Parameter Effects and Solving Linear Equations in Dynamic, Linked, Multiple Representation Environments

Pao-Ping Lin and Che-Jen Hsieh

The concept of function is one of the most important, but also one of the most complicated, concepts in the teaching and learning of mathematics. Solving linear equations is another important topic related to functions. The purposes of this paper are to describe computer environments which can be used in the exploration of parameter effects upon functions and their relationship to linear equation solving, to contrast the graphical representations of solving one-unknown linear equations and systems of two-unknown linear equations, and to consider some interesting questions that can be explored in the environments.

Multiple Representations of Functions

Many researchers (Janvier, 1987; Kaput, 1987, 1989; Lesh, Post, & Behr, 1987) have described distinct representational systems of functions that occur in mathematics learning and problem solving. These systems include formulas, tables, graphs, and verbal descriptions. The function graph is viewed as a key component in understanding the relationships among the various representational forms of a function. Creating computer environments which link multiple representations of functions has also been an attractive endeavor for many educators (Confrey, 1990; Lesh, 1987; Lin & Olive, 1992; Schwartz, Yerushalmy, & Harvey, 1990). Thomas (1975) investigated students' acquisition of the concept of function and hypothesized that students acquire the concept in three stages: a function as a process of assignment, a function as a translation between different representational settings, and a function as an object that can be manipulated.

One of the best ways to conceptualize a function as an object that can be operated on is through the manipulation of its graph. The objectification of functions was considered by Lin and Olive (1992) in their design of dynamic, linked, multiple representation environments, built on

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the *Geometer's Sketchpad* [*GSP*] (Jackiw, 1991) using a Macintosh computer. The environments described in this paper are based on these same considerations. Users can operate on the graph of a function in two ways, through the transformation approach and the formula-graph approach. In the transformation approach, variations of the graph are shown dynamically on the computer screen. The function graph and the referencing coordinate system can vary independently of each other. By using the mouse to drag specified points on the screen, the function graph can be translated, stretched, or shrunk. The coordinate axes can be varied by changing either the unit length, the position of the origin, or the angle of the x and y-axes. The function formula displayed on the screen varies dynamically as changes to the graph or referencing coordinate system are made. In the formula-graph approach the user employs a control device to change the coefficients of a function displayed on the screen in a general form (e.g., $y = ax^2 + bx + c$). The user may also alter the referencing coordinate system. The variations of the function formula and the referencing coordinate system can be done inde-
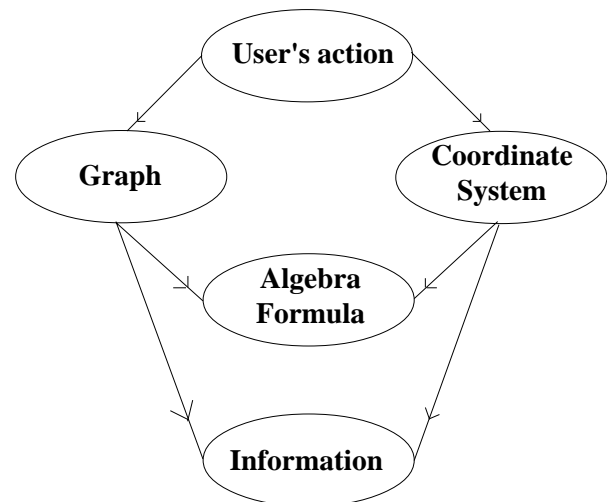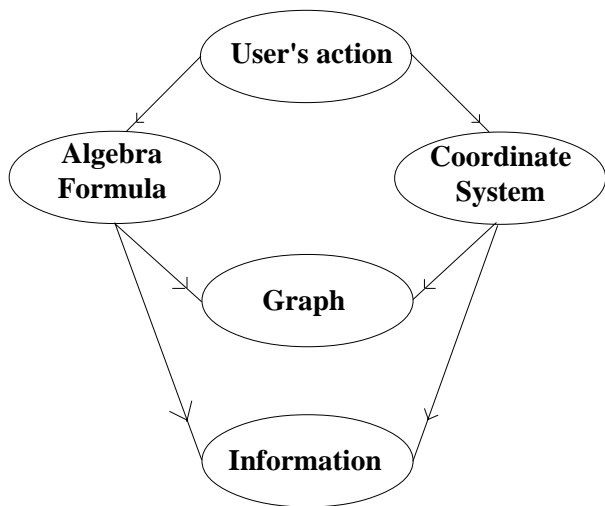

Figure 1: The transformation approach.

pendently of each other. Both affect the function graph in ways that are dynamically displayed on the screen.

The transformation approach and the formula-graph approach offer valuable improvements over the more common parameter approach, in which the user specifies coefficient values for function formulas and the computer draws the resulting graphs on the same coordinate system. In both the transformation approach and the formula-graph approach, the dynamic display capabilities of *GSP*

Figure 2: The formula-graph approach.

to $y = f(x) © m$, and $y = g(x) © m$, where $©$ is the operation and m is a constant or function. These functional operations, which are the valid equation-solving operations applied to both sides of an equation, preserve the solutions of the original equation, though extra solutions may be obtained. Actually, when considering polynomial functions, these operations can be viewed as combined changes of the coefficients of the original functions $f(x)$ and $g(x)$. For example, adding $x^2 + x + 2$ to $x + 2$ changes the coefficients of $x + 2$ from 1 for x and 2 for the constant, to 1 for x^2, 2 for x and 4 for the constant (i.e., $x^2 + 2x + 4$). This leads to the consideration of the effects of operating on a single term as a variation of only the coefficient of that term. Combinations of these coefficient variations are equivalent to more complicated functional operations. We will concentrate on simple operations on polynomial functions in this paper.

