The Study of the Unknown: Taking Geometry to New Dimensions

Bridget Arvold

Spirited classroom discussion occasionally drives both students and teacher toward new horizons.

Michael	Sure, we can summarize the various relationships between points, lines, and planes.
Yolonda	[Using pencils to illustrate…] Well, lines can coincide, they can intersect, or they can be ... what's the word ... skew?
Fritz	Yeah, and lines have to be non-coplanar to be skew. They need space.
Beth	[Using a pencil and a book to illustrate...] And a line can lie in a plane, can intersect a plane, or can be parallel to a plane. A line and a plane are never skew.
Suzie	But ... but ... [mumbling something to herself.]
Beth	But what Suzie?
Suzie	It's just that ... if lines are one-dimensional and they need three dimensions to be skew, does that mean that a line and a plane need a fourth dimension to be skew?
Andre	Hey, yeah. And two planes would need the fifth dimension right?
Michael	Wait. I don't know what you mean.
Suzie	See, one plus one is two, but to be skew you need an extra dimension. So two lines need the third dimension.
Tia	I get it. A two dimensional plane and a one dimensional line ... two plus one is three ... so to be skew they would need an extra dimension, so they'd need the fourth, right?

My initial response to this discussion was total amazement. These students were already finding patterns and formulating conjectures. I had no idea whether or not there were answers to the questions raised but was eager to find out. We decided to integrate "dimensions" into our year long investigation of geometry.

Mathematics researchers from a nearby university assisted by providing exciting ideas, a stimulating reading list, and actual classroom instruction. As the students read the classic *Flatland* (Abbott, 1952) and the comic book *The Adventures of Archibald Higgins: Here's Looking at Euclid* (Petit, 1985), they learned to "see" from different perspectives. Molding clay into various shapes introduced us to geometric topology and similarities between a coffee cup and a torus (donut). Our vocabulary grew as we investigated the local and global geometries of different surfaces. Students designed games based on the jungle gym-like model of three-manifolds in *The Shape of Space* (Weeks, 1985). This in turn spurred students to investigate the four-dimensional cube also known as the hypercube or tesseract. They made drawings and built models of these hypercubes. In a workshop facilitated by Jeff Weeks, students collaborated to construct a model of a slice of hyperbolic space.

The students were eager to relate almost everything in the regular curriculum to the fourth dimension. Amazingly, this seemed to make the traditional content much easier for most of them to understand. The students' grasp of typically challenging concepts such as locus of points and relationships between areas and volumes of similar figures was remarkable.

As a six-week project, students continued to research topics they felt were related to the fourth dimension and then molded their ideas into unique creations. The students' expressions of the fourth dimension were shared with their families and others during an evening program. Through poetry, sketches, monologues, and presentations, students shared their ideas. The fourth dimension was related to black holes, the Bermuda triangle, dreams, extra-sensory perception, communications, religion, unidentified flying objects, and more. One student, Zach Veilleux, presented his interpretation of the fourth dimension in a photo essay using time-lapse photography. One of his photos appears in Figure 1.

At year's end I wondered whether this study of dimensionality was worth the additional time and energy it required. I believe that it was worth every second; the number of positive outcomes, both mathematical and attitudinal, had not been evident in previous classes. These outcomes included improved reasoning skills, increased awareness of connections within mathematics and between mathematics and real life, and a greater depth of understanding of geometric concepts. Students came to view mathematics as a dynamic discipline that enables them to explore the real world and unknown worlds, and which allows for creativity in expressing individual

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interpretations of mathematical phenomena.

Perhaps the most exciting outcome of this unit was the enthusiasm that the students showed for the mathematics they were studying. The students were motivated to pose their own questions and to attempt to answer them by exploring and reading related materials. Students shared their enthusiasm with friends by bringing them to the resource-rich classroom after school. This unit sparked an interest in learning about dimensionality that has continued for almost four years. Not only have I continued to study dimensionality and develop workshops for students and teachers grade K-16, but some of the students from this class are still investigating the fourth dimension as well.

Figure 1: One student's representation of four-dimensional space.

While in Mrs. Arvold's geometry class (9th grade), we studied the concept of higher dimensions. We were assigned a project in which we were to somehow represent a dimension higher than the third. I had no idea what the fourth dimension would look like, but I was interested in photography, so I decided that camera tricks might somehow work. I shot a roll of film, using mostly double or triple exposures. The photo at the left is a double exposure of my father. I decided that in the fourth dimension, you would probably be able to see all sides of an object at one time, hence the multiple exposures from several angles.

- Zach Veilleux

(Zach is now a freshman at Wooster College, Wooster, OH.)

