Assessing Students' Beliefs About Mathematics

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The beliefs that students and teachers hold about mathematics have been well-documented in the research literature in recent years. (e.g., Cooney, 1985; Frank, 1988, 1990; Garofalo, 1989a, 1989b; Schoenfeld, 1987; Thompson, 1984, 1985) The research has shown that some beliefs are quite salient across various populations. These commonly held beliefs include the following:

- Mathematics is computation.
- Mathematics problems should be solved in less than five minutes or else there is something wrong with either the problem or the student.
- The goal of doing a mathematics problem is to obtain *the* correct answer.
- In the teaching-learning process, the student is passive and the teacher is active. (Frank, 1988)

It is generally agreed that these beliefs are not "healthy" in that they are not conducive to the type of mathematics teaching and learning envisioned in the *Curriculum and Evaluation Standards for School Mathematics* [*Standards*] (NCTM, 1989).

There appears to be a cyclic relationship between beliefs and learning. Students' learning experiences are likely to contribute to their beliefs about what it means to learn mathematics. In turn, students' beliefs about mathematics are likely to influence how they approach new mathematical experiences. According to the *Standards*, "[Students'] beliefs exert a powerful influence on students' evaluation of their own ability, on their willingness to engage in mathematical tasks, and on their ultimate mathematical disposition." (NCTM, 1989, p. 233)

This apparent relationship between beliefs and learning raises the issue of how the cycle of influence can be broken. The type of mathematics teaching and learning envisioned in the *Standards* can provide mathematical experiences that will enrich students' beliefs about mathematics. Thus, mathematical experiences provide one place where intervention can occur; however, it may also be advantageous to intervene at the other point in the cycle, namely students' beliefs. The *Standards* suggest that the assessment of students' beliefs about mathematics is an important component of the overall assessment of students' mathematical knowledge. Beliefs are addressed in the tenth standard of the evaluation section, which deals with assessing mathematical disposition. Mathematical disposition is defined to include students' beliefs about mathematics. It is recommended that teachers use informal discussions and observations to assess students' mathematical beliefs (NCTM, 1989). Although teachers' awareness of students' mathematical beliefs is important, it may be equally important for students to be aware of their own beliefs toward mathematics.

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One medium for bringing students' beliefs to a conscious level is open-ended questions. As students ponder their responses to such questions, some of their beliefs about mathematics will be revealed. As groups of students discuss their responses to these questions, some students' beliefs will likely be challenged, leading to an examination of these beliefs and their origins, and, possibly, to the modification of these beliefs.

This article presents some open-ended questions that can be used to address students' beliefs about mathematics. These questions have been used by the author with elementary, junior high, and senior high school students; preservice and inservice elementary, junior high, and senior high school teachers; and graduate students in mathematics education. The questions have been culled from a variety of sources and do not represent original ideas of the author. Each question is followed by a summary of typical responses from the aforementioned groups. The responses from the various populations were strikingly similar, which is not surprising since the beliefs held by these groups are generally quite similar. In some cases, possible origins of the belief or possible avenues for further discussion are included.

Questions to pose

Questions can be presented to students in a variety of formats. Students can be given a question or series of questions to ponder for homework, or they can be assigned to gather responses to questions from an adult, an older student, and a younger student. Some questions can be posed as journal writing entries, while others can be presented for class discussion with no prior preparation on the part of the students. It is important, however, regardless of the manner of presentation of the questions, that students receive some feedback about their responses.

This feedback may come in the form of a class or small group discussion, or it may be in the form of questions or comments from the teacher in the students' journals.

If you and a friend got different answers to the same problem, what would you do?

The most common answer to this question is that the students would both rework the problem. When asked what this accomplishes, the students reply that one person might find an error in his or her work. Another popular answer is that it depends on who the other student is. If the other student is perceived to be "smarter," then the tendency is to accept that person's answer. On the other hand, if the student feels mathematically superior to the other person, he or she will stick with the original answer. Only on rare occasions do students suggest that both students could have a correct answer. When this option is presented to students, they tend to think of examples where both students have the same numerical value represented in a different form (i.e., $\frac{1}{2}$ and $\frac{3}{6}$, or 0.5 and 1:1). It is very unusual for students to consider possibilities where the two answers are completely different but equally correct, as is often the case in problem solving. This provides support for Frank's (1988) findings that students perceive mathematics as a search for *the* one right answer.

If you were playing "Password" and you wanted a friend to guess the word "mathematics," what clues would you give? ("Password" clues must be one word and may not contain any part of the word "mathematics.")