Solving systems of linear equations can also be represented graphically. The process of Gaussian elimination or row operations on a matrix can be illustrated by dynamic graphic variations. These solution processes can also be thought of as combinations of coefficient variations.

Based on the framework for the formula-graph approach on which Lin and Olive (1992) built their multiple representation environments, we have constructed several environments for the exploration of parameter effects on linear and quadratic functions, and for the use of graphical representations to illustrate the process of solving linear equations. Dynamic, linked, multiple representations of function formulas, graphs, and tables are the focus of these environments as well.
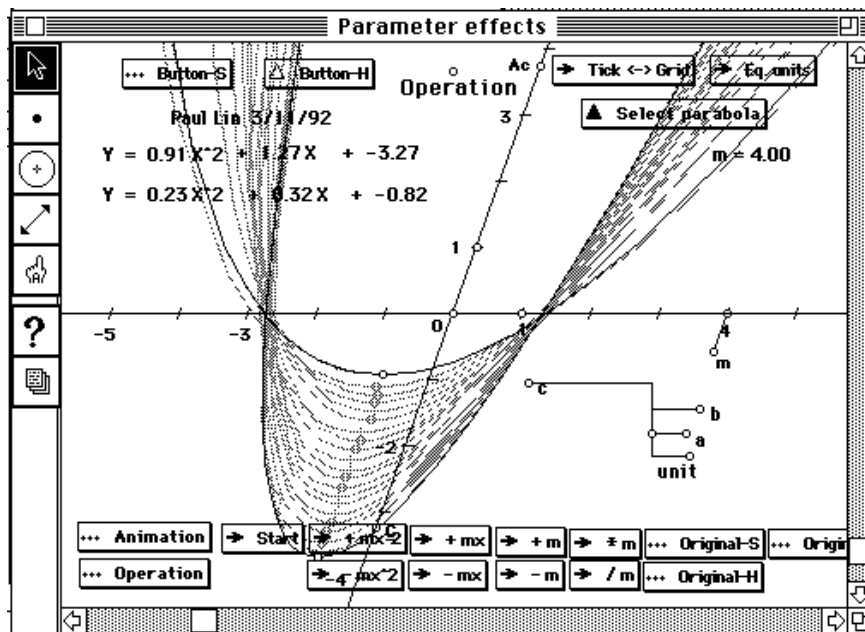
offer users a better view of the process of variation. The result is an improved visual and intuitive understanding of the links between a function's graph, algebraic formula, and referencing coordinate system. Also, the transformation approach gives the user an opportunity to explore how graphical manipulations can affect a function's formula. Manipulating graphical representations of functions helps to develop a sense of functions as objects that can be operated on. The relationships of the components in both the transformation and the formula-graph approaches are shown in Figure 1 and Figure 2.

The information components in these two figures are the data related to the environments, such as point coordinates, segment measures, and line slopes. The relationships are dynamic and linked; that is, as the user acts on either of the "parent" components of the environments, the effects on the three "children" components are displayed continuously and immediately.

Solving linear equations and parameter effects

Graphical representations of one-variable polynomial equation-solving processes are closely related to the effects of varying parameter values. When using graphs to illustrate the process of solving the equation $f(x) = g(x)$, the graphs of the two functions $y = f(x)$ and $y = g(x)$ are drawn first. The solutions to $f(x) = g(x)$ are the x-values of any points where the two graphs intersect. Adding or subtracting a function on both sides of $f(x) = g(x)$, or multiplying or dividing both sides of the equation by a constant function will change the two functions

Figure 3: Dividing a function by m on an oblique coordinate system in the Parameter effects environment.

Description of the Environments

The *Geometer's Sketchpad* is a dynamic, geometric construction kit which takes full advantage of the mouse interface of the Macintosh computer. By clicking and dragging tool icons, the user can construct points, lines, line segments, rays, or circles. Geometric figures can be constructed simply by connecting their components or by using the Construct menu. These figures can be dynamically transformed by simply dragging one or more of their component parts. All geometric relations are preserved during these changes. Measurements of segment lengths and arc lengths, areas, perimeters, and angles can be obtained and used for calculations. Measurements and calculations change as figures are transformed. Because *GSP* software was designed for the purpose of geometric study, it does not provide coordinate system and function graphing capabilities. We developed analytic coordinate systems and function graphs based on the geometric construction functions *GSP* (version 2.0) provided.