Activities for Exploring Dimensionality

The development of visualization skills and the formation of concepts of dimensionality are fostered by engaging students in simple spatial activities. The following activities are only a few of the successful ones that I have used. Many more can be found in the references listed.

The 3-D Pull

This activity was used with second graders to help them move between the second and third dimensions.

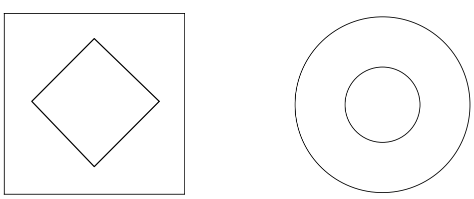

Figure 2: Squares and circles.

Draw two concentric circles of different radii on a plain piece of paper (Figure 2). Stare at the smaller circle until you can visualize it as the top of a very tall tower. Hold this image in your mind for a while, and then let the smaller circle slide back onto the paper. Now push the smaller circle away so that it becomes the bottom of a cylindrical hole. Let it slide back to the second dimension. Extensions of this activity involve using shapes other than circles. For example, draw squares with the sides of the two squares non-parallel (Figure 2), and try to visualize a twisting and pulling of the smaller square up into the third dimension.

In The Blind

I have used this activity with fourth grade students, middle and high school students, prospective elementary teachers, and in-service teachers. It facilitates moving between dimensions and introduces the opportunity for conjecturing about a higher dimension.

Make a blind so that students cannot see the objects you will be placing on an overhead projector. Place three-dimensional objects on the overhead and number them. The following objects work well: a cylinder placed on its base, a cone placed on its base, a hemisphere placed on its great circular region, a penny, a small marble, and a cylinder on its side. Turn on the overhead, and ask students to identify each object and write a word or two to describe each one. Then have students draw exactly what they see on the overhead. When all students are finished, facilitate a sharing of ideas, first in small groups and then as a class. The discussion should include the fact that the marble and the cylinder placed on its side have fuzzy outlines on the overhead projection. This leads beautifully into a discussion of sight recognition (Abbott, 1952). Next, while the overhead projector is on, slowly rotate each object allowing students to see the changing two-dimensional representations. Finally, reveal each object and ask students to sketch a three-dimensional representation of the object on paper. Having samples of each of the objects for each group of students aids the students in the sketching activity.

Patterns and the Fourth Dimension

Euler's study of the relationship between the number of vertices, edges and faces of a polyhedron may have inspired this pattern recognition activity which is found in many discussions of the fourth dimension. This activity gradually takes the participant from zero dimensions to four dimensions. A chart such as the one in Figure 2 can be used for conjecturing about the number of vertices, edges, faces, and spaces in a hyper-cube.

	Dim	Corners	Edges	Faces	Spaces
Point	0				
Segment	1				
Square	2				
Cube	3				
Hypercube	4				
Hyper - Hypercube	5				

Figure 3: Chart for recording information about surfaces in various dimensions.

In my experience, the age of the participants is directly related to the time in which they produce a credible conjecture and a convincing argument. Second and fourth graders explained how they visualized the growth from one stage to the next whereas older students and adults related their initial focus on number patterns in the table before visualizing the changes in the edges and faces. Therefore, it may be wise to postpone the organization of data in a table as long as possible.

Life On Other Worlds

Imagining life on different surfaces is a favorite activity with kindergarten students through mathematics researchers. The response time for different groups proves quite surprising.

The activity involves a young lady and her friends, each living on an unknown surface. The objective is to discover the nature of each of the surfaces using minimal information. It is helpful to have the following materials available to students: spheres, narrow strips from computer paper, life-savers, full sheets of construction paper, markers, tape, 1" x 11" strips of clear transparency film, and copies of little Mo (Figure 3) drawn on a 1" x 1" square of clear acetate.

Students are given the following information:

1. Mary lives in a square house on a smooth surface. She walks out her front door and walks in a straight line (intrinsically straight) until she finally comes to the back door of her house. On what surface(s) might she live?
(A discussion of the meaning of "smooth" as no sudden changes, like the edges of a cube, is necessary. A discussion of the meaning of "straight" might involve students going outside and walking straight over a hill. Another option is to try to put a narrow strip of computer paper straight on the surface of a sphere.) Mary decides to test her conjecture by walking straight out her side door. She never returns. Do you need to change your conjecture? If so, why? If not, what might a new conjecture be?

2. Sally is very saddened by Mary's disappearance, so she decides to wear a bright yellow safety line. She goes out her front door, walks straight and returns to her back door. The next day she wears a red safety line and walked straight out her side door. She continues walking straight until she returns to the other side door. What is the nature of Sally's surface?

