Advanced Mathematics from an Elementary Standpoint

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The National Council of Teachers of Mathematics (NCTM, 1991) suggests that "central to the preparation for teaching mathematics is the development of a deep understanding of the mathematics of the school curriculum and how it fits within the discipline of mathematics" (p. 134). For NCTM, this means preservice teachers should study a broad range of mathematics. Teachers should be able to see the connections between mathematics and other school subjects and with out-of-school situations where mathematics could be applied. The Council argues that mathematics teachers must study higher mathematics at an advanced level and revisit the content of elementary mathematics.

The point of view of the Committee on the Mathematical Education of Teachers (1991), a committee organized by the Mathematical Association of America, is consistent with the ideas expressed by NCTM. The Committee points out that substantive changes in school mathematics will require corresponding changes in the mathematics component of teacher education programs. The Committee argues that the content of mathematics courses offered in those programs must reflect "the rapidly broadening scope of mathematics itself" (p. xi). According to the Committee, the ideal mathematics teachers for the 1990s are those who

- possess knowledge and have an understanding of mathematics that is considerably deeper than that required for the school mathematics they will teach,
- enjoy mathematics and appreciate its power and beauty,
- understand how mathematics permeates our lives and how the various threads within mathematics are interwoven. (p. xiii)

In addition, the Committee lists more specific features in every section of the document. The Committee considers it important that teachers "appreciate the development of mathematics both historically and culturally" (p. 1). For example, Standard 6 states

> The mathematics preparation of teachers must include experiences in which they:
>
> - explore the dynamic nature of mathematics and its increasingly significant role in social, cultural, and economic development;
> - develop an appreciation of the contributions made by various cultures to the growth and development of mathematical ideas;
> - investigate the contributions made by individuals, both female and male, and from a variety of cultures, in the development of ancient, modern, and current mathematical topics;
> - gain an understanding of the historical development of major school mathematics concepts. (p. 9)

In another popular document, *Everybody Counts*, published before the two documents mentioned above, the National Research Council (1989) points out that effective mathematics teaching can be achieved if teachers' knowledge has appropriate pedagogical and mathematical foundations. In that document, the following is recommended:

> Those who would teach mathematics need to learn contemporary mathematics appropriate to the grades they will teach, in a style consistent with the way in which they will be expected to teach. (p. 64)

What the members of this council have in mind is a way of teaching that encourages active engagement with mathematical ideas, concepts, and structures.

Ideas similar to the one presented above have been also proposed in other countries. For example, in England the committee for school reform headed by Cockcroft (1985) recommended the following:

> Students should not be admitted to first method courses in mathematics unless their degree courses have contained a substantial mathematical component. (p. 208)

To summarize, it can be said that there is general agreement that mathematics teachers should have a strong background in elementary mathematics and that they should study some advanced mathematics. For some mathematics educators this means preservice teachers should study elementary mathematics from an advanced standpoint. This idea is not new. Two books by Felix Klein (1939 and 1945) were written for this purpose. The title of this paper is based on the titles of Klein's books[1]. Moise (1963), following Klein's lead, published a book of elementary geometry from an advanced point of view for mathematics students in college. His purpose was to invite students to reexamine elementary geometry. For Moise, reexamination means to clean up behind the introductory courses, furnishing valid definitions and

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valid proofs for concepts and theorems that were already known, at least in some sense and in some form.

To some extent, the idea of providing teachers with a strong mathematical background has not been fulfilled in most teacher education programs in the United States and in many other countries around the world. This somewhat problematic situation should be resolved. There is another way to think about the mathematical component of teacher education programs; namely, the introduction of advanced mathematics from an elementary standpoint. By advanced mathematics, I mean mathematics that has been created during the twentieth century, especially during and after World War II. By an elementary standpoint, I mean that teachers of mathematics should be presented a survey of advanced mathematical ideas without the intention of doing formal mathematical research.

Before introducing this approach, a few words about how mathematics is conceived in this paper are necessary. There are many ways of cataloguing or classifying mathematical sciences. A relatively recent classification distinguishes between theoretical and experimental mathematics. These two categories are independent of the aims of the research, for example, whether they are "pure" or applied is not relevant. On the other hand, this classification does not refer to a division of labor between a person working in an office and another working in the field. An experimental mathematician might carry out mathematical experiments with a computer as well as with paper and pencil. The essential difference highlighted by this classification is that between deduction and induction. That is, the experimental mathematician deduces conclusions from observations while the theoretical mathematician infers conclusions from an axiomatic system. Experimental mathematics is to some extent similar to what D'Ambrosio (1985) has called ethnomathematics. This categorization seems to be more fruitful for mathematics education than the classical classification of mathematics into pure and applied.