Three environments will be described: the parameter effects environment for linear and quadratic functions, an equation-solving environment for one-variable linear equations, and an equation-solving environment for systems of two-variable linear equations. The power of these environments lies in their manipulatable, dynamic, linked, and multiple representation properties. It is difficult to describe these properties without using the computer. With the provided figures and the reader's imagination, we hope that the following explanation and the examples can give readers an overview of the design of these environments and an appreciation of their power.
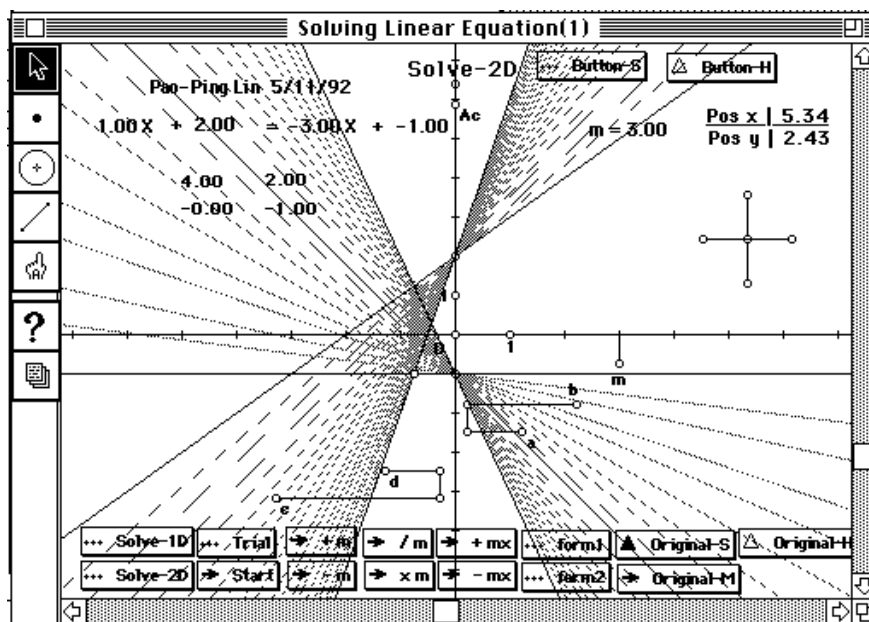
There are basically three different groups of elements in the environments— the coordinate system, the formulas, and the graphs. Each group has its own control devices, which can be points, vectors, or buttons. For convenience, all of the objects or sets of objects on the screen can be hidden or moved.

The Coordinate System

All three environments have "variable" coordinate systems (Figure 3). The user can drag the unit point on either axis to rescale it. A button named **EQ. units** can make the *y*-unit the same as *x*-unit. The user can choose to display grid lines instead of tick marks. Because the angle formed by the coordinate axes is controlled by dragging point Ac on the *y*-axis, the coordinate system can be rectangular or oblique. By dragging the origin, the user can translate the whole coordinate system. All related graphs and information displays are adjusted continually as changes are made. For example, the tick marks adjust themselves according to any changes to the unit lengths. However, any point constructed independently of the coordinate system will not move as the coordinate system is modified. Only the information about its position will be adjusted.

The Formulas

The environments contain either one or two sets of vectors which control the coefficients of the function equations. The letters $(a, b, c...)$ denote the end points of the vectors (Figure 3, 4). In the vector set, vector lengths are determined by the attached unit segment or by the unit length of the coordinate system (Figure 4). Vector signs are established by the position of the end points: those to the right have positive signs, those to the left have negative signs. The user can control the values of the coefficients a, b, c,.. in the function formulas by horizontally dragging the endpoints of the corresponding vectors. Modifying the vectors causes a simultaneous change in the function equations. The corresponding graphs will also change immediately in the parameter effect environment. In the equation-solving environments, coefficient modifications are held until the user **STARTS** the equation-solving process (Figure 4).

The Graphs

The graphs are drawn according to the function formula and the coordinate system. In the parameter effects environment (Figure 3), the function formula

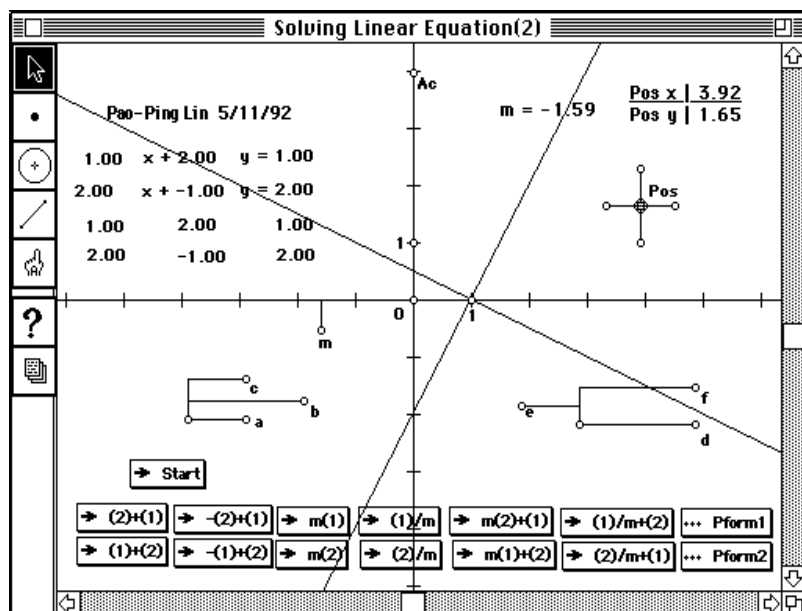Figure 4: Solving linear equations in the Solve-2d mode.

Figure 5: Solving a system of linear equations.

can have the form $y = bx + c$ or $y = ax^2 + bx + c$. In the one-unknown equation solving environments (Figure 4), the graphs are determined by the functions on each side of the equation. They may be constant, linear, or quadratic functions. In the two-unknowns equation-solving environment (Figure 5), the function formulas must both be linear. The graphs are the two lines corresponding to the two equations, drawn according to the results of the equation-solving process.

Buttons for the Functional and Equation-solving Operations

In each environment, the value of the constant m corresponds to the position of a segment attached to the x-axis. This constant can be changed by dragging the segment along the axis. Functional and single variable equation-solving operations are controlled by buttons that represent addition, subtraction, multiplication, and division by the value of m, as well as addition and subtraction of mx or mx^2 (see Figures 3 and 4). In the two-variable linear equation-solving environment (Figure 5), buttons imitate matrix row operations on the coefficients of the function formulas. For example, when the **-(2) + (1)** button is activated, the second equation is multiplied by -1 and the product is added to the first equation.

