A Model of Secondary Students' Construction of the Concept of Function

Melvin R. Wilson

The National Council of Teachers of Mathematics (1989) referred to the concept of function as "an important unifying idea in mathematics" (p. 154). This is not a new statement. Recommendations have been made throughout this century about the importance of functions in secondary school mathematics. For example, statements similar to the one made by NCTM were made in 1904 by the German mathematician Klein (Hamley, 1934, p. 52), in 1921 by the National Committee on Mathematical Requirements of the Mathematical Association of America (Hedrick, 1922, p. 191), and in 1963 by the Cambridge Conference on School Mathematics (Buck, 1970, p. 237).

In connection with statements emphasizing the importance of functions have come recommendations regarding how the function concept should be taught. Many recommendations early in the century focused on the importance of using real world situations to informally acquaint students with functions (Breslich, 1928). During the middle of the century, prominent mathematics educators believed that functions should be introduced as sets of ordered pairs or arbitrary mappings between sets; that is, that beginning instruction about functions should take place within a formal mathematical structure (May & Van Engen, 1959). This recommendation has had lasting effects, as evidenced by the structural presentation used in most high school algebra textbooks today (Cooney & Wilson, in press).

Some of the more recent recommendations regarding how to best teach secondary students about functions have been based on consideration of students' cognitive processes in constructing concepts about functions. For example, Sfard (1989) observed that students first develop an *operational* conception of function, in which they think of the computational processes associated with functions, sometimes followed by a *structural* conception, in which they think of functions as objects. She proposed that mathematical concepts like function should not be introduced by means of structural descriptions, such as that described by the definition of function as a set of ordered-pairs, but rather by operational descriptions, such as the definition of function as a dependence of one varying quantity on another. Dreyfus and Eisenberg (1982) similarly suggested that functions be introduced in such a way that students' *intuitions* and experiences be utilized. Dubinsky, Hawkes and Nichols (1989) proposed a model for the learning of functions by college students. In the context of this model they suggested that certain computer activities might assist students in constructing function concepts.

The purpose of this article is to propose a model describing how secondary students construct function concepts as they relate to formal, algebraic definitions of function. Certainly students do not wait until high school to begin constructing function concepts. Although formal mathematical treatment of the function concept usually occurs first in a beginning algebra course, students have many opportunities to explore functional relationships, that is, relationships between varying quantities, much earlier than this. Piaget, Grize, Szeminska, and Bang (1977) even found that many pre-schoolers have an implicit understanding of functional relationships. Children's pre-secondary experiences with functional relationships occur both in and out of school and provide them with important intuitions about mathematical functions. Further, all students do not arrive at their first algebra course equipped with equivalent intuitions nor do all students have the same patterns of thinking about functions once they begin formal instruction about functions. However, there are enough common elements among students' conceptions that I believe is worthwhile to explore and discuss. This article attempts to serve that purpose.

The proposed model describes various conceptions high school students might have of functions as well as some of the mental operations performed by students to build these conceptions. This model is based on my interpretation of research reports about students' understanding of functions (e.g., Dubinsky, Hawkes, & Nichols, 1989; Sfard, 1989; Thomas, 1975; Vinner and Dreyfus, 1989; and Leinhardt, Zaslavsky, & Stein, 1990) and upon my own experiences in teaching and observing high school students.

Theoretical Framework

This article focuses on the mental operations performed by high school students in constructing the concept of function. The theoretical perspective upon which this model is based relies heavily on ideas from Gestalt psychology and Constructivism. I use Gestalt theory to provide a framework describing various stages of conceptual development, and constructivist theory to describe how students think about specific tasks related to functions, as well as how students might progress through the stages.

Gestalt Psychology

Proponents of this psychology claim that individuals naturally develop a holistic view of concepts.