3. Larry walks straight out his front door with a blue line attached, and enters his back door. Larry then attaches a green line and walks straight out his side door. He returns to the other side door. He calls Sally on his intergalactic phone and states that his surface is the same as hers. Mary asks if he planted a flag at the pole. Larry is puzzled. Mary explained that she placed a flag at the point where her yellow and red safety line intersected. Larry is astonished because his lines never crossed. Explain.

4. Curly walks straight out his front door and returns to enter straight through his side door. On what surface(s) might Curly live?

5. Mo lives in the center of town. Mo walks straight out her front door and continues until entering town on the opposite side. Mo suddenly feels very strange. A friend came up to shake hands but put the wrong hand out. Then Mo noticed that the letters on

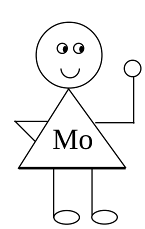

Figure 4: Little Mo.

a billboard were written backwards or reversed or something. Mo entered the house and the nameplate on the door said WO instead of MO. What is happening? What conjectures can you make about Mo's surface? At this point distribute a transparency strip and a little Mo (Figure 4) to each participant. Help participants write "MO" on the strips with their markers, then tape the strips to form Moebius strips, and slide little Mo around from MO to WO. This might lead to the study of reflections or other Moebius strip activities.

Conclusion

Exploring the topic of dimensionality with students and teachers has been very rewarding. I have noticed that this type of dynamic and relational learning affords greater initial understanding of basic concepts. Spending less time reviewing basic concepts allows more in-depth discussions and the uncovering of more content. Moreover, using this type of instruction and venturing into a topic not generally found in most textbooks has forced me to become increasingly dependent upon outside resources, especially university faculty. This collaboration has turned my planning into a learning experience and has helped me view myself as a teacher, a mathematician, and a researcher.

References

Abbott, E. A. (1952). *Flatland: A romance of many dimensions* (6th ed.). Mineola, NY: Dover.

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Annotated Bibliography

Abbott, E. A. (1952). *Flatland: A romance of many dimensions* (6th ed.). Mineola, NY: Dover.
This classic science fiction narrative takes the reader to worlds of different dimensions: Pointland, Lineland, and Flatland. It is appropriate for high school students and adults.

Banchoff, T. F. (1990). *Beyond the third dimension: Geometry, computer graphics, and higher dimensions*. New York: Scientific American Library, Freeman.
This colorfully illustrated coffee table book is loaded with computer graphics and a mathematically sophisticated yet informal discussion of the fourth dimension.

Banchoff T. F. (1990). Dimension. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 11-59). Washington, DC: National Academy Press.
This book presents visions of five strands of rich mathematics which can be used from early childhood through college study. The chapter includes activities promoting the visualization of dimensions and the discovery of patterns, especially those relating to the hypercube.

Petit, J. P. (1985). *The adventures of Archibald Higgins: Here's looking at Euclid*. Los Altos, CA: Kaufman.
The informal comic book style lures young readers while encouraging them to develop a questioning mind and visualize worlds of different dimensions.

Rucker, R. (1984). *The fourth dimension*: *Toward a geometry of higher reality*. Boston: Houghton-Mifflin.
This book is filled with simple diagrams and includes a lengthy discussion of time and the fourth dimension.

Weeks, J. R. (1985). *The shape of space: How to visualize surfaces and three-dimensional manifolds*. New York: Marcel Dekker.
This resource fills in many of the gaps left by current geometry and topology curricula. Clear definitions, novel exercises (with answers provided), and numerous illustrations bring sophisticated mathematics within the reach of high school students.

Problem Solutions

Fencing Problem

The land should be enclosed by a semi-circle to maximize the enclosed area. The diameter of the semi-circle is the natural barrier, and the fencing is used for the circumference. The area of the enclosed region is determined by noting that the circumference, C, of the semi-circle is C = 100 = πr. The radius of the semi-circle can be expressed as $^{100}/_{\pi} = r$. Thus the area of the semi-circle is $A = {}^{1}/_{2}\pi r^2 = {}^{1}/_{2}\pi({}^{100}/_{\pi})^2 = {}^{100^2}/_{2\pi} = {}^{100^2}/_{2\pi} \approx 3183$ square feet .

100!

One zero appears at the end of 100! for every product of 2 and 5 in the prime factorization of 100!. Thus, the task is to count the number of factors of 2 and of 5 in 100!. There are 20 multiples of 5 in the numbers 1-100, and the numbers 25, 50, 75, and 100 each have an additional factor of 5. This gives a total of 24 factors of 5 in 100! Clearly, there are at least 24 factors of 2 in 100!. Therefore, there are 24 zeros at the end of 100!.