Experimental mathematics as a human activity in general is as old as other intellectual activities. Experimental mathematics as an experimental activity on mathematical objects such as numbers, equations, functions, random numbers, or geometrical figures is a recent innovation (Hammersley & Handscomb, 1964). An example of experimental mathematics is Monte Carlo methods. These methods deal with experiments on random numbers. Monte Carlo methods have been used to study problems as diverse as the growth of insect populations, the performance of nuclear reactors, and the "traffic" of telephone exchanges.

The concern of theoretical mathematics with abstraction and generalization is at the same time one of its main strengths and an inherent weakness. This inherent weakness is that the more general and formal the language, the less likely the mathematics is to furnish a numerical solution to a particular problem in a specific context (Hammersley & Handscomb, 1964). The presentation of advanced mathematics promoted in this paper is meant to be done from the perspective of experimental mathematics.

What are the aims for the introduction of advanced mathematics in teacher education programs? I have four main objectives in mind: (1) to communicate the idea that mathematics is an unfinished science, (2) to begin studying the history of contemporary, post-modern mathematics, (3) to convey the notion that the development of mathematics is affected by the socioeconomic and historical context in which it occurs, and (4) to introduce mathematical ideas that might influence research and practice in mathematics education.

It is the purpose of this paper to present four areas of advanced mathematical research that could be introduced in teacher education programs from the elementary point of view advocated here. These four areas are fuzzy set theory, subjective probability, search theory, and Voronoi diagrams. I do not include fractal geometry because this topic has already been treated by several mathematics educators in journals like the *Mathematics Teacher* and other publications. The same could be said about chaos. In contrast, the topics listed above have received little or no attention from mathematics educators[2].

I am interested in pointing out some of the implications that each one of these areas of research in mathematics might have for mathematics education as an intellectual field. The presentation is intended to go beyond the mathematics to a perspective of the mathematics by a mathematics educator.

Fuzzy Set Theory

The first work about fuzzy set theory was published by Zadeh in the United States in 1965. The basic principle in fuzzy set theory is not hard to understand. It is based on a redefinition of the membership function of an element and a given set in a specific universe. In classical set theory, given a set, a particular element will either belong or not belong to the set. Hence given a set *S*, its membership function is defined as follows $f(x) = 1$ if $x \in A$ and $f(x) = 0$ if $x \notin A$. In fuzzy set theory, the situation is different because there are cases in which it cannot be decided very easily if a given element belongs or not to a given set. Such cases are frequently found in classifying species in biology. In fuzzy set theory, given a set F and any element s in a specific universe of discourse (e.g., arbitrary collection of objects, concepts or mathematical constructs), a degree of membership d, where d is a real number in the closed interval [0,1], is assigned to s by a membership function *f*. Now we have defined a fuzzy subset *U* of *F* following Zadeh's definition.

Fuzzy set theory has been applied to many areas of research outside of mathematics. One area of application has been in psychology, where research using fuzzy set theory has been done in the recognition and formulation of mathematical problems. This means that the study of fuzzy sets is not only important from a mathematical perspective, but could be of importance for looking at some problems in mathematics education from a different perspective.

The use of fuzzy set theory implies that one should abandon Aristotelian logic as a normative guide to the study of cognition (Kochen, 1975). That is, we need to accept the theory that people do not necessarily have to establish or base their reasoning on dichotomies. For example, given a measure *x*, one does not have to play only with the possibilities of *x* being large or not large if one follows a fuzzy reasoning strategy.

Fuzzy set theory could be presented as an example of a theory that has been developed from different perspectives and applied to many areas in different mathematical communities. A collection of selected papers written by Zadeh and edited by Yager, Ovchinnikov, Tong, and Nguyen (1988) provides an excellent source of basic concepts, ideas, and applications. The collection is organized into three major areas: formal foundations, approximate reasoning, and meaning representation. For a current survey of applications, activities, and developments in fuzzy sets a good source is the *Journal of Fuzzy Sets and Systems*.

