Using Problem Fields as a Method of Change

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The task of the comprehensive school is to train students to cope with as many of the many-sided problems in their future lives as possible. The teaching of mathematics should be developed so that the knowledge and skills learned are useful for most students. Up to the present time, mathematics has often been taught as an end in itself; that is, mathematics for mathematics' sake (Martin, 1986). Some changes, however, are now occurring in the conceptions of how mathematics should be taught.

Communication in mathematics learning

Traditionally, people have thought that there were no connections between language and mathematics and that mathematics created its own language to satisfy its needs. The 1980s, however, gave way to an increased interest in the meaning of language and communication for mathematics teaching and learning. This increased interest is apparent in many documents concerning mathematics teaching and learning that were published during that period (Cockcroft, 1982; Department of Education and Science and the Welsh Office, 1989; NCSM, 1989; NCTM, 1989). One of the central themes in *Mathematics Counts* (1982) is language and communication. This report demands, among other things, that emphasis be placed on "discussion between teacher and pupils and between pupils themselves" (Cockcroft, 1982, p. 71) when teaching mathematics to students of all ages.

Constructivism emphasizes the learner's contribution to the learning process. Teachers adhering to the theory of constructivism are compelled to listen to their students and follow their thinking processes as well as try to understand their students' thinking. One method of achieving these goals is to use discussion as an element of teaching. Students very often have preconceptions (or misconceptions) about the concept to be learned. Teachers can try to understand the students' ways of thinking by listening to discussions between them; then they can use their students' perceptions as a starting point for their teaching (Schoenfeld, 1987).

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When teaching mathematics in the comprehensive school, the students' mathematical activity should form an essential part of teaching (Cockcroft, 1982; Spiegel, 1982; Walsch, 1985; Department of Education and Science and the Welsh Office, 1989; NCTM, 1989). In the Cockcroft Report (1982) it is stated that there is not only one definitive style for teaching mathematics; instead "approaches to the teaching of a particular piece of mathematics need to be related to the topic itself and to the abilities and experiences of both teachers and pupils" (p. 71). Nevertheless, some elements such as appropriate practical work should be present in the mathematics teaching of all students. In the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) the role of the students is described using the following action verbs: investigate, explore, describe, develop, use, apply, invent, relate, model, explain, represent, and validate.

The constructivist view of learning stresses the student's contribution to the learning process. Qualitatively successful learning, so-called deep learning, can only happen when a student is actively working with new subject matter. This leads to emphasizing a student's own activities. Thus in a learning situation, there should be opportunities for the students to originate activities.

Problem fields as a method

In order to implement the changes described above, it is generally suggested that the teaching of school mathematics should be realized in an open way. This can happen through the use of open-ended tasks in mathematics classrooms. Open-ended tasks can stem from a set of problems that are somehow connected; such a set of connected problems will be called a *problem field*. One can generate a problem field from most problems by changing the conditions given in the task (see Pehkonen, 1986).

Two examples of problem fields will be described below (for more examples see Pehkonen, 1988a, 1988b). These examples represent problem fields that have been developed for use in heterogeneous classes in Finnish comprehensive schools. In each problem field the difficulty of the problems ranges from very simple problems that can be solved by all of the students to more difficult problems that only the more advanced students will be able to solve.

A characteristic of problem fields is that they are not bound to a particular educational level; instead, they are suitable for mathematics teaching from the primary grades to the level of teacher education. The role of the easier problems is to reinforce the problem solving persistence of students. The most important aspect of all of the problems is the way in which they are introduced to a class: *A problem field ought to be given gradually to students, and continuation should be related to the students' solutions*. The answers to the problems are less important; therefore, several of the questions posed in this article are left unanswered. The level to which a teacher takes a problem field depends on the responses from the students.

An example: Polygons with matchsticks

The following problem field was developed from a matchstick problem with which I was familiar. Through use in courses involving preservice and inservice teachers this problem has gained more "flesh around the skeleton." Such a network-like system of experiences with a problem field is important for its successful use. In the following discussion, I will provide one possible sequence of questions for this problem field along with some comments about expected responses from students.