To use the equation-solving environments, the user first has to set the desired formula coefficients, then to initialize the equation with the **START** button. After this has been done, sequences of operations on the equation can be performed. The resulting matrix of coefficients is displayed under the original equation (Figures 4 and 5). In all the environments, the user can activate the **ORIGINAL-M** button to keep the current graph before an operation, the **ORIGINAL-S** button to show it after the operation, and the **ORIGINAL-H** button to hide it. The calculated results for any related values have at most three decimal places of accuracy.

Application of the Environments

Parameter Effects Environment

Two modes are provided in this environment for studying the effects of varying the coefficients upon the function graph. In the animation mode, the user can determine the a, b, or c values or use the **ANIMATE** buttons to let the computer continuously change the values of the coefficients. As the values of the coefficients are changing, the graph varies correspondingly. The basic task for the user is to figure out what effects variations in the values of a, b, and c have on the graphs. How can these effects be explained? Is the explanation unique? With the aid of *GSP*'s Trace locus command, the path taken by selected points can be tracked as they move across the screen, allowing new insights into how the function graph is changing. For example, tracing the whole graph when varying the value of b (Figure 6), we can see that the vertex point Y seems to trace a parabola.
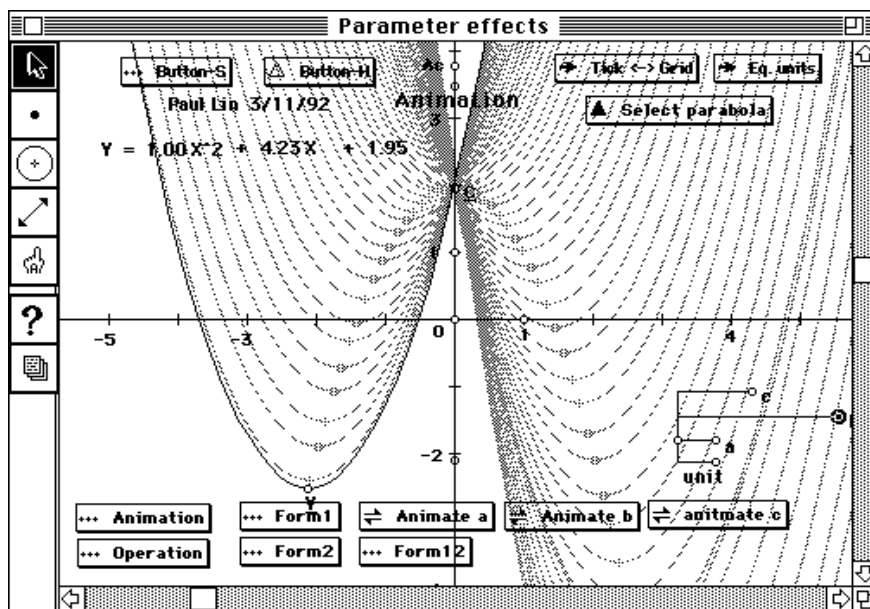

Figure 6: The effects of varying b in the equation $y = ax^2 + bx + c$.

What is the equation for that parabola? How can we explain it visually or algebraically? Similarly, if we take any point on the parabola and trace its locus as the formula coefficients vary, what is the result? What is the equation for this graph? When a equals zero, what is the locus of a point on this line as b or c varies? Can we find its formula? When the value of b is allowed to vary, will any point remain fixed? Such a fixed point is called a "screw-eye" (Goldenberg, 1991) and can be explored easily using this environment (see Figure 6).

Students can also study the effects of varying a, b, and c in the formula $y' = a(x + b)^2 + c$ and contrast the relationships in this form with those in the formula written as $y = ax^2 + bx + c$. The user can obtain the graph of $y = ax^2 + bx + c$, or the graph of $y' = a(k+x)^2 + c$, or both (Figure 8). The dynamic way in which the continuous variation of the a, b, and c parameters can be explored is unique to our environments.

The second mode provided in the parameter effects environment is the operation mode. Here the user can operate on the function formula by adding or subtracting a constant, linear term, or quadratic term, or by multiplying or dividing by a constant term. On the screen, the graph is changed gradually from the original to the graph resulting from the operations on the function formula. Kaput (1989) used terms such as "tilts" (slope changes) and "bends" (multiplications by linear expressions) to describe the
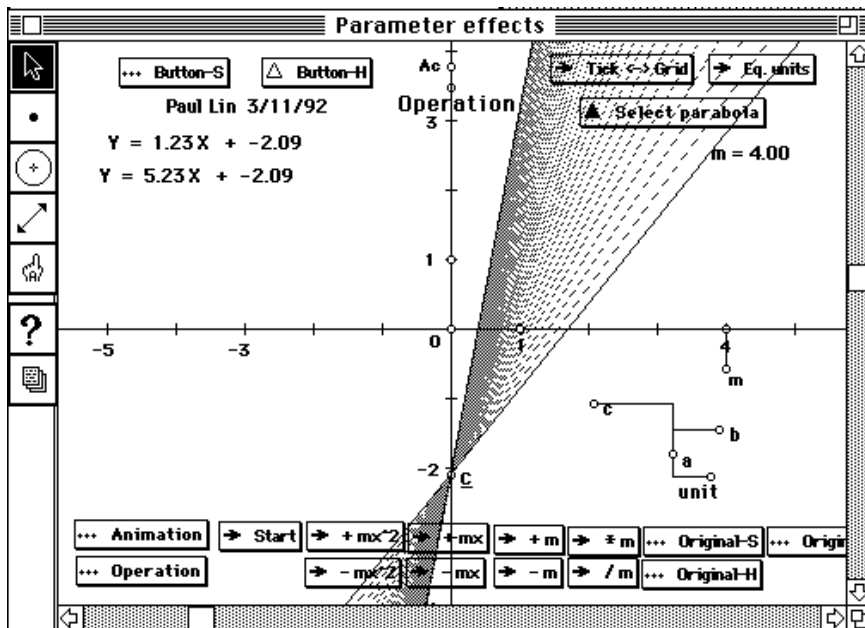

Figure 7: The effect of varying b in the equation $y = ax^2 + bx + c$ with $a = 0$.

process of building up a function equation to match a given function graph. The operation mode of the parameter effects environment can be used to describe dynamically the tilting and bending processes, as well as others described by Kaput. Figure 7 shows the slope changing by adding to the linear term when $a = 0$. Figure 9 shows the graph changing process of adding a squared term ($4x^2$ in this case) to a linear function by gradually increasing the value of the coefficient a to 4.

