Behind Closed Doors

Michelle Benedict and Gerald Thompson

Many students and teachers have followed with great interest the continuing saga of the game show problem controversy presented in four different issues of *Parade* magazine in the "Ask Marilyn" column written by Marilyn Vos Savant. The problem was posed to Marilyn by Craig F. Whitaker of Columbia, Maryland in the following way:

> Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to switch your choice? (Vos Savant, September 9, 1990, p. 22; December 2, 1990, p. 28)

Marilyn claims that you should switch because switching doors will result in a $2/3$ probability of winning. Most people tend to believe that the probability of winning is $1/2$. In fact, readers have claimed that the answer is $1/2$ with tremendous amounts of emotional energy expended in the process. One reader even went so far as to tell Marilyn, "You are the goat!" (Vos Savant, February 17, 1991, p.12).

The probability of winning given that you switch is in fact $2/3$ as indicated by Marilyn. When those readers who are students or teachers try computer or classroom simulations of the problem they discover, using the Law of Large Numbers, that the statistical probability converges to $2/3$. One amazed reader said to Marilyn, "You'll have to help rewrite the chapters on probability" (Vos Savant, July 7, 1991, p. 28).

Actually, this problem does not require a new chapter on probability but it does require, as an excellent starting place, an "old" chapter on probability, which can be found in any good introductory probability and statistics textbook. In such, refer to the sections on conditional probability, independent events, and dependent events. Conditional probability is denoted with symbols like $P(A \mid B)$ which represents the probability of A given B and is defined to be the probability that both A and B happen divided by the probability that B happens, denoted $P(A \cap B)/P(B)$. Events A and B are independent if and only if $P(A \cap B) = P(A)P(B)$; so for the independent events A and B the probability that both A and B happen is simply the product of the separate probabilities. Thus if A and B are independent, $P(A \mid B) = P(A \cap B)/P(B) = P(A)P(B)/P(B) = P(A)$. Hence, intuitively speaking, events are independent from the perspective of conditional probability if conditioning on one of the events does not affect the probability of the other event.

Treating independent events as if they are dependent and vice versa are common "intuitive" errors made by students. For example, in three flips of a fair coin, given that the coin comes up showing heads on the first two flips, students may feel that surely the third flip will most likely be tails. After all, it seems only "fair." Actually, the coin has no memory of being flipped those prior times and hence the coin will not cooperate with that sort of mistaken intuition. In symbols, we see that

$P(\text{heads on third flip} \mid \text{first two flips heads}) = 1/2 = P(\text{heads})$.

The other case, treating dependent events as if they are independent, leads to errors like the ones in the game show problem.

Let us now turn our attention to the subtleties of the game show problem. An old Chasidic saying, "the hand, held before the eye, can hide the tallest mountain," gives us a metaphor for the error. The fact that the contestant will eventually be staring at exactly two doors, and behind one of those doors is a car and behind the other is a goat blinds Marilyn's readers to the actual probability question. It is true that if a person randomly selects one door from two available doors, then the probability of selecting a particular door is $1/2$. Many of the readers feel that what happened on the first door selection is independent of what will happen next. The rules of the game, however, are cleverly concealing an initial branching step that changes the problem subtly but significantly from the simple random selection of one door from two into a problem where choosing the car on the second step, the switching step, depends on what you pick in the first step. Thus the game show problem has a "sleight of pen" twist to it.

First, let us try to explain why the probability of winning given that you switch is $2/3$. Instead of focusing on doors, which gets a bit confusing, focus on randomly selecting a prize from the following list:

George the goat, Fred the goat, and a car.

You could pick George, Fred, or the car with equally likely probabilities, namely $1/3$ each. After you pick a prize, the game show host must reveal, and remove from the list,

Ms. Benedict is an assistant professor in the Department of Mathematics and Computer Science at Augusta College. She has done research in applied mathematics and currently teaches courses in probability and statistics.

Dr. Thompson is a professor in the Department of Mathematics and Computer Science at Augusta College. His primary interest is combinatorial geometry.