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Individuals form perceptual *fields*, or frames of reference, in which they organize information about particular concepts. Learning involves establishing and reorganizing these perceptual fields. A well formed perceptual field is referred to as a *gestalt*. Attributes of a gestalt are simplicity, symmetry, and beauty. A gestalt is flexible and dynamic. An individual might isolate parts of the field, and relate these parts back to the whole, but learning does not involve a piece by piece acquisition of connected ideas (Hilgard, 1956). This article describes a model with four perceptual fields describing how secondary students think about functions. The fields will be described in detail in a later section.

Constructivism

While Gestalt psychology focuses on organization of concepts, constructivism focuses on the mental operations used by individuals in constructing concepts. Constructivist epistemology offers a view of knowledge that is distinct from the more traditional logical positivist perspective popular among those who classify themselves as cognitive scientists. Central to constructivist philosophy is the notion of the nature of reality and how one comes to know it. Constructivism focuses on the mental operations used by individuals to construct concepts. These constructions are the only "reality" to constructivists. "To know is . . . an activity of the subject and knowledge is a construction in the truest sense of the word" (Furth, 1981, p. 15). For constructivists, a thing in the world is not an object of knowledge until the individual interacts with it and constitutes it as an object. In other words, constructivism makes no assumptions about external reality, of which knowledge is a static "copy."

> Piaget (Furth, 1981) proposed that an individual's knowledge can be understood in terms of *schemes*. Furth describes a scheme as a coordination and organization of *adaptive action*, considered as a behavioral structure within the organism, such that the organism can transfer or generalize the action to similar and analogous situations. (p. 44, italics added)

Schemes involve the coordination of external acts. However, the scheme is not the act. Rather, the external act is a manifestation of the scheme.

The notions of *assimilation* and *accommodation* are central to this idea of active and adaptive knowledge. A scheme consists of a *situation*, which triggers an activity, ending in a result (Steffe, 1989). When faced with a problem or task, an individual calls up a scheme for dealing with that task. A scheme involves *mental* operations, not physical ones. The situation refers to how the individual assimilates or interprets a problem or task. In other words, the situation involves the individual's *understanding* of the problem. An individual's understanding of a problem is dependent upon the perceptual field used by the individual to organize thinking about that problem. The action is a generalized action that can be adapted to particular applications of the scheme. The result refers to "the sequel to the activity," (von Glaserfeld, 1980, p. 81) or what happens as a result of carrying out the activity.

Learning occurs when the learner makes adaptations or *accommodations* in his or her scheme(s). An accommodation occurs when the learner attempts to neutralize a *perturbation* resulting from information conflicting with his or her current schemes (Steffe, 1989). A *functional* accommodation is one that occurs in some application of a scheme. This is the most common type of accommodation, and usually involves a change in the activity stage of an individual's scheme. Functional accommodations lead to *internalization* of actions. Internalized actions can be *re-presented*, that is, re-generated mentally (von Glaserfeld, 1989) without the presence of sensory material. Furth (1981) refers to schemes involving internalized actions as *figurative* schemes.

A *metamorphic* accommodation, which does not necessarily occur in any particular application of the scheme, involves a restructuring of the first part of the scheme, the situation. In other words, metamorphic accommodations cause an individual's concept to be restructured, and in turn, the perceptual field in which that concept is organized. Metamorphic accommodations lead to *interiorization* or reinteriorization of actions. Interiorized actions are "purely conceptual acts whose results are abstract" (Cobb, 1983, p. 49). Furth (1981) refers to interiorized actions as *conceptual operations*. He classifies schemes based on conceptual operations as *operative* schemes.

One characteristic of operative schemes is that the mental operations can be reversed. That is, the individual can see as a result that with which he or she starts. Figurative schemes do not have this characteristic. As will be seen, this is an important attribute in differentiating between the stages of understanding for the concept of function as described in this article.

