

Baked Beans and Spaghetti

Robert Aldred, Peter Hilton, Derek Holton, and Jean Pedersen

To Andrew, a budding young mathematician

In this paper we show that there is more than one way to skin a rabbit, by giving a couple of approaches to solving the problem: How many towers of all heights can be made if you have available a certain specified number of baked bean cans and spaghetti cans, but need not use them all in any one tower?

We show the relevance of this problem, and more especially the techniques used to solve it, for problem-solving in the classroom.

1. Introduction

This problem arose while one of us was working with some children with special needs. The researcher discovered that they enjoyed stacking cans, one on top of another, to make reasonably high towers. All the cans were the same size. For what follows we will assume that cans of a given type (e.g. baked beans) are indistinguishable from each other.

One of the mathematical tasks that the researcher injected into this activity was to find the number of different towers that could be built with no more than two baked bean cans and two cans of spaghetti. One of the students made progress toward a solution by recording the

number of towers. This number is just $\binom{4}{2} = 6$, which was what he got.

However, more than six towers can be made with two baked bean cans and two spaghetti cans if we allow the height of the tower to vary. For instance, there are two towers of height one, four towers of height two, six towers of height three, and six towers of height four. So altogether we can make 18 towers using no more than two each of two different kinds of cans. These 18 towers are given in Figure 1 using a representation that the student invented.

					B	S	B
		S	B	S	B	S	B
S	B	S	S	B	B	S	S
				B	S	B	S
S	B	S	B	B	S	S	B
B	S	S	S	B	B	B	S
B	B	B	S	S	S	B	B

Figure 1: Possible towers using no more than two each of two different kinds of cans.

In an attempt to generalize, the researcher then asked each of the other authors, “How many such towers can be made with k baked bean cans and k spaghetti cans?”. Well, that’s not quite true on two counts. First, because the others are all sophisticated mathematicians, he asked, “How many 0-1 sequences can be constructed using at most k zeros and at most k ones?”. Second, because he didn’t want to be laughed at for asking such an elementary question, he first looked up a few books on combinatorics to see if it was a standard exercise. When he couldn’t find it, he plucked up the courage to ask his colleagues.

Why don’t you try to solve the problem before reading further?

We all decided that the problem sounded interesting but suggested that in addition to the towers which actually had cans in them, it might be nicer mathematically to

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include the tower that has *no* cans. This is the tower omitted in Figure 1 but included as ‘?’ in Figure 2.

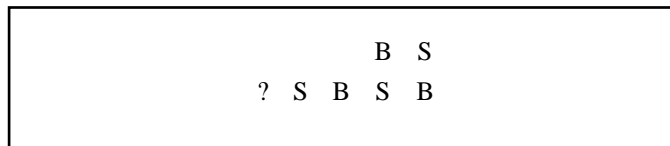


Figure 2: The five towers with one can of each type available. The tower with no cans is denoted ?.

We agreed, too, that it would save long-winded sentences later, if we called a tower of *h* cans an *h*-tower. Figure 1 clearly contains 1-towers, 2-towers, 3-towers and 4-towers but the 0-tower is omitted. In general, if we have *k* cans of each type, then *h* lies between 0 and $2k$. (You can see that we found the formulation in terms of towers and cans more appealing than the 0-1 sequence formulation.)

In the following sections we only describe explicitly the ‘finished product’. As with so much in mathematics, the final presentation does not reflect all our work. In our case, we first laboriously developed a recurrence relation to describe the quantity t_k , the number of towers which could be made with *k* cans of each kind available. In other words, we obtained a formula expressing t_k in terms of t_{k-1} . This formula involved some rather mysterious sums of sequences of binomial coefficients, and the summing of these sequences presented some difficulty. Eventually, we found a closed formula for t_k —and it was unexpectedly simple.

Now we begin the job of refining our methods to eliminate the mystery and to explain why the final formula took on the form it did. In Section 2 we describe what seems to us the most natural and easily understood argu-

ment which yields the formula $t_k = \binom{2k+2}{k+1} - 1$. The argument has a nice geometric flavor, too, involving the summing of the entries in a ‘fundamental rhombus’ of Pascal’s Triangle (see Figure 3). Then in Section 3 we ask ourselves—and answer—the inevitable question, “Why should the number of towers, t_k , be just one less than a certain binomial coefficient?” As was loudly proclaimed in Hilton, Pedersen, and Ross (1987), ‘In mathematics there are no accidents,’ so there must be some way to relate the various towers to selections of $(k+1)$ objects from an assembly of $(2k+2)$ objects. We describe such a way in Section 3. Actually, the material of Section 3 was developed before that of Section 2, since we already had the formula for t_k before we had the best argument for obtaining it.

