semiplanes (Enriques, 1911).

Contemporary conceptions

The twentieth century saw two main developments of the concept of angle. One was influenced by Hilbertian formalism, and the other by Klien's Erlangen's Program. Hilbert (1902) in his book *The Foundations of Geometry* defined angle as follows:

> Let a be any arbitrary plane and h, k any two distinct half-rays lying in a and emanating from the point O so as to form a part of two different straight lines. We call the

system formed by these two half-rays h, k
an angle (p. 13).

Hilbert then proceeded to define the interior and exterior of an angle. Like Euclid, Hilbert's system did not recognize zero and straight angles, nor angles greater than two right angles. This system did not allow one to produce theorems about the sum of the angles of concave polygons. However, this system is followed by many contemporary mathematicians and used in school mathematics of some countries.

Contemporary definitions of angle, following Bourbaki's reductionist view of geometry, make use of an algebraic approach influenced by the work of Klein. Dieudonné (1964), for example, defined the group of angles of rays as an additive group isomorphic to the group of plane rotations. Through the use of this definition, and extensive use of the dot product, he quickly derived the main trigonometric formulas. Choquet (1964) also used this last approach, but commented that for an algebrist, the group of angles should be the quotient of the group of even isometries by the group of translations. This notion of angle also found its way into school mathematics curricula, especially those influenced by the New Math. Both Dieudonné and Choquet concluded that by making extensive use of linear algebra, most contemporary geometric knowledge could be constructed without the notion of angle.

These approaches had no influence, however, on angles of contact. Although the paradoxes associated with them were solved by Cantor, he did not propose a process to measure them. During the twentieth century several ways to compare the angles of contact have been found.

Kasner was interested in this issue and proposed, following Newton's advice, that the measurement of horned angles should be based on the curvatures of the curves. He proposed (Kasner,1945) that the measure of a horned angle between two curves with curvatures g_1 and g_2 be given by

$$M_{12} = \frac{(\gamma_2 - \gamma_1)^2}{\dfrac{\delta\gamma_2}{\delta s_1} - \dfrac{\delta\gamma_1}{\delta s_2}}$$

Kanser shows that this measure is invariant under the group of conformal transformations.

An alternative way to measure horned angles was proposed later by Waismann (1959). His starting point was that "intuition suggests that we should fix our attention on the horn-shaped space between the two curved lines, and ask whether these spaces cannot be compared to one another" (p. 221). As we can divide any two horn-shaped angles by a straight line, we only need to consider mixed angles between a straight line and a curve. Waismann proposed that we could order all these angles considering that an angle is greater than another if, when the straight lines and the vertices of both angles coincide, this angle extended beyond the other, and he showed that this was a well-ordered set.

In particular, if we take the contact angles between a circle and its tangent, then we have a way of actually measuring the angles. We just need to define the unit of measurement of the angle that a circle of radius 1 makes with its tangent, and take 1/r as the measure of the magnitude of the angle of any other circle of radius r with its tangent. We can define multiplication of an angle of amplitude 1/r by an integer n, as being the angle with amplitude n/r and radius 1/r. Addition between two angles of amplitudes 1/r and 1/r' is an angle of amplitude 1/r + 1/r' (Waismann, 1959). As we have seen before, no matter how big r is, the contact angle will always be smaller than any acute angle, which means that we are not using measures of the same type, or as Waismann put it, we are dealing with what he called ultrareal numbers.

This process of measuring contact angles was probably inspired by Klein who in a lecture given in 1908 outlined an approach similar to Waismann (Klein, 1939). In fact, he went further. He argued that if we consider all the analytic curves through a point O, we can develop each curve in a power series whose coefficients depend on the derivatives of successive orders. This development allows us to order all the analytic curves through that point. Given two such curves

$$y_1 = a_1x + b_1x^2 + g_1x^3 + \ldots$$

$$y_2 = a_2x + b_2x^2 + g_2x^3 + \ldots$$

we shall say that the angle of the first curve with the x axis is greater or less than that of second curve according as $a_1 > a_2$ or $a_1 < a_2$. If $a_1 = a_2$ then we will analyze the next coefficients and proceed the same way.

To conclude

Human concern about directions appeared very early, embodied with mystic, social, practical, and intellectual concerns. The concept of direction, or more precisely, of the difference in directions, was present in early cultures. Attempts to actually provide a measure of a difference of directions succeeded both in construction and in astronomy. Both Egyptians and Chinese, for example, used a process involving ratios, that anticipated trigonometry and made no use of angles, to determine what the Egyptians called the "seked" of a pyramid. Although the Babylonians had no word for angle, they developed a quantitative process that involved the use of a background of stars to measure differences in direction.

The Greeks were the first to investigate several possible definitions of angle. But their angle, as Euclid's

Elements, was concerned with their everyday space, and so their concept of angularity was related to the notion of a corner formed by two lines or two surfaces. The paradoxes that this approach produced lasted until the seventeenth century, where the invention of the notion of curvature provided a more reliable approach to the problem of mixed angles. The paradoxes remained in place but there was a new mathematical concept that avoided them. Only recently can we approach these angles more confidently using more refined mathematical devices.

The interesting point is that in spite of the Bourbakist formal approach that “algebrized” rectilinear angles, the problems are not over. In fact, the very question of the nature of an angle was raised by physicists who, in dimensional analysis, were trying to axiomatize the measure of angles (Krantz, Luce, Suppes, & Tversky, 1971).

> Angle is the bastard quantity in dimensional analysis, about which everyone seems a bit uncomfortable (…) It is said to be dimensionless (because it can be defined as the ratio of two lengths, namely, the arc subtended to the radius of the circle) and also to be extensively measurable (…) and therefore has a unit. In listing the dimensions of physical quantities, it has to be included in the units but usually it is omitted from the dimensions; so, for example, angular velocity is reported in radians per second, but supposedly it has only the dimension T-1. Something is bizarrely wrong (p. 455).

Although the paradoxes produced by the intermixing of an Archimedian with a non-Archemedian system have been solved, new issues are now developing having to do with the meaning of measuring angles. In a way, the sense that “something is bizarrely wrong” is a sign that mathematics is a living field, continually searching for new paradigms (Kuhn, 1970).

From the point of view of a mathematics educator, it is interesting to note that these several developments of the concept of angle have their counterparts in contemporary school mathematics. In fact, there are several kinds of angles currently used in schools: (1) Euclid’s and Hilbert’s definitions of angle are included in some curricula; (2) in some parts of the world, angles are associated with rotations; and (3) angles as a measure of periodic events are used in high school mathematics.

An innovative concept of angle, the amount of turn, has been used in Turtle Geometry. In this geometry, a line segment is the path of a turtle on a computer screen, and an angle is the command that tells the turtle how much it should turn to draw the next line segment. Interesting new theorems arise from this approach, namely the "Exterior Angle Theorem", that says that the sum of the exterior angles of a polygon is always 360 degrees (one degree is the unit of a turtle turn) (Abelson & diSessa, 1980). It will be interesting to see how the concept of angle continues to develop and challenge existing conceptions.

Note

1. This is the second part of a two-part article. Part one was featured in the Summer 1990 issue of *The Mathematics Educator*.

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