Constructivism maintains that no set of experiences can *cause* individuals to interiorize actions. It is impossible to predict the circumstances under which an individual will make accommodations. An accommodation comes as a natural response to a problem situation for which application of an individual's current schemes does not result in solving the problem. Initial neutralization of a perturbation may come as an insight while thinking about a problem. However, all modifications will not be permanent. Permanent modification may occur while an individual is not even thinking about the problem. The first evidence of accommodation might appear in a later problem situation in which the learner must again use a representation of a given scheme as material in some other scheme(s).

The Function Concept

The concept of function is very complex. There are several reasons for this. First, there are many common ways to represent functions, including graphs, formulas, tables, mappings, and descriptions. Meaningful understanding requires individuals to construct multiple representations as well as operations for transforming from one representation to another. Second, the notion of function involves many other concepts. A few of the sub-concepts associated with it are domain, range, inverse, and composition. Other concepts closely related to function are quantity, variable and ratio. It is difficult to discuss functions without referring to some of these sub-concepts. Third, there are several accepted definitions for function (e.g., dependence relation, rule, mapping, set of ordered-pairs). Although these definitions are equivalent (or nearly equivalent) mathematically, they differ conceptually (see e.g., Vinner & Dreyfus, 1989).

A Model for the Construction of Function

We can only hypothesize concerning what goes on in students' minds as they learn. In the constructivist way of thinking, we can only describe what we think students' mental operations are. This does not mean we should not attempt to describe students' mental processes. Such descriptions serve as templates or prisms through which we may look and gain valuable insights into how we might improve teaching and learning.

The current model about how secondary students construct the concept of function proposes a succession of cognitive stages that culminate to yield a meaningful understanding of and competence in working with functions. That is, it discusses how secondary students might *construct* a *gestalt* of functions. The model consists of four perceptual fields, or general ways of organizing ideas about functions. A learner's progression among the four fields is not necessarily linear. Further, the first two fields, although at a lower cognitive level than the later fields, do not represent a beginning point for functional thinking. Rather, they describe convenient starting points for analyzing how students think about functions of the secondary curriculum (i.e., algebraically defined functions). In describing the model, I describe attributes of each field and then discuss mental operations students might engage in progressing through the various fields.

The examples I use in describing the model will involve linear functions, but similar examples could be generated using other types of functions. The students described in the examples are hypothetical, that is, they represent a collage of individuals I have observed, rather than specific cases.

Four Perceptual Fields

<u>Field I</u>: A function is an *expression*. Many students identify functions exclusively with algebraic expressions, equations, or formulas. These students may be able to perform operations on two or more expressions, and may even be able to apply an algorithm in constructing an equation's graph. But these students do not understand the relationship between the expression and the values obtained from the expression. There is little or no understanding of the process that transforms a value of the independent variable into a new value. These students consider the function to be an object, but have not yet constructed properties associated with the function to make it a dynamic object (see field IV below).

<u>Field II</u>: A function is an *action*. A student with this conception is able to substitute numbers for variables into a formula or expression, and perform calculations to obtain numerical values. Images of the traditional "function machine" with input and output come to mind when describing this field. The student may be able to identify independent and dependent variables or input and output, but is only beginning to understand the dynamic nature of the process involved in transforming one variable into the other.

<u>Field III</u>: A function is a *process*. A student who thinks of a function as a process is able to think about taking a quantity and transforming it into another. The student understands the relationship between dependent and independent variables and relationships among various representations of a function (expression, table, graph). In addition to simply being able to substitute numbers into an expression, he or she understands globally how changes in one variable signal corresponding changes in the other. Additionally, the student understands operations for evaluating functions not represented by a simple formula. This process conception of function is particularly important for understanding functions not directly representable by algebraic expressions, such as circular or logarithmic functions.

<u>Field IV</u>: A function is a *dynamic object*. This field incorporates each of the other three. The student may think of the function as an expression, but unlike the Field I conception, he or she sees the expression related to a process. He or she also relates the function with other important properties, such as its graph, its domain and range, and understands the relationships among these properties. A key feature of this field is that the student is able to consider various aspects of the function, drawing on the most pertinent properties in solving problems. In other words, he or she is able to relate any important feature to the whole. This field describes a "gestalt" for the function concept. It describes an organized, logical and rich concept of function.