The animation mode and the operation mode provide two different routes to the same result. Varying coefficients can be seen as a sequence of continuous simple function operations. Conversely, operations on functions can be thought of as a coefficient varying process, too. This environment provides students with a unique tool for observing the links between algebraic expressions, graphs, and coordinate systems.

Solving one-variable linear equations

The linear equation-solving environment uses three modes to operate on equations of the form $ax + b = cx + d$. In the trial mode (Figure 10), the value of x corresponds to the position of the point X on the number line. The value of each side of the equation, referred to as "left" and "right," is determined by substituting the value of x. The value of x is dynamic, as are the values of the expressions containing x. The value of x which
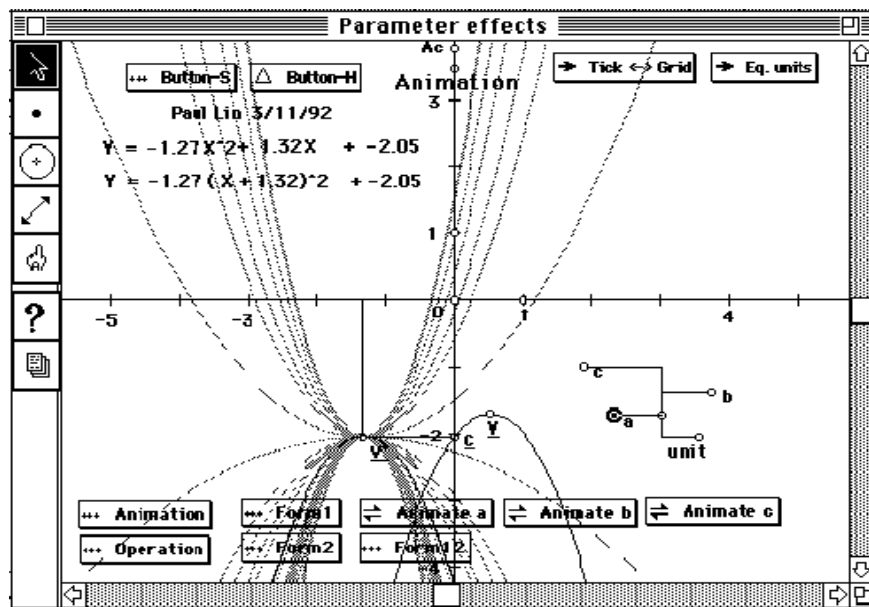

Figure 8: Comparison of $y = ax^2 + bx + c$ and $y = a(x + b)^2 + c$.

causes "left" and "right" to converge to the same value is the solution to the equation.

In the 1-D mode (Figure 11), a vector addition method is used. This mode is generalized from representations of equation-solving processes such as that shown in Figure 12. In the 1-D mode, ax, b, cx, and d are represented by vectors; the left hand side of the equation is represented by the sum $ax + b$, and the right hand side by the sum $cx + d$. The ticks on the ax and cx vectors correspond to the absolute value of x. As buttons are used to transform the equations by adding, subtracting, multiplying, and dividing, the vectors representing $ax + b$ and $cx + d$ are always equal. The solution has been found when the user obtains the coefficients $a = 1$, $b = 0$, $c = 0$, $d = 1$ (or $a = 0$, $b = 1$, $c = 1$, $d = 0$). The **FORM1** and **FORM2** buttons can be used to switch between the matrix or equation representation of the results.

In both the trial and 1-D modes, we can ask students how the direction of the vectors is determined. Is the solution positive or negative? How can we change the coefficients to get the same solution, a negative solution, or a positive solution? In what direction will the "left" and "right" vectors move when we change x in the trial mode? For what values of x will the "left" vector end on 0 or swing in the negative or positive direction? When x moves, will the two vectors always move in the same direction?
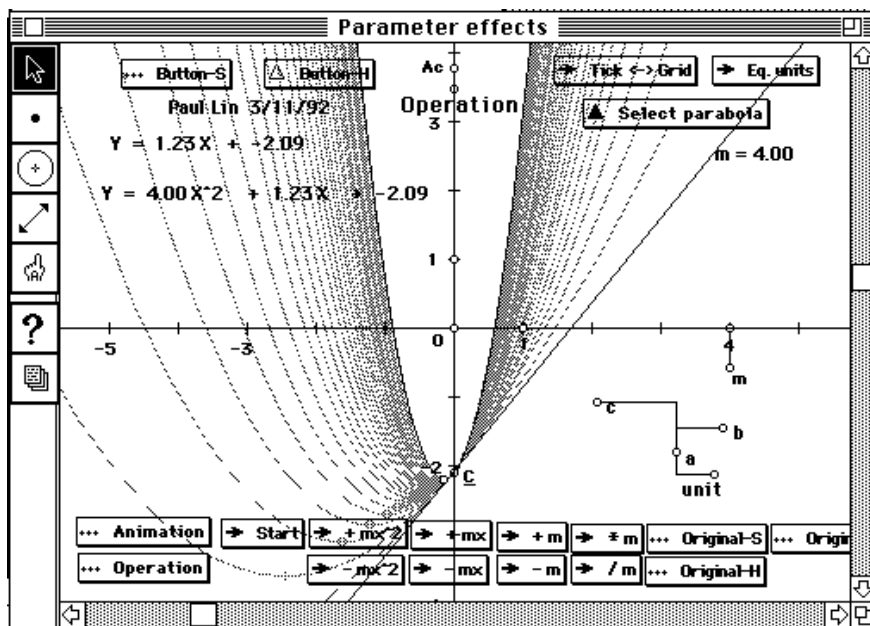