Fuzzy set theory research was also possible, in part, due to the support of the U. S. Army, Navy, and Air Force. Therefore, it provides an example of a branch of mathematics whose development has been strongly influenced by issues not related to the so called internal logic of mathematics. These issues are part of what some sociologists of science have called extra-epistemic arenas.

Subjective Probability

The central claim in the subjective approach to probability is that probability does not exist (de Finetti, 1974). De Finetti argues that we should abandon the belief in the existence of objective probabilities like we did with the superstitious beliefs about Phlogiston, the Cosmic Ether, and so on. The idea is that a probability assigned to an event does not exist without the knower assigning that probability to the event. Experiences with the event will help the knower to assign a "better" probability to it.

The subjective probability *p* of an event is given by the ratio $p = S/T$, where *S* is the amount that one is willing to lose if the event does not occur and *T* is the amount that one will have if the event occurs. For example, in the context of boxing, statements about "odds" instead of the language of probability are used: The odds are "1 to 9" in favor of Boxer A. Now, for an individual making a bet, this statement means the following: If she bets $1,000 on Boxer A and wins, then the gain is $9,000. But if she loses, the loss is $1,000. In summary, after the fight, she either has gained $1,000 + 9,000 = $10,000, or she has lost $1,000. Therefore, the probability *p* that Boxer A wins is

$$p = 1{,}000/(1{,}000 + 9{,}000) = 1{,}000/10{,}000 = 1/10.$$

This example was taken from a secondary school Italian textbook (Castelnuovo, Giorgi, & Valenti, 1987).

The ideas in subjective probability might influence mathematics education practice and research. For instance, subjective probability might be introduced in teacher education programs and in high schools. In Italy, where de Finetti was born, some elementary ideas about subjective probability are introduced in high school. This introduction provides students with another approach, along with the objective and statistics approaches, to the study of random phenomena. It provides teachers with the opportunity to learn that more than one mathematics can be used to deal with problems related to random phenomena.

The introduction of subjective probability might have an impact on the way in which mathematics educators conceptualize children's notions of probability. Instead of comparing children's solutions of a probability problem to the objective probability that already exists, the solutions could be seen as the subjective probability assigned to the event by the children according to the information the children have about the event.

Search Theory

Search theory, like many post-modern mathematical ideas, was created during World War II. It emerged as a part of operational research, a branch of mathematics developed during World War II, in the work realized by the Antisubmarine Warfare Operation Research Group (ASWORG) (Chudnovsky & Chudnovsky, 1989). Documents reporting elaboration of search theory were declassified by the military in 1958 and therefore became available for study outside of the military. Major problems of search theory were, and are, related to the detection and destruction of enemy "elements" (armored tanks, army vessels, airplanes, etc.). More specifically, the basic problem of search theory during the war was the design of screening formations to protect the United States' convoys in the Atlantic (Koopman, 1979). Other authors have pointed out a different origin of search theory. According to Aigner (1988), search theory has its origins in "Shannon's work on entropy of experiments and his noiseless coding theorem" (p. 1). Aigner says that the connection between search theory and coding was not well understood until the sixties. This connection can be formulated as follows:

> The average time needed to locate an unknown object equals the average length of an optimal code which, in turn, is roughly the information or uncertainty encountered at the start of the search. (Aigner, 1988, p. 1)

According to Koopman (1979), one of the founders of search theory, the situations to be studied and resolved by search theory are characterized by three data:

> (1) the probabilities of the object under search being in its various possible positions; (2) the local detection probability that a particular amount of local searching effort (time spent, concentration of searching units, and the like) should detect the target; and (3) the total amount of searching effort available. (Koopman, 1979, p. 527).

Problems in search theory can be divided into two basic categories: search for an immobile (or stationary) hider and search for a mobile (or moving) hider. These two kinds of problems could be solved in two different contexts or "search spaces." These contexts are bounded spaces or unbounded domains. In general, all problems solved by search theory have one aspect in common: the objective is to find an unknown object in the shortest time and with as little cost as possible (Aigner, 1988).

One of the oldest and most widely known search problems is that of finding a counterfeit coin among a group of legal coins. The situation is the following: in a group of n coins there is one and only one counterfeit coin which is assumed to be heavier than the rest. The problem can be solved using an equal arms balance. Try to solve it for the case $n = 9$.