Twelve matchsticks will be needed to concretize the problems. The starting situation is the following:

- With 12 matchsticks one can make a square with an area of 9 au (au = area units) as in Figure 1. (Any abbreviations needed, such as au, can be explained in the context of the situation.)

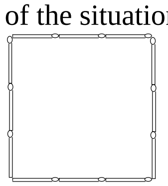

Figure 1: A square with 12 matchsticks.

From this situation, a sequence of problems, in other words a problem field, has been developed. In each problem, the perimeter of the polygon should be made up of 12 matchsticks.

- Can you use 12 matchsticks to make a polygon with an area of 5 au?

In order to provide slower students with more time to think, the faster ones can be asked to find another solution and perhaps even a third solution. Figure 2 shows some possible responses from students.

- How many different polygons of 5 au can you make with 12 matchsticks? Can there be more than ten different solutions?

The students will probably find many of the solutions. But there are still more complicated solutions which they probably will not find. How many of them can be found when the whole class is working together?

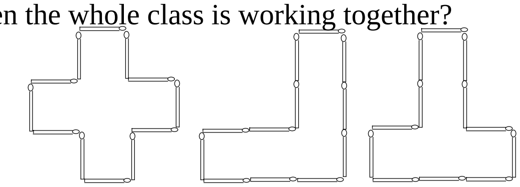

Figure 2: Polygons with area 5 au.

- Is it possible to use 12 matchsticks to make a 6 au (7 or 8 au) polygon?

The solutions in Figure 3 can be found easily. But are there any other solutions in each case? How many different solutions can be found?

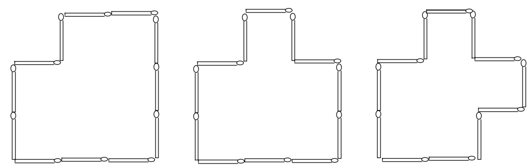

Figure 3: Polygons with areas of 8, 7, and 6 au, respectively.

- Is it possible to use 12 matchsticks to make a 1 au (2, 3, or 4 au) polygon?

With the technique of the earlier solutions where the area could be calculated using rectangles, one can rather easily construct one polygon with an area of 4 au, but polygons with smaller areas might seem impossible to find. Nevertheless, at the junior high school level it should be possible at least in principle to find a general solution, the parallelogram, where the area is not calculated using rectangles. After finding this solution one can still ask for another solution for polygons with areas smaller than 4 au. How many possible solutions are there altogether?

- Is it possible to use 12 matchsticks to make a polygon whose area is greater than 9 au?

This question has been posed for homework in courses offered for preservice and inservice teachers. Thus far, it has not been solved.

Another example: Number Triangle

The origin of this problem field is a British activity called Arithmogons. While teaching a Grade 7 class in Helsinki, Finland, I used a form of an Arithmogon to improve the students' mental calculation skills and problem solving skills in a nontraditional way. My form of Arithmogon led us to an interesting problem sequence called Number Triangle. In the frame of this problem field, the students practiced mental calculations with negative numbers, among other things, as part of six different lessons spread out over a few weeks.

The following problem has been used as a starting point for this problem field:

- What numbers should be placed in the blank circles in the triangle in Figure 4 so that the sum of the three numbers on each side is equal?

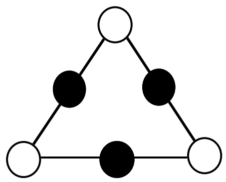

Figure 4: Number Triangle.

There are many solutions for this problem thus the following questions may be asked:

- Can you find another solution?
- How many different solutions could there be?

Although students can find many different solutions, they are usually not able to see that there are actually an infinite number of solutions.

- Is it possible to use negative numbers in the circles?

Although students in junior high school initially disagree, after working through some examples, they will usually find that it is possible to use negative numbers.

- Can you find a solution where the triangle's side sum (i.e., the sum of the numbers on the same side) will be 80?

Students at the junior high school level usually solve this kind of problem by a trial-and-error strategy.

- Which numbers are possible values for the triangle's side sum?

Most students will suggest whole numbers here although other values are possible.