Then, in Section 4, we show how our approach and our results may be generalized in certain obvious ways. We do not at all favor generalization for its own sake. But where

the generalizations answer questions which are quite as natural as their originals, and where the proofs are scarcely, if at all, more difficult, then those generalizations should be presented and analyzed.

We close with a section in which we explain how, in our view, this case history of a mathematical adventure exemplifies the typical strategies, and difficulties, involved in finding and solving promising problems.

2. Pascal’s Proof

One would expect that a problem concerning the number of towers that could be made, having available an equal number of baked bean cans and spaghetti cans, should have something to do with the binomial coefficients and, consequently, that it might be helpful to look at Pascal’s Triangle. Not only is this the case but, surprisingly, this approach yields the answer to our question almost immediately.

Since we want to preserve the symmetry between the roles of baked beans and spaghetti, in this section we use an extended notation for the binomial coefficients. Instead

of the usual $\binom{n}{p}$ we will write $\binom{n}{p,q}$, where $p+q=n$.

Now notice that in the expansion of $(B+S)^n$ the binomial coefficient $\binom{n}{p,q}$ is the coefficient of the term

$B^p S^q$ so that it simply counts the number of ‘words’ (of length *n*) with *p* B’s and *q* S’s. Each of these ‘words’ corresponds in an obvious way to an *n*-tower. Hence we

see that $\binom{n}{p,q}$ really counts the number of *n*-towers built with exactly *p* baked bean cans and *q* spaghetti cans¹.

In Figure 3 we will call (p,q) the *coordinates* of the entry $\binom{n}{p,q}$ and, as in coordinate geometry, identify

points with their coordinates. Now notice that if you want to know the number of different towers that could be made with *k* baked bean cans and *k* spaghetti cans available then all you need to do is add together all the entries contained within the parallelogram (rhombus), including its boundary, having vertices $(0,0)$, $(k,0)$, $(0,k)$ and (k,k) (see Figure 3). This might seem a formidable task but, in fact, it can be done in $(k+2)$ straightforward steps, yielding a delightfully simple answer.