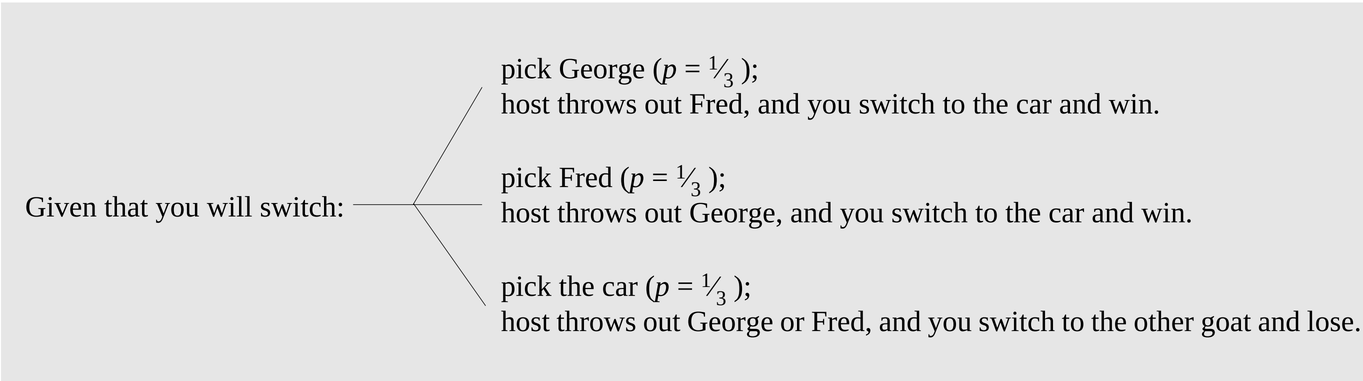

Figure 1: A tree diagram of possible outcomes given that an individual always switches doors.

one of the goats, and (a fine detail that is easy to overlook) *he host cannot touch your initial prize*. If you were so "lucky" as to select a goat on the first stage of the game, which has a $\frac{2}{3}$ probability of happening since you could select either George or Fred, you force the host to reveal and remove the only remaining goat from the list, leaving the car behind the untouched door. If you were so "unlucky" as to pick a car on the first stage, the probability of which is $\frac{1}{3}$, the host has a choice of revealing and removing either George or Fred leaving the untouched door with the other goat. If you decide to switch, the problem of winning simply involves choosing a goat at the first stage of the game which has $\frac{2}{3}$ probability of happening. This forces the host to remove the only remaining goat from the list allowing you to then switch to the door guaranteed to have a car. That is, in the switching strategy, to win you first have to "lose" (i.e., choose a goat) and to lose you first have to "win" (i.e., choose a car). Figure 1 is a tree diagram for the problem showing that the conditional probability P(win | switch) $= \frac{2}{3}$.

Interestingly enough, it is easy to get the answer $\frac{1}{2}$ as the solution to a new but related and more probabilistic problem that allows for a switching probability. In this version, the probability of winning will vary from $\frac{2}{3}$ to $\frac{1}{3}$ depending on your likelihood of switching with $\frac{2}{3}$ being the answer given that the contestant randomly switches or not with a 50% probability.

Suppose the novice contestant has a fixed probability of switching doors which is known to be p. That is, given a choice of switching, P(contestant will switch) = p and P(contestant will not switch) = $1 - p$. What is the probability that the contestant wins the car? In this problem, the contestant can win in one of two ways. The contestant can switch and win, or the contestant can decide not to switch and win:

$$\text{P(wins car)} = \text{P(switches and wins)} + \text{P(does not switch and wins)}.$$

Now for the "and" probabilities, we use the conditional equation after having solved for $P(A \cap B)$, namely $P(A \cap B) = P(A|B)P(B)$. So P(wins) = P(wins | switches)P(switches) + P(wins | does not switch)P(does not switch).

We have just seen that P(wins | switches) = $\frac{2}{3}$. Since winning the car without switching would simply involve selecting the car at the first stage, P(wins | does not switch) = $\frac{1}{3}$. Hence, P(wins car) = $\frac{2}{3}p + \frac{1}{3}(1 - p)$.

We claim that the solution of this version of the problem should make everyone happy. When the probability of switching is 1, the probability of winning becomes $\frac{2}{3}$, which was Marilyn's solution since she said the contestant should definitely switch. By adding a random switching probability of 0.5, P(wins) = $\frac{2}{6} + \frac{1}{6} = \frac{1}{2}$, which was Marilyn's readers' favorite solution. This pleases the intuition of the reader who is more focused on the random selection of one of the two doors at the end. Finally, for the worst possible game strategy, if the probability of switching is 0, the probability of winning becomes $\frac{1}{3}$.

The game show problem is very similar to the famous "Lady or the Tiger?" problem. In the "Lady or the Tiger?" problem, the reader will once again be faced with decisions that seem to have equal probabilities associated with them. Once again, because of an initial selection step, the dependent events may seem to be independent. Instead of "Should you switch or not?" the question becomes "Should the princess wait in Room A or not?" One version of the problem is illustrated by Figure 2.