The four most common answers are, predictably, add, subtract, multiply, and divide. Other clues include numbers, problems, operations, calculate, hard, and subject. These responses suggest that students tend to view mathematics as synonymous with arithmetic. This revelation can lead nicely into a discussion of other branches of mathematics, the types of problems that are posed in the branch, and the types of tools used such as ruler, compass, graphing calculator, etc. This discussion, along with some classroom activities in other branches of mathematics such as geometry, probability, and statistics, can help dispel the students' belief that mathematics is merely computation.

If given a choice, would you prefer to have a) one method which works all the time or b) many methods which work all the time when solving a problem?

Most students indicate that they would prefer to have one method for solving a problem because they do not have to remember as much as if they had multiple methods. This response suggests that students perceive memorization as a major component of mathematical learning. Some students, however, will indicate that they would prefer to have several methods from which to choose when solving a problem because they can check their answers using a different method. Other students point out that sometimes one method is more efficient for a particular problem than other methods. For example, the quadratic formula is a method that will find the root(s) of any equation of degree two; however, if faced with an equation such as $x^2 + 4x + 4 = 0$, factoring would be significantly more efficient. It is also likely that some students will understand one method better than others and that all students are not likely to understand the same method. This question is one that students can debate among themselves, providing examples of mathematical situations which support their opinions.

Is it possible to get the right answer to a mathematics problem and still not understand the problem? Explain.

Unfortunately, students are able to obtain correct answers without understanding what they are doing all too often. How many elementary students can perform the "invert and multiply" algorithm for dividing fractions but cannot tell a story to go along with the number sentence? How many first-year algebra students can find the roots of quadratic equations such as $x^2 - x - 6 = 0$ but cannot explain what "$x = 3$, $x = -2$" means in terms of the graph of the equation? After pondering the initial question for awhile, many students admit that this is how they function in their mathematics classes most of the time. Students can generate examples of mathematical tasks that they perform by rote without understanding the reasons for the steps they perform. This can provide opportunities for meaningful reteaching of concepts by the teacher or for interesting research projects for individual students. This type of reteaching is often the focus of the first mathematics content course taken by preservice elementary teachers. This course typically consists of the conceptual underpinnings of the four basic operations on subsets of the real numbers. Class discussions are often punctuated by comments such as "So that is why the decimal point goes there!" Discussions of this nature can help students see that there is more to studying mathematics than merely obtaining correct answers.

How do you know when you have correctly solved a mathematical problem?

Reworking the problem, checking with the teacher or a classmate, looking in the back of the book, working backwards (in the case of arithmetic) or plugging in values (in the case of algebra) are common answers to this question. Seldom do students suggest that they check to see if the answer makes sense in the context of the initial problem. This provides an opportunity to introduce a problem such as the one given in the Third National Assessment of Educational Progress:

> An army bus holds 36 soldiers. If 1128 soldiers are being bused to their training site, how many buses are needed?

This problem was given to 45,000 high school students, and one-third of them responded that the answer was 31 remainder 12 without checking to see if such an answer made sense in the context of the problem (Schoenfeld,1987). All too often the "check back" stage of the problem solving process is neglected, either because students are in a hurry to finish the problem or because they do not perceive checking back as an essential component of problem solving. Many students equate checking back with checking numerical computations. They do not perceive looking for generalizable results, examining the efficiency of the method used, looking for additional answers, or identifying the underlying mathematical concept(s) as part of the problem solving process. This limited view is likely due to their prevailing belief that the objective of working a mathematics problem is to obtain an answer.

What subject(s) is mathematics most like? least like? Why?

The most popular answer to this question is that mathematics is most like science because it involves memorizing formulas and working with numbers. Virtually all other subject areas fall into the category of being least like mathematics. Nevertheless, during a class discussion, something interesting usually occurs. Someone may say that mathematics and music are not alike, but a student who has some musical background may reply, "Oh yes they are!" The student will go on to explain that music involves patterns, counting beats, using fractions to determine how long to hold a note, time signatures, and a variety of other mathematical concepts. A similar discussion usually ensues about art. This discussion provides a nice opportunity to talk about the Golden Ratio and its uses in art and architecture. Some students will claim that mathematics is not similar to studying a foreign language, while other students will contend that mathematics *is* a foreign language! In both French and mathematics, it is necessary to adopt the conventions of the language, learn new vocabulary, and determine how isolated words or ideas connect to form meaningful sentences or concepts. In both instances, practical experiences in the real world help to refine newly acquired skills and concepts.