Figures 1 and 2 illustrate relationships among the four fields, and show three ways an individual might progress through them.

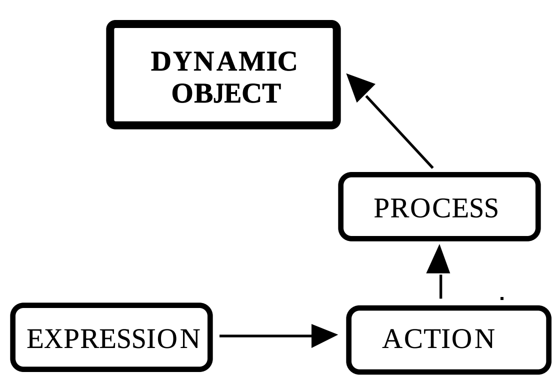

Figure 1: Path for student beginning with an expression concept.

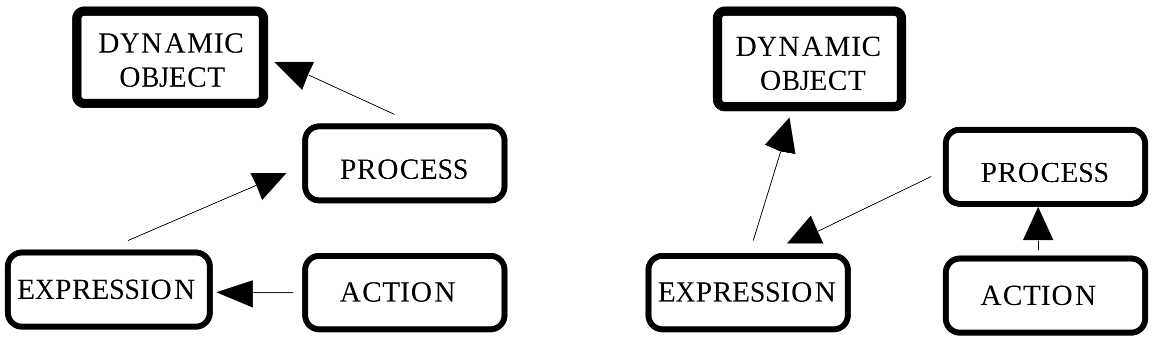

Figure 2: Possible paths for student beginning with an action concept.

The first two fields (expression and action) lie at the lowest cognitive level along a hypothetical continuum measuring understanding of the function concept. Most students develop either an action concept or expression concept soon after beginning formal instruction about functions. Research has shown that most students' pre-secondary conceptions are closer to what I have defined as an action concept (see e.g., Greeno, 1988). A process concept represents a higher level of thinking about functions, and a dynamic object concept represents the highest cognitive level. Further, a process concept is an extension of an action concept, so students generally develop an action concept before they develop a process concept. An expression concept is not required in order to develop a process concept, but both an expression concept and process concept are prerequisites to obtaining a dynamic object concept.

Mental Operations

Many activities have potential for revealing how students think about functions. Graph construction, graph interpretation, and graph translation (see Leinhardt et al., 1990 for examples) allow students to use schemes associated with their concept of function. The following discussion describes possible schemes used by students to graph linear functions. These examples illustrate that even though two students may appear to behave similarly when faced with a given task, their ways of thinking about the task may be quite different.