First apply the Christmas Stocking Theorem,

$$\sum_{j=0}^k \binom{r+j}{r,j} = \binom{r+k+1}{r+1,k}, \quad (1.1)$$

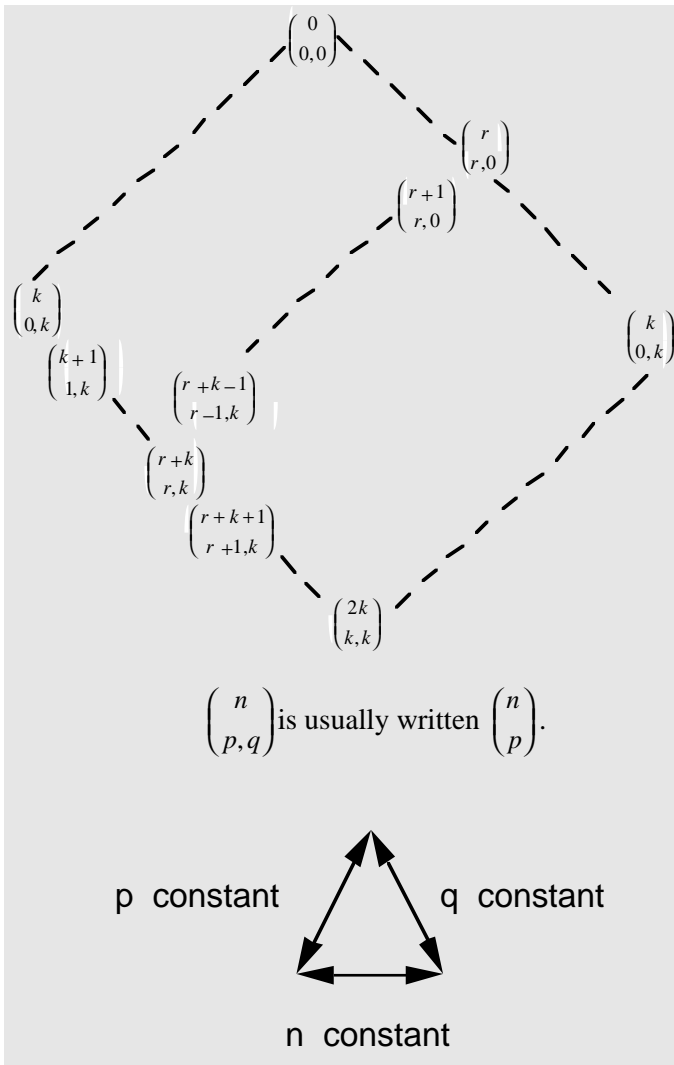


Figure 3: The entries contained within the parallelogram.

to the successive rows of the parallelogram in the direction in which the coordinate p is constant (that is, for $r = 0, 1, \dots, k$).

In Figure 4, where a typical application of (1.1) is shown enclosed in the classical Christmas Stocking shape, the underlined entries give the row sums of all $(k + 1)$ rows of the parallelogram in the direction where the coordinate p is constant. Thus it is the sum of the underlined numbers that provides the answer to our question. With a little reflection, we see that we need just one more application of the Christmas Stocking Theorem, this time in its dual form, and summing to $(k + 1)$ instead of k ,

$$\sum_{j=0}^{k+1} \binom{s+j}{j, s} = \binom{s+k+2}{k+1, s+1}. \quad (1.2)$$

Then (1.2), with $s = k$, tells us that the sum of all the underlined entries in Figure 4 plus the extra entry $\binom{k}{0, k}$ is $\binom{2k+2}{k+1, k+1}$, as shown in Figure 5.

Since, of course, $\binom{k}{0, k} = 1$, this shows that the number of towers (including the 0-tower) which can be made with k baked bean cans and k spaghetti cans available is

$$\binom{2k+2}{k+1, k+1} - 1.$$

3. Zeros and Ones

But at this stage, although there was jubilation that we now had a really revealing solution of the problem (remember it was not our first), we nevertheless had the feeling that the solution was not quite complete. There

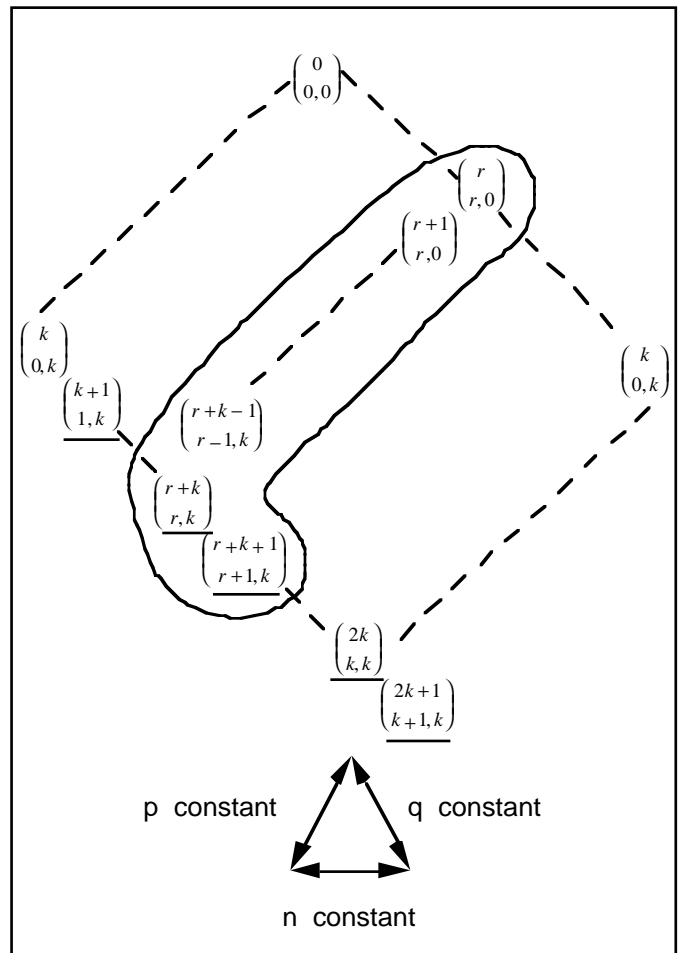


Figure 4: Application of the Christmas Stocking Theorem.