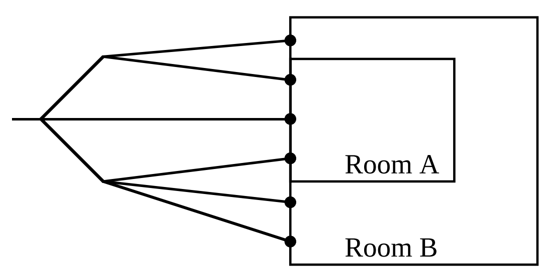

Figure 2: Possible paths for the "Lady or the Tiger?" problem.

Imagine that Figure 2 shows paths through the garden of an exotic castle. Each path leads to a door which opens into one of two rooms in the castle. The rooms are labeled in the diagram as Room A and Room B. As legend has it, there was a princess in the castle who fell madly in love with a young man. Unfortunately, the king, father of said princess, despised the young man. In fact, the king decided to try to do away with the young man. The king told the princess that she should wait in one of the rooms shown above. In the other room, the king would place tigers. The young man would then be forced to pick a path through the garden and open one of the doors into one of the two rooms. If the young man found the room with the princess, he would get to keep the princess. If, however, he found the room with the tigers, then the tigers would get to keep him. The princess is now supposed to choose which room she wants to use for herself and which room is to be used for the tigers. Which room should she choose for herself to maximize the probability of being found?

Once again we see a problem for which there seems to be a 50% probability for randomly selecting a "prize." There are, after all, three paths to doors into Room A and three paths to doors into Room B and at any decision step we assume the random selection of a branch of one of those paths. It is, however, once again the initial branching process that changes the probability of finding each room so that one is more likely than the other. By numbering the doors from top to bottom and placing the values of the conditional probabilities of selecting each branch (Figure 3), we see that

P(Room A)

= P(door 2) + P(door 3) + P(door 4)

= P(I and b) + P(II) + P(III and c)

=P(b | I)P(I) + P(II) + P(c | III)*P(III)

$= (\frac{1}{2})(\frac{1}{3}) + (\frac{1}{3}) + (\frac{1}{3})(\frac{1}{3})$

$= \frac{11}{18}$ which is $> \frac{1}{2}$.

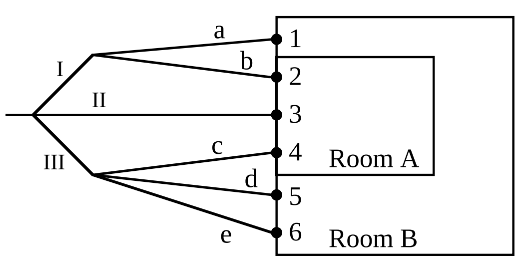

Figure 3: Labeled paths representing the conditional probabilities for the "Lady or the Tiger?" problem.

Clearly, the princess should choose to wait in Room A. The initial branching makes selecting Room A more likely. At the first branching, regardless of the path selected, it is possible to choose a door into Room A; however, if the middle path is selected, it would not be possible to find Room B. These facts make the selection of Room A more likely. As in the game show problem, the initial selection cannot be disregarded to form a new problem involving simple random selection of one door from six, illustrated in Figure 4.

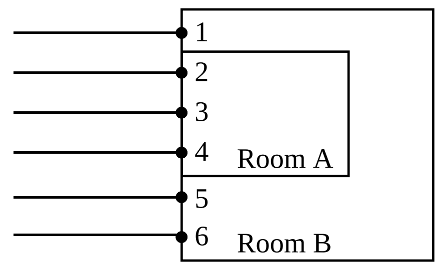

Figure 4: Paths disregarding initial branching.

One wonders whether the young man actually found the room with the princess. Perhaps he found a car.

Ideas for the Classroom

For explaining the difference between dependent events and independent events and for experiencing the idea of conditional probabilities in the classroom, draw branching tree diagrams for probability experiments with multiple steps. For some of these experiments, make the second step dependent upon the outcome of the first step. For example,

Roll a fair die. If the outcome is more than 3, you lose. If the outcome is 3 or less, then flip a fair coin. If you get heads, then you win; otherwise, you lose.

Follow these activities with other experiments where the steps are independent of each other. For example,

Flip a fair coin a first, second, and then a third time. If you get heads on the third flip, you win.

First consider probabilities like P(lose) for each game. Then consider conditional probabilities by conditioning on the first step like P(lose | odd on die) and P(lose | heads on first flip). Observe what happens to P(lose) for the different experiments when conditioning is added. Such classroom projects are fun and may help students apply the concept of independence appropriately in the future.

References

Vos Savant, M. (1990, September 9). Ask Marilyn. *Parade*, p. 22.

Vos Savant, M. (1990, December 2). Ask Marilyn. *Parade*, p. 28.

Vos Savant, M. (1991, February 17). Ask Marilyn. *Parade*, p. 21.

Vos Savant, M. (1991, July 7). Ask Marilyn. *Parade*, pp. 28, 29.