For other subject areas, it is less likely that students will provide an explanation of how the subject is similar to mathematics. This explanation usually has to be initiated by the teacher, and then students contribute their own ideas. For example, history and mathematics are alike because just as we cannot change history, we cannot alter certain mathematical facts (at least not without changing a great deal of mathematics). The Boston Tea Party occurred in Boston, not Orlando, and we cannot change that. Similarly, the probability of any given event must be between zero and one, inclusive, and we cannot change that. (It is best to use an example that relates directly to the topic being studied when this discussion takes place.) Although a great deal of history is past and cannot be changed, history is being made right this very moment, just as new mathematical knowledge is constantly evolving. Students tend to perceive mathematics as a static discipline in which everything has already been created or discovered; however, the frontiers of mathematical knowledge are being pushed farther every day. Also, past historical events affect the way we live our lives today while the mathematics that has already been discovered or created shapes the mathematics that we work with today. For example, the bombing of Pearl Harbor in 1941 affects the current location of the United States naval fleet; namely, a majority of the fleet is not kept in the same harbor. Likewise, the fact that there are infinitely many prime numbers affects the modern day work of cryptographers, who develop coding schemes using prime numbers.

There are many similarities between mathematics and language arts, but they require some thoughtful consideration to uncover. Spelling, grammar, and mathematics are alike in that they have certain rules that must be memorized and followed. A key difference, however, is that rules in spelling and grammar are frequently broken, while rules in mathematics are generally universally applicable (except for cases involving zero in certain instances). Literature and mathematics are alike because two people may read the same story or poem and come away with entirely different messages. In mathematics, two people may interpret a problem differently, and thus may get different answers or may have different approaches to the problem.

Describe someone in your class or school who you think is mathematically talented.

This discussion needs to be handled with some tact so as not to hurt anyone's feelings. Encourage students not to use names of individuals but rather to describe characteristics of that person that show evidence of mathematical talent. Many students, particularly elementary students, will say that mathematically talented people can do mathematics quickly. They are the students who raise their hands first to answer a question, finish a test first, or advance the farthest in the "Around the World" flashcard game. This response likely stems from students' belief that mathematics problems should be done quickly. Older students will often indicate that mathematically talented people are more logical and analytical and can do things in their heads. This view may explain some students' reluctance to draw diagrams or write down information to solve a problem. In addition to the aforementioned characteristics of mathematically talented people, students will invariably mention stereotypical physical or personality characteristics of such people. The next question provides an opportunity to examine these stereotypes.

Can you think of any television characters who are mathematically talented?

The first answers from students are usually characters who fit the archetypical "nerd" image. Characters such as Steve Urkel from *Family Matters* and Arvid from *Head of the Class* come readily to mind. These characters are equipped with pocket protectors, eyeglasses, briefcases, and white socks, and they are the intellectual giants of the situation comedies on which they appear. With a little bit of prompting, however, students can usually think of another popular, prime-time television character who is mathematically talented but who does not fit the "nerd" stereotype. Dwayne Wayne of *A Different World* is a college student majoring in mathematics who is good-looking, popular, fashionably dressed, and well-respected by both his peers and his teachers. Several episodes of the show have dealt directly with mathematics, including episodes where Dwayne tutored students who were struggling with their mathematics classes. Another episode found Dwayne enrolled in a poetry class in which he felt his intellectual talents were being wasted and could be better spent on coursework in his field. Once students begin thinking along these lines, they can think of other characters and non-fictional people who are mathematically talented but who do not fit the negative stereotype.

Close your eyes and try to picture a mathematician at work. Where is the mathematician? What is the mathematician doing? What objects or instruments is the mathematician using? Open your eyes and draw a picture of what you imagined.

Ask students a variety of questions about the mathematician they imagined. Was their mathematician male or female? How old was their mathematician? What was their mathematician wearing? What did their mathematician look like? In what types of activities was the mathematician engaged? Were there other people around?

This activity is sometimes used in science education to help students overcome stereotypes about scientists. The results of the activity are quite similar regardless of whether it is done using a scientist or a mathematician. In the case of the mathematician, the students generally picture an older male with grey Einstein-like hair, wearing glasses, and sitting at a desk. He is usually using pencil and paper, books, a calculator or computer, and sometimes a ruler. The mathematician is often in a nondescript room, and there are no other people around. These observations suggest that students view mathematics as a solitary endeavor that is carried out in a place very different from their everyday surroundings. They also apparently view mathematics as a male-dominated discipline.