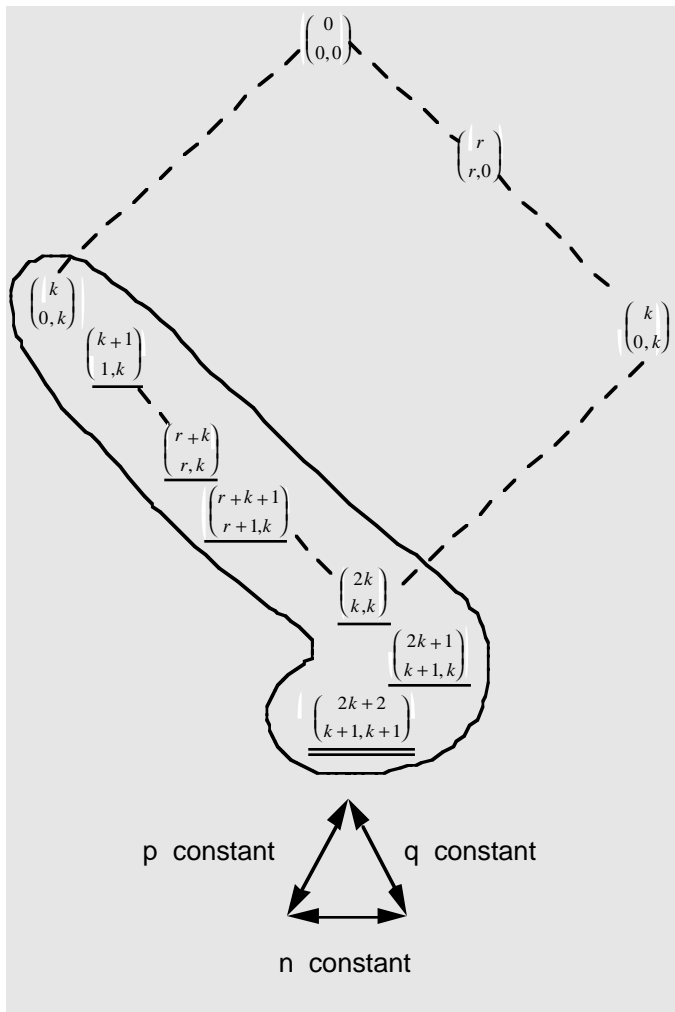


Figure 5: Sum the singly underlined entries to get the number of towers that can be made with k baked bean cans and k spaghetti cans.

were two worries really. The first and minor one was the presence of the -1 in the answer. Surely there was some reason for this. However, it was more perplexing that, although the number of towers that could be made having k baked bean cans and k spaghetti cans available had turned out to be essentially a single binomial coefficient, we had no explanation of this remarkable fact—only a mathematical proof! There just *had* to be a way of setting up the problem so that it became clear why a binomial coefficient

—and, in particular, the binomial coefficient $\binom{2k+2}{k+1}$ —was involved. Recall our conviction that “in mathematics there are no accidents” (Hilton, Pedersen, & Ross, 1984).

The sort of thing we had in mind was the method used to find the number of solutions of the equation $x + y = 5$, where x and y are non-negative integers. This number can be found by using sequences made up of five 1’s and one 0. All such strings give a solution to the equation, using the convention that 111110 represents $x = 5, y = 0$; 111101

represents $x = 4$ and $y = 1$; 111011 represents $x = 3$ and $y = 2$; and so on. There is a one-to-one correspondence between the sequences and the solutions. Hence we get the binomial count $\binom{6}{1}$ for the number of solutions. This method generalizes for equations with three unknowns. For instance, the sequence 11101011 represents the non-negative solution $x = 3, y = 1, z = 2$ in the equation $x + y + z = 6$. The total number of such solutions is $\binom{8}{2}$. (For more details and extensions of this, see Bose & Manvel, 1984, p.47ff.)

The question that we then asked ourselves was, “How can we show, just using 0-1 sequences, that the number of 0-1 sequences with k zeros and k ones available is just one less than $\binom{2k+2}{k+1}$, that is, the number of 0-1 sequences using exactly $(k+1)$ zeros and $(k+1)$ ones?” The answer we give here did not fall out immediately when we asked the question. What we hide below is at least a week of on and off again deliberation.