Figure 9: Bending a line by adding $4x^2$ to $y = 1.23x - 2.09$.

In the 2-D mode (Figure 4), the two lines $y = ax + b$ and $y = cx + d$ are drawn. The x-value of their intersection is the solution to the equation $ax + b = cx + d$. In both the 1-D and 2-D modes, the user can study the relationship between algebraic formulas and their graphical representations. These different representations give students ways to visualize equation transformations and help them focus on the processes as well as the products of the equation-solving process. The environments also allow students to observe the invariance of solutions under the equation solving transformations (Lesh 1987).
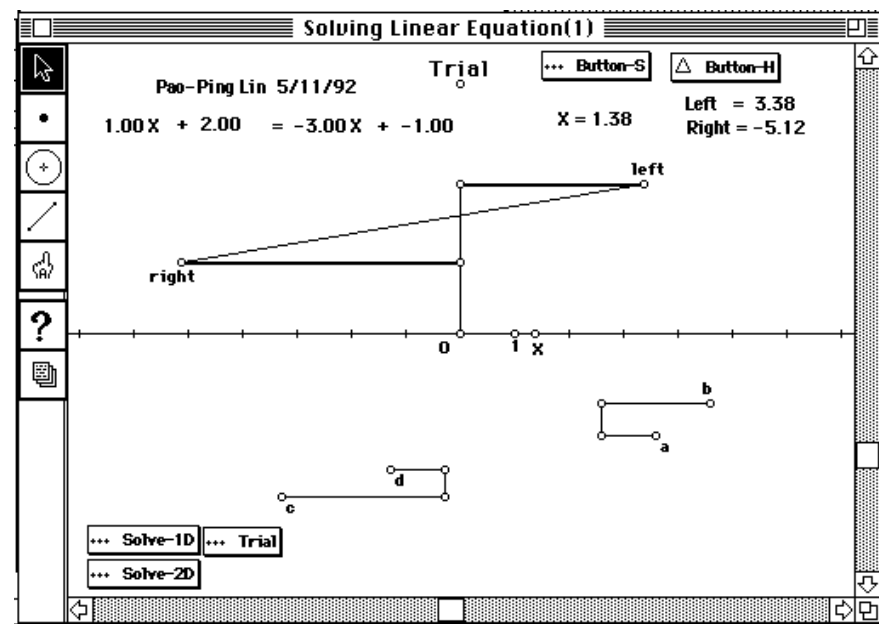

Figure 10: Solving linear equations in the Trial mode.

Solving two-variable linear equations

Two modes are provided for solving a system of two linear equations in this environment. In the Solve1 mode (Figure 13), the user can manipulate the graphs of the two equations in order to study the relationship between the function graphs and their corresponding equations. Dragging point C1 or C2 changes the slope of the two lines, thereby changing the coefficients in the function equations; dragging point S1 or S2 is the same as multiplying or dividing the equations by a constant; dragging the intersection point translates the two lines at the same time. Users can set up and solve the system of equations by manipulating the graphs, then contrasting these operations with the button operations in Solve2 mode.
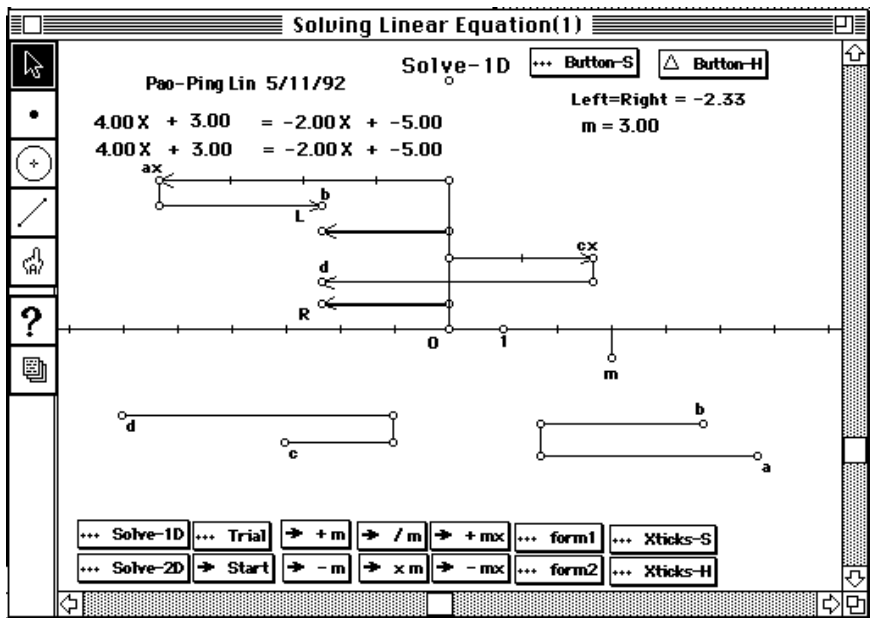
Figure 11: Solving linear equations in the Solve-1D mode.