Suppose each of four hypothetical students, Emily, Adam, Pat, and Diane, are faced with the task of graphing a linear function, say the function represented by the equation $y = -2x + 3$. Suppose Emily thinks of a function as an expression (Field I), Adam thinks of a function as an action (Field II), Pat as a process (Field III), and Diane as a dynamic object (Field IV). What behaviors might each exhibit in solving this problem? Emily and Diane may activate schemes which lead them to do something like the following: identify the slope and y-intercept of the graph, plot the y-intercept (the point (0, 3)), count up two units and over one unit from the point (0, 3) (arriving at the point (-1, 5)), plot that point, and finally construct the line joining these two points. On the other hand, Adam and Pat may each activate a scheme that leads them to substitute several numerical values for x into the equation, compute corresponding values for y, construct a table of x and y values, plot several of the points listed in this table, and finally construct a line containing them. All four students are able to "correctly" perform the task. Adam and Pat use the same procedures for completing the task, as do Emily and Diane. But how might the four students differ in their thinking? That is, how might their mental schemes for operating on this problem be different?

To Emily (expression concept), the equation $y = -2x + 3$ is the function. She has developed procedures for graphing linear equations which involve the steps described above. Thus, when she is given the task of graphing a certain linear function, her linear function graphing scheme is activated. To Emily, the algorithm performed in obtaining the graph is nothing more than a sequence of discrete steps needed to obtain an answer (the graph). The algorithm has little dynamic quality. Similarly, the graph is just a static object (like the function), and has no relationship to the original equation. Her graphing scheme is figurative because Emily can re-present the (internalized) actions associated with it in order to solve the problem described above. However, the actions have not been interiorized.

Adam (action concept) thinks of a function as an action. When faced with the problem of graphing this linear equation, he immediately thinks about plugging in numerical values for x. To manage the data, he constructs a table, and then transfers the data from the table to the graph. But in Adam's mind the graph is still not a dynamic description of the process involved in transforming one variable into another. Rather, it is the result of performing an action. Like Emily, the actions associated with Adam's graphing scheme have been internalized but not interiorized. Although he thinks about the action of graphing, he does not understand the relationship between the points plotted and the original equation and therefore probably could not identify points that satisfy the equation, even though he has just found and plotted several. This task would involve reversing the graphing scheme, something that can be done only after interiorization of actions associated with the scheme.

For Pat (process concept), the function has become a process as well as an action. Pat has a scheme similar to Adam's (action), but it involves interiorized actions. He could use the scheme as material for operat ing in another scheme. For example, if a problem were posed in which he was asked to identify several points whose coordinates satisfy the function, he could take a re-presentation of data generated for the table, or information from the graph, as material in a scheme to generate a list of ordered pairs of points satisfying the equation. Another evidence of the operative nature of Pat's graphing scheme would be his ability to reverse the graphing process, and obtain a table or equation which represented a given linear function whose graph were given.

Like Emily (expression concept), Diane (dynamic object concept) thinks of a function as an object. But her scheme for operating in this situation is much richer than Emily's. She could operate like Adam and Pat if she wanted to, but in this situation she activates the most efficient scheme for solving this particular task. Unlike Emily's static view of the function and graph, Diane thinks of the function as a dynamic process as well as an object. To Diane, the graph is a way of depicting the relationship between the variables. The actions associated with her graphing schemes have been interiorized as well as internalized. She not only can re-present the actions, but can use this re-presentation as material for operating in other schemes. For example, she could explain how the slope of the graph she constructed illustrates how one variable changes as the other changes, or how the *y*-intercept is related to the equation. Like Pat (process concept), she could reverse the process, that is, generate an equation or table of values for a linear function if given its graph.

Construction of Perceptual Fields

The situation described above tells how four students with different conceptions of function are likely to think about a specific function task. Following is a discussion of what types of mental operations might be required for them to restructure their schemes, and in turn their perceptual fields, regarding the concept of function.

<u>Expression-Action</u>. Both Emily (expression) and Adam (action) have figurative function graphing schemes. That is, the actions associated with their schemes have been internalized but not interiorized. Although their concepts of function differ, their ways of operating (mentally) in the above situation are similar. Their schemes consist primarily of algorithms. Since functional accommodations account for the learning of algorithms, developing either scheme would involve functional accommodations to their existing knowledge about number operations or variables.