Actually, our first stab at a solution, which involved a recurrence relation, had produced $t_k = 2\binom{2k+1}{k+1} - 1$. When we started looking for 0-1 sequences here we took $(k+1)$ ones and k zeros and tried to use the first one that appeared as a marker. For instance, take the 0-1 sequence 011010011 with five ones and four zeros (here $k = 4$). If you throw away everything up to and including the first 1, you get 1010011. If we let 1 be a spaghetti can and 0 a baked bean can, this sequence could stand for the tower SBSBBSS. There was a problem, though. If we kept everything after the first 1, we had always to have k 1’s in the counted sequence. This was equivalent to having k baked bean cans in every tower. How could we count towers with fewer than k baked bean cans?

So we went back to $t_k = \binom{2k+2}{k+1} - 1$ and sequences with $(k+1)$ zeros and $(k+1)$ ones. Someone had the bright idea of keeping everything past the first one and before the last zero. So 001**1010101** would be truncated to 10101 and would represent the tower SBSBS. This notation looked good because it enabled us to have a variable number of zeros or ones, up to a maximum of k each.

Could we establish a one-to-one correspondence between these truncated sequences and the towers? If this was to be possible, somehow we would have to have an extra sequence because of the -1 in $\binom{2k+2}{k+1} - 1$. To see how this comes about, consider the example below (Figure 6) with $k = 2$.

If you check all this carefully you will see that in fact every tower appears, once and only once. Further, we *do* have an extra sequence—the very last one in our list, which has no subsequence (not even an empty one like 001011 has) between the first one and the last zero!

It was all downhill from there. We now only had to show in general that there was a one-to-one correspondence between the set of 0-1 sequences with $(k + 1)$ zeros and $(k + 1)$ ones, with the sequence 00...011...1 excluded, and the set of towers made from k baked bean cans and k spaghetti cans. We leave it to you to fill in the details.

111000	SSBB
110100	SBSB
110010	SBBS
110001	SBB
101100	BSSB
101010	BSBS
101001	BSB
100110	BBSS
100101	BBS
100011	BB
011100	SSB
011010	SBS
011001	SB
010110	BSS
010101	BS
010011	B
001110	SS
001101	S
001011	?
.....	
000111	!!

Figure 6: The relationship between truncated sequences and towers with $k = 2$.

4. Further Generalizations

There are two obvious directions for further generalization. In the first place, why insist that there are an *equal number* of baked bean and spaghetti cans? In the second place, why insist that there are only *two* types of can?

We will be content to discuss the first generalization, leaving the reader to ponder the second. We claim that the treatment by means of Pascal's Triangle (Section 2) generalizes almost effortlessly. We suppose there are k baked bean cans and l spaghetti cans available and, by an

identical argument to that used in Section 2 (see Figure 7, which should be compared with Figure 5), we conclude that the sum of the entries in the parallelogram with vertices at $(0,0)$, $(k,0)$, $(0,l)$ and (k,l) is

$$\binom{k+l+2}{k+1, l+1}^{-1}. \quad (4.1)$$

This then is the number of towers (including the 0-tower) which can be made with k baked bean cans and l spaghetti cans available.

But the argument given in Section 3 also generalizes immediately to this case. For given any legitimate tower (or string)

$$\boxed{\text{some string of zeros and ones}} \quad (4.2)$$

drawn from k 0's and l 1's, we may encase (4.2) in a longer string, corresponding to a $(k + l + 2)$ -tower

$$0 \dots 0 1 \boxed{\text{same string of zeros and ones}} 0 1 \dots 1, \quad (4.3)$$