Search theory is currently applied to many scientific and practical problems. Problems that require a search for an unknown object are found in practically all areas of human activity. Some of these areas are search and rescue, surveillance, exploration for minerals, medicine, space science, and industry. A book edited by Haley and Stone (1980) provides a good survey of the applications of search theory.

The study of search theory from an elementary standpoint will provide teachers with an example of a mathematical theory that was elaborated very recently and that was developed out of the needs of the U.S. Navy in World War II. Also, search theory provides an example of a mathematical theory that was not available to the entire community of mathematicians at the time of its elaboration. It was available to them only after the Navy decided to declassify it.

Voronoi Diagrams

The diagrams of Voronoi were originally introduced in 1850 by the mathematician P. G. L. Dirichlet (Boots, 1986). Some authors consider Descartes to be the inventor of Voronoi diagrams in 1644 (Klein, 1989). These diagrams were given a straightforward treatment by the mathematician G. Voronoi in 1908.

The basic Voronoi diagram problem is the following: Given a set of points $S=\{p_1, p_2, ..., p_n\}$, find every region X_i in the Euclidean plane containing p_i such that the points in X_i are closer to p_i than to any other point in S. This definition of the problem is based on the notion of distance. Klein (1989) proposes a definition of the problem based on the concept of bisecting curves. Figures 1 and 2 are examples of Voronoi diagrams for $n = 2$ and $n = 3$, that is, for $S = \{p_1, p_2\}$ and $S = \{p_1, p_2, p_3\}$ respectively.

Voronoi diagrams have been applied to a wide range of problems in such areas as archeology, geography, biology, ecology, crystallography, geology, and molecular physics. For example, these diagrams have been applied to solve the problem of partitioning a given urban area into regions to maximize the availability, in terms of distance, to a particular service. Another interesting application of Voronoi diagrams can be found in the area of geometric crystallography (Engel, 1986). Several names have been used to describe these diagrams: Dirichlet tessellations or

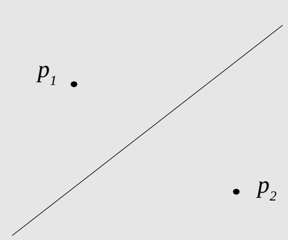

Figure 1: Voronoi diagram for two points.

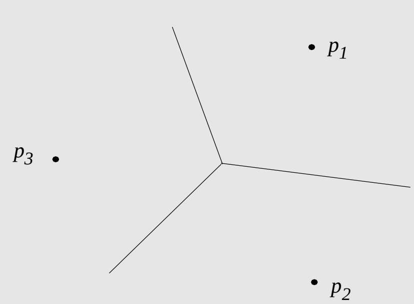

Figure 2: Voronoi diagram for three points.

tilings, Wigner-Seits cells, Thiesen polygons, and regents graphics in art.

Voronoi diagrams provide a good example of a mathematical idea that has been around for a long time, that has been "rediscovered" several times, and that has been usedin a wide range of applications. The exposure of mathematics teachers to this idea may cause them to reflect on their views about the nature of mathematics.

Conclusion

That mathematics education is affected by the ways in which mathematics educators conceive of mathematics is a widely accepted statement. Research on teachers' beliefs suggests that college mathematical experiences have little influence on teachers' conceptualizations of mathematics. I think this is true in part because colleges have not provided prospective teachers with opportunities for experiencing mathematics in ways that are substantially different from those they have experienced previously. The advanced mathematics from an elementary standpoint approach proposed in this paper is intended to change that. Its main purpose is to provide instances in which preservice and inservice teachers will have an opportunity to encounter a mathematics that is exciting, alive, growing, and full of surprises. They will also encounter a mathematics that is not clean and pure from external influences. This "new" mathematics might challenge teachers' beliefs about mathematics, promote reflection on those beliefs, and eventually, lead them to a change of perspective.

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Notes

1. In 1947 the mathematician H. Rademacher gave a series of lectures at Stanford University. These lectures were edited by D. Goldfeld and published in 1982 under the title: *Higher Mathematics from an Elementary Point of View*. The title and content of this paper have no relation to this book.

2. Issues 34 and 37 of *Consortium*, the newsletter of the Consortium for Mathematics and its Applications (COMAP), published in 1990 and 1991, respectively, included articles about Voronoi diagrams.

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