Do you suppose McDonald's has a mathematician on its corporate staff? What might that person do for McDonald's?

The initial reaction of most students is that a mathematician is employed to help with inventory and accounting tasks. A common response from elementary students is that a mathematician is needed to keep track of how many hamburgers have been sold so that the signs on the golden arches that proclaim "*x* billion hamburgers sold" will be accurate. These responses again suggest that students are considering only the computational aspect of mathematics. To stimulate additional thought, the teacher can pose some real-world questions: How does McDonald's decide where to build a new restaurant? How does McDonald's decide on new food products to offer? How are the promotional games created? These questions open the doors for discussions about data collection, statistics, and probability.

What businesses in our town might employ a mathematician? What would the mathematician do?

Responses to these questions vary depending on the businesses in the town, but students generally have a limited view of the career opportunities for mathematicians. These questions can be used to initiate a discussion of careers in mathematics and careers which use mathematics. Students can interview townspeople who are mathematicians or who use mathematics in their jobs to gather information about various careers. Students can share their findings with the class or the entire school through oral reports, written reports compiled into a class book, pictures, murals, and video or audio tapes. Carpenters, architects, nurses, engineers, scientists, actuaries, pharmacists, statisticians, and operations researchers are among the people who use a great deal of mathematics in their careers.

Concluding remarks

The preceding questions, individually or collectively, cannot provide teachers with definitive information about each student's beliefs about mathematics. Nevertheless, such questions and discussions, coupled with observations of students' interactions in mathematical settings, can provide teachers with valuable information about the beliefs that influence their students' study of mathematics. Students' beliefs about mathematics are manifested in the classroom in whether and how they ask and answer questions, work on problems, and approach new mathematical tasks. The assessment of students' beliefs about mathematics can help teachers plan instruction and structure the classroom environment so as to help students develop more enlightened beliefs about mathematics and mathematics learning (NCTM, 1989).

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Problem Solutions

Page 6: Band around the Earth

Circumference = $2\pi r$. The new circumference is $2\pi r + 10$, or we could say that the new circumference is $2\pi(r+x)$, where x is the uniform height of the band above the earth. Hence $2\pi r + 10 = 2\pi(r+x) = 2\pi r + 2\pi x$. So $10 = 2\pi x$, and $x = 10/(2\pi) \approx 1.6$ feet. Therefore, choices a) and b) are true.

Page 10: Aye, aye, sir

All complex numbers can be written as $x + iy = r(\cos q + i \sin q) = r e^{iq}$. Let x = 0 and y = 1, then r = 1 and $q = \pi/2$. Hence $0 + 1i = 1(\cos(\pi/2) + i \sin(\pi/2)) = 1e^{i\pi/2}$. Therefore, $i^i = e^{(i\pi/2)i} = e^{-\pi/2} \approx 0.207879576...$

Page 10: Subsets

Take any element in the set. It either appears or does not appear in a given subset. Therefore, the element is a member of exactly half, or $2n - 1$, of the subsets. The element is used as an addend $2n - 1$ times. This is true for every element of S. Thus, $(2n - 1)(1 + 2 + 3 + 4 + ... + n)$ is the desired sum.

Page 10: Points to ponder

For each of the three cases, assign points A and B vertices and O orthocenter of triangle ABC. Construct segment AB. Construct line l_1 perpendicular to AB and through O. Construct line l_2 through B and O. Construct line l_3 through A and perpendicular to l_2. C is the intersection of l_1 and l_3. Each case will construct the same point.

If P_1, P_2, and P_3 are distinct non-collinear points, either exactly one or all three cases will produce a solution. One solution occurs only when a pair of lines P_1P_2, P_1P_3, or P_2P_3 are perpendicular.

If two points are concurrent and assigned to the orthocenter and vertex, then the solution is all points on the line through the concurrent orthocenter and vertex and perpendicular to the obtainable side of the triangle.

Page 10: Regions of a trapezoid

Consider labeling the regions as below. Let h be the height of the trapezoid. Let b_1 be the length of the base of the trapezoid containing the point N, and let b_2 be the length of the other base. Then the area of the trapezoid is the sum of the areas of I, II, and III, which equals $\frac{1}{2}h(b_1 + b_2)$. So $I = (\frac{1}{2})(\frac{1}{2}h)b_2$, and $III = (\frac{1}{2})h(\frac{1}{2}b_1)$. Hence, $I + III = (\frac{1}{4})b_1h + (\frac{1}{4})b_2h = (\frac{1}{2})$ area of the trapezoid. Therefore, I + III = II.