- How could you vary this problem?

There are many possible variations at hand. One of the simplest is to vary the mathematical operation being used.

Where do problem fields originate?

When teachers are using open-ended problems in their teaching practices, it may be difficult to find enough ready-made problem fields for their classes; therefore, being able to vary a given problem to form a problem field is a valuable skill for teachers to possess. The following examples will be used to discuss how to vary a simple problem or mathematical puzzle. By varying the conditions of the starting point and the goal state, an individual can produce new problems and thus develop a problem field.

Example: Matchstick Puzzle.

Consider the pattern made with 24 matchsticks in Figure 5. The following puzzle can be used as a starting point:

- Take away four matchsticks, so that there are five squares left.

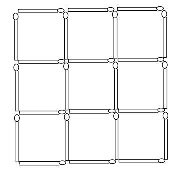

Figure 5: The starting point for the Matchsticks Puzzle.

When someone has found a solution, the teacher can ask for other possible solutions. Some students will probably ask whether they are going to count only the small squares (9 squares) or all of the squares (14 squares). This immediately raises another problem. From the original puzzle, a problem field can be generated by asking, for example, the following questions:

- What are the different amounts of squares that you can have left if you take four matchsticks away?
- Is there an amount of squares from 0 to 14 that can not be attained by taking four matchsticks away?

The following provides other possible variations:

- In how many different ways can you have five squares left if n matchsticks (n = 0, 1, 2, …) are taken away?

We can also continue systematically:

- What different amounts of squares will we have left if one matchstick is taken away? What about taking two matchsticks away? three matchsticks away? and so on.
- How many different ways can we have one square left if a certain amount of matchsticks are taken away? How can we get two squares? three squares? and so on.

Example: 25 Way Maze.

As another example of how to vary a problem, the number maze problem in Figure 6 will be discussed:

- Find a path through the number maze so that the sum of the points is 25. Each piece of the path can be traveled only once.

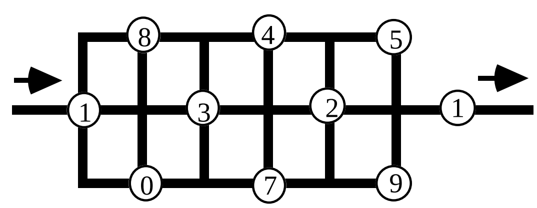

Figure 6: 25 Way Number Maze.

The task of varying this puzzle has been presented to several groups of preservice teachers during their training. First, the Number Maze was given to the teachers. After solving the problem, the teachers were asked to generate a new problem by varying the Number Maze problem. Usually, they needed some time to "warm up" but then they exposed several possibilities for variations. Here are some suggestions for varying the puzzle:

- Change the number domain (e.g., negative integers).
- Change the mathematical operation (e.g., use a different basic operation).
- Place algebraic expressions in some places (e.g., replace 8 with 8*x*).
- Change the required sum to a different value.
- Change the layout of the maze (e.g., a three-dimensional model on a cube).
- Change the question (e.g., How many different sums can you find? What is the sum on the longest or shortest path? What is the smallest or largest sum? Are there values between the smallest and largest sum which can not be attained? What are the smallest and largest numbers one can get by concatenating the numbers in a path one after the other to form a single number?).

Final remarks

I have two main reasons for introducing problem fields to preservice and inservice teachers. These reasons are to describe to teachers how to deal with problem fields and to give them an idea of how students feel when they are solving problem fields. Usually, the teachers have liked the problems and have enjoyed the way that we have dealt with them in class.

With regard to the creation of problem variations in a mathematics class, it may be motivating to let the students create some of their own variations to a given puzzle. Then the entire class can try to solve some of those variations.

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From Dr. Wilson's Notebook . . . A Band around the Earth

A band tightly encircles the earth. If we splice a 10 foot piece into the 25,000 mile band and evenly distribute the slack around the globe, which of the following are true?

a) A fly could crawl under the band.

b) An adult could crawl under the band.

c) An adult could walk under the band without ducking.

This problem came from the files of Dr. James Wilson, Head of the Department of Mathematics Education at The University of Georgia.