In the Solve2 mode, the user can set up the equations by dragging the vector endpoints which control the coefficients of the equations $ax + by = c$ and $dx + ey = f$. This environment provides a graphical illustration for the equation-solving process and the relationship between function equations and graphs. The equation manipulation operations, controlled by button activation, are actually row operations on the matrix of system coefficients (Figure 14). The graphical representation of the final solution of a system of equations is two lines, parallel to the two axes in the coordinate system.

Button operations in this environment can help the user make distinctions between functional operations and equation solving transformations, which may confuse many students. Functional operations change the coefficients of other terms, but the coefficient of y never changes; that is, the sum of two functions $y = f(x)$ and $y = g(x)$ is $y = f(x) + g(x)$. But in the two-variable equation solving transformation, the coefficient of y is changed. The result of adding the first equation in the system, $y = f(x)$, to the second equation, $y = g(x)$, is $2y = f(x) + g(x)$. This environment provides an opportunity for students to compare the operations.

Rotation or Stretching: Pitfalls of Visual Learning

Every user should know that in a dynamic, visual learning environment such as *GSP*, the computer definitions of geometric objects may or may not be exactly the same as those in Euclidean geometry. For example, if A, B, and C are three collinear points, we know that line AB, line BC, and line AC all represent the same line in Euclidean geometry. In *GSP*, however, although the lines are coincident, we can still distinguish between them and the points on the lines may behave differently.

For a linear function, $y = bx + c$, we can define many coincident lines. In Figure 15, we define three lines which are the graphs of $y = 0.36x - 2.73$ (note that the correspondence between equation and lines is not one-to-one). Line 1 is defined by the slope b and the intercept c; line 2 is defined by the point $(-\%, 0)$ and the intercept c; line 3 is defined by the slope b and the point $(-\%, 0)$. The three lines are coincident on the line BC in Figure 15. When we add $-5x$ to this function, the three lines seem to behave the same way as they move to $y = -4.64$ and $x = -2.73$. However, when we choose points P1, P2, P3 on the lines L1, L2, L3, respectively, and trace their loci as the lines move to the new position, we can see that their behavior is actually very different. The loci of the three points show us that the lines either stretch vertically or horizontally, or "curve" in a hyperbolic route to the new destination. Actually, the corresponding points in the variation processes are determined by the definition given to the lines and the internal structure of *GSP* regarding points on a line. We need to consider first the transformation caused by adding mx to the function equation.

Generally speaking, when we talk about functional operations, we obtain new functions as results of the operations. For example, adding $y = 4x$ to $y = 1.23x - 2.09$, yields $y = 5.23x - 2.09$ (Figure 7). We tend to say that the variation from $y = 1.23x - 2.09$ to $y = 5.23x - 2.09$ is not a rotation but a vertical stretching. Since the addition is a functional operation, we need to consider the mapping of (x, y_1) and (x, y_2) in the original functions to $(x, y_1 + y_2)$ in the
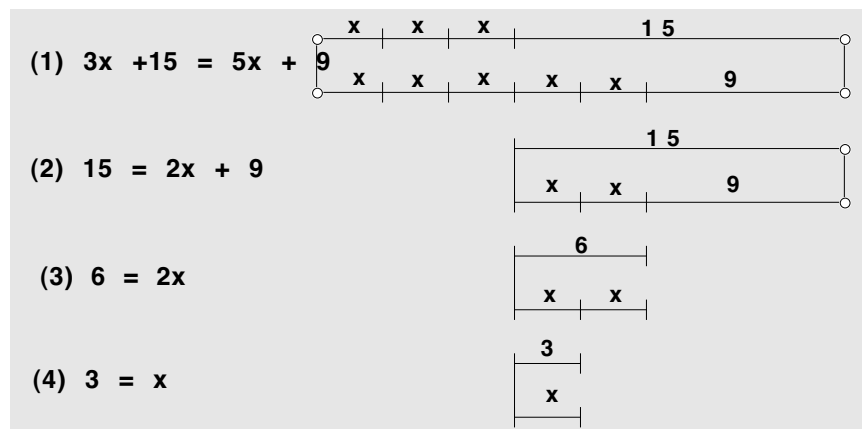

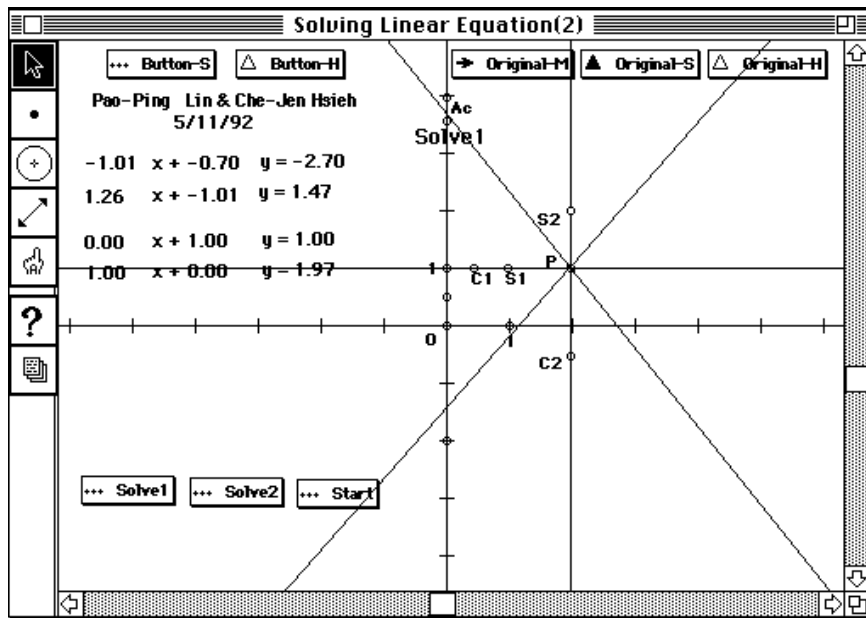Figure 12: Graphic illustration of a linear equation-solving process.