<u>Process</u>. The restructuring necessary to construct a process concept of function is different from an expression-action restructuring. The process concept of function is an extension of the action concept and constructing a process concept involves interiorizing (as opposed to internalizing) schemes associated with an action concept, thereby reorganizing the way tasks are assimilated. This interiorization requires metamorphic accommodations, since the student must change the situations (mental) of his or her schemes, rather than simply changing the activities triggered by the situations.

To illustrate this idea we return to the example of the task of graphing a linear function. By applying his graphing scheme, Adam, who has an action concept of function, can successfully construct the graph of a given linear equation. However he is unable to reverse the process and construct a formula corresponding to the graph of a given linear function, whereas Pat, who has a process concept, can successfully complete such a task. The reason for this is that Pat can use a re-presentation of his graphing scheme as material for operating in another scheme (a "reverse graphing" scheme), while Adam cannot. The metamorphic accommodation necessary for Adam to interiorize his graphing scheme, and use its re-presentation as material in other schemes, will involve restructuring his concept of function to include the idea of a process as well as an action. A task of identifying the equation corresponding to a given graph might motivate such a reorganization.

<u>Dynamic Object</u>. Construction of a dynamic object concept of function requires a reorganization of an individual's expression concept. This reorganization cannot occur, however, until after an individual has first developed a process concept of function. This is because a dynamic object concept of function involves seeing functions as processes as well as objects. That is, the processes associated with the function are viewed as part of the object. To better understand this idea, consider the concept of number. One child might need to perform the process of counting to 10 in order to understand 10, while a second child could count to 10, but would not have to count to understand 10. The second child's concept of the "object" 10 includes the process of counting to 10, although this counting process does not need to be activated each time. Similarly, an individual with a dynamic object concept of function would not necessarily have to generate the processes associated with a particular function in order to understand the function, although such processes could be generated if necessary. Thus, the construction of a dynamic object concept requires reorganization of an individual's expression and process concepts.

Interiorization requires metamorphic accommodation, and in order for accommodation to occur, an individual must encounter a task for which his or her present schemes are insufficient for operating. An individual with process and expression concepts might construct a dynamic object concept of function in the context of a problem or set of problems involving the application of his or her graphing scheme(s). For example, to consider how various parameters of linear functions (e.g., m and b in equations of the form $y = mx + b$) affect their graphs, an individual must be able to consider a function on a set of functions. The set of linear equations may be thought of as the domain of the function, and the set of graphs of the equations as the range. Since the functions themselves are being operated on, one must be able to see each function as an object which points to a process. In other words, the graphing scheme must be used as material in a scheme used to identify properties of families of functions. To accomplish this task an individual might revise his or her concept of function to include the idea of a function as a dynamic object.

Metamorphic Accommodations

We must remember that no set of experiences can cause students to interiorize actions. It is impossible to predict the circumstances under which a student will make a metamorphic accommodation. An accommodation comes as a natural response to a problem situation for which application of an individual's current schemes does not result in solving the problem. For example, one way for Adam (action concept) to restructure his graphing scheme to include a process concept might be for him to neutralize the perturbation caused by his inability to identify the equation corresponding to a given function's graph. Initial neutralization of Adam's perturbation may come as an insight while thinking about the problem. However, this modification probably will not be permanent. Permanent modification may occur while Adam is not even thinking about the problem. The first evidence of his accommodation might appear in a later problem situation in which he must again use a re-presentation of his graphing scheme as material in some other scheme.

Every teacher has experienced the frustrations of waiting for students to make accommodations. Teachers' statements such as "but we just discussed that yesterday" and "I know you understood this last Friday" reflect frustrations which result when students only temporarily modify their schemes to include more sophisticated concepts. On the other hand, as learners we probably have had experiences in which a particular concept seemed to make sense after a period of time. We may not even consciously think about the concept for a time, but upon returning to it, it suddenly makes sense. In these instances a metamorphic accommodation has occurred.