Figure 13: Solving system of linear equations by dragging the two lines to horizontal and vertical and moving points S1, S2 to unitize the coefficients in the Solve1 mode.

the coordinate system. We hope that these environments can provide new directions for the teaching and learning of functions.

Notes

resulting function. Since the behavior of the points on a line is determined by the internal structure of *GSP*, sometimes the function operation does not give us any clue about ways to interpret the variation process. For example, when solving linear equations $2x + 2y = 1$ and $2x - y = 2$, subtracting the first one from the second one, we obtain $2x + 2y = 1$ and $-3y = 1$ (Figure 14). What is the process of the transformation which maps equation $2x - y = 2$ to $-3y = 1$ and keep the $2x + 2y = 1$ unchanged? Is it a rotation of line $2x - y = 2$ to $-3y = 1$? Is it a stretching? Or is it neither a rotation nor a stretching? Visually, it is a rotation only if we consider a line as a whole. We leave these questions for the reader to consider.

Summary

In this paper we have described three manipulatable, dynamic, linked, multiple representation environments for studying parameter effects of linear and quadratic functions and solving linear equations. These environments are related directly to the mathematical topics currently taught in the high school. We have described the environments, designed to help students understand parameter effects through dynamic, graphical movement or change of the equation graph; the graphical interpretation of equation solving process; and relationships between function equations, function graphs and

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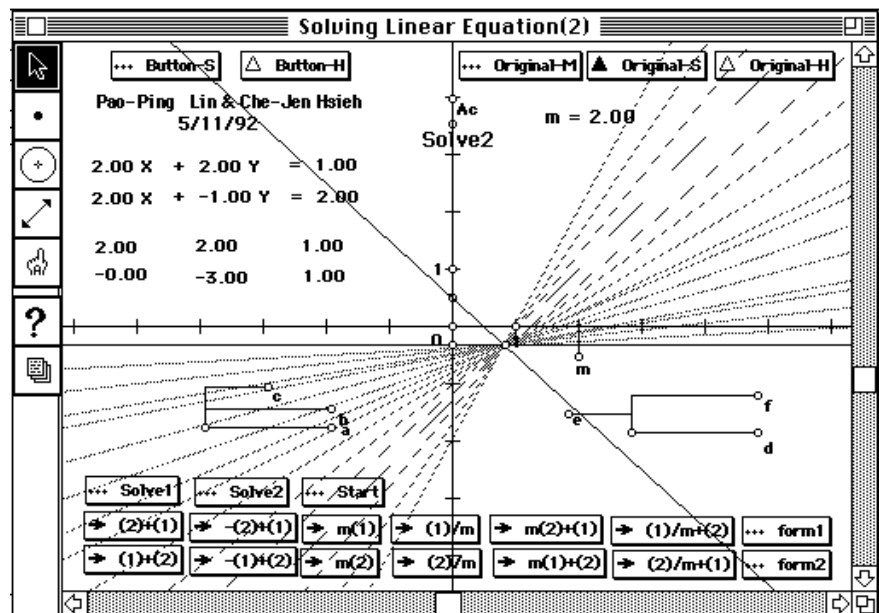
Figure 14: Solving the system of linear equations $2x + 2y = 1$ and $2x - y = 2$ by subtracting the first equation from the second.

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Figure 15: One of the pitfalls of visual learning.

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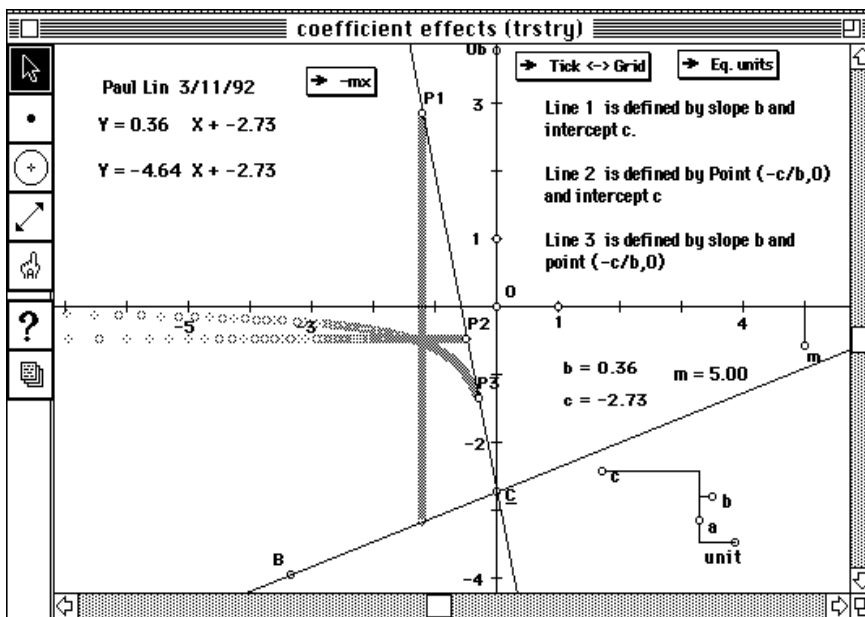
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