Metamorphic accommodations are unpredictable but they do not necessarily occur at random. If students have many opportunities to neutralize perturbations they are more likely to experience permanent modifications in their schemes. In this model, teaching becomes a process of providing students with repeated experience in solving problems which allow them to make modifications in their existing schemes. Steffe (1989) refers to these as *engendering* experiences because they engender or encourage accommodations. But the teacher must be patient as students make constructions. Time is required for students to make major cognitive reorganizations. If teachers do not allow students time to construct concepts, students may become frustrated.

Summary and Conclusions

A model describing how secondary students might construct various function concepts has been presented. In this model, students make accommodations in function fields, each of which describes a stage of cognitive development of an individual's concept of function. Students think of functions as: (1) expressions, (2) actions, (3) processes, or (4) dynamic objects. The fourth field (dynamic object) describes a gestalt for the concept of function. That is, it is the most organized, flexible and useful way to think about functions.

Since a dynamic object concept of function is the most sophisticated and last concept to develop, it would be ineffective to first introduce students to functions as objects (e.g., sets of ordered-pairs). This is not a conclusion unique to this article (see e.g., Malik, 1980; Sfard, 1989; Vinner & Dreyfus, 1989), yet a glance at almost any beginning algebra textbook reveals that functions are usually introduced as sets of ordered pairs or arbitrary mappings between sets. A more dynamic definition, such as the definition of function as a dependence relation defined by a rule, would be more appropriate for beginning secondary instruction about functions. Emphasizing such a definition would help students to build their conceptions of function on existing knowledge about number operations and intuitions gained through everyday experience with relationships between quantities. It would also make the consideration of functions found in the real world, such as those in physics, biology and social science, a more natural part of secondary instruction about functions since most functions in these areas describe dependence relations. In other words, a less formal approach would make it more natural for students to develop formal notions of function by building on less formal but more natural intuitions and experiences.

Finally, it is insufficient for mathematics education research to focus only on the teaching activity or only on the learner's behavior. Focusing just on the teacher somehow implies that it is the teacher who "causes" learning. On the other hand, by focusing only on student behavior one might overlook the fact that two students exhibit the same behavior but have radically different understandings. A more complete model of learning includes an analysis of how the learner constructs his or her mathematics. In this model, the teacher's role, the student's behavior, and the student's thinking are all considered. Decisions about curriculum and teaching methods should be based on knowledge about how students make mathematical constructions. In other words, students' mathematics should be what teachers use to teach new mathematics. Students' mathematics should be what determines appropriate mathematical activity. Teachers must be aware of their own concepts, but only as points of reference.

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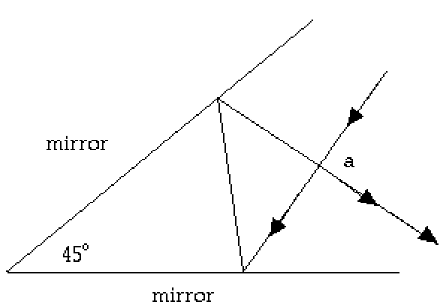
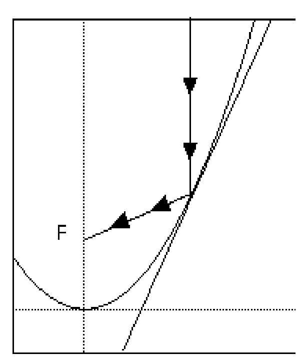
A Little Light Reading

The angle mirror is an optical device for making right angles. Two mirrors are set at an angle of 45 degrees to each other. Prove that the device causes a ray of light to leave the device at a right angle to the direction from which it entered. Prove angle a equals 90 degrees.



The Reflecting Telescope Theorem

A ray of light striking a parabolic mirror is reflected as from the tangent to the curve. Prove: Any ray of light parallel to the axis is reflected to the focus. Find the coordinates of the focus. Point F is the focus.



From the desk of Dr. William D. McKillip, a professor of mathematics education at The University of Georgia.