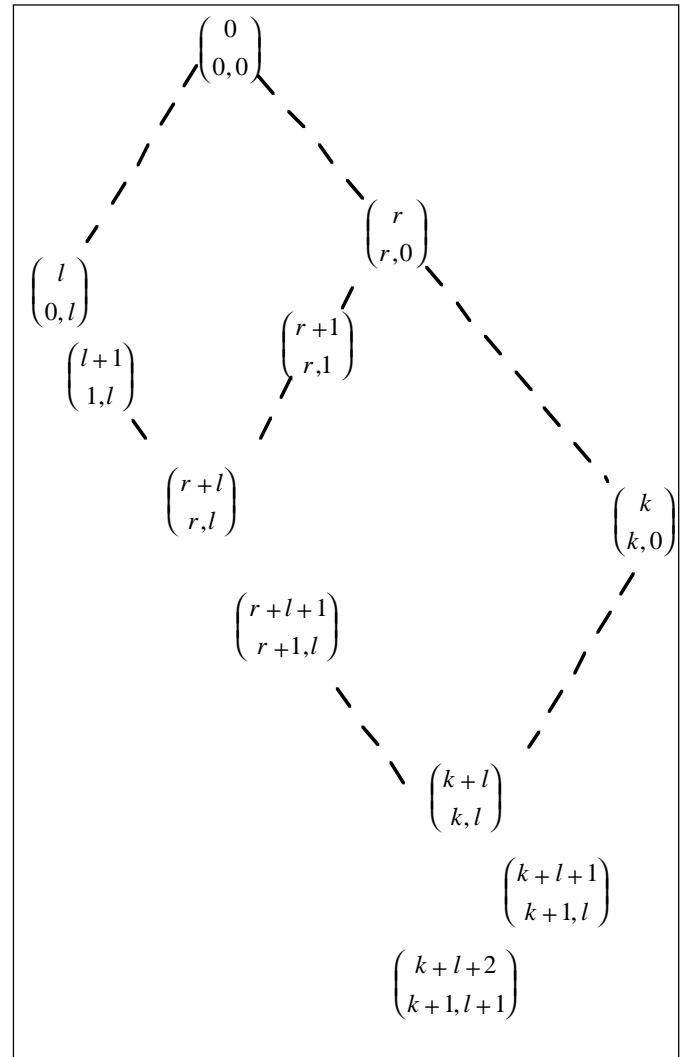


Figure 7: Parallelogram with vertices at $(0,0)$, $(k,0)$, $(0,l)$, and (k,l) .

consisting of $(k + 1)$ 0's and $(l + 1)$ 1's, where we exhaust the available 0's at the beginning and the available 1's at the end. It is again clear that every $(k + l + 2)$ -string made from $(k + 1)$ zeros and $(l + 1)$ ones occurs once and once only in (4.3), with the exception of the string $0 \dots 0 1 \dots 1$, which does not occur at all. Thus, again, the number of

towers like (4.2) is $\binom{k+l+2}{k+1, l+1} - 1$.

One further generalization clearly deserves mention. We found a method (via the Christmas Stocking Theorem) in Section 2 for summing the entries in the parallelogram with vertices $(0, 0), (k, 0), (0, l), (k, l)$. Suppose now we consider *any* parallelogram in Pascal's Triangle with sides parallel to the 'coordinate axes', i.e. with vertices at $(a, b), (a+k, b), (a, b+l), (a+k, b+l)$. Then, by looking at Figure 8 and remembering the answer we got above (when the vertices were at $(0, 0), (k, 0), (0, l), (k, l)$), you should see that the sum of the entries in the given parallelogram is

$$\binom{a+b+k+l+2}{a+k+1, b+l+1} - \binom{a+b+k+1}{a+k+1, b} - \binom{a+b+l+1}{a, b+l+1} + \binom{a+b}{a, b} \tag{4.4}$$

When you are trying to see how we got this, don't forget that our parallelograms include their boundaries. Try verifying (4.4) in a particular but not special case, say, with $a = 3, b = 2, k = 2,$ and $l = 1$. You may also be intrigued to see just how (4.4) generalizes (4.1).

The coordinates of the four binomial coefficients entering into formula (4.4) are exhibited in Figure 8, with appropriate signs. The diagram is a useful mnemonic for remembering the formula—and should enable you to create a sensation at your next party!

5. The Mathematical Process

We want to emphasize here the processes that were at work in solving this problem. They are all processes that mathematicians use every day. They are also the processes that are present in any good problem-solving exercise. We list them here because of their relevance to current mathematics teaching (see National Council of Teachers of Mathematics, 1989).

The first step was to *find a problem*. This created no difficulty at all. The problem just popped out of some innocent work with some elementary school children. If you are alert, non-trivial mathematical problems come up all over the place. It takes very little thought to invent worthwhile problems of your own. For example, see Holton (1990), where a whole series of problems based on the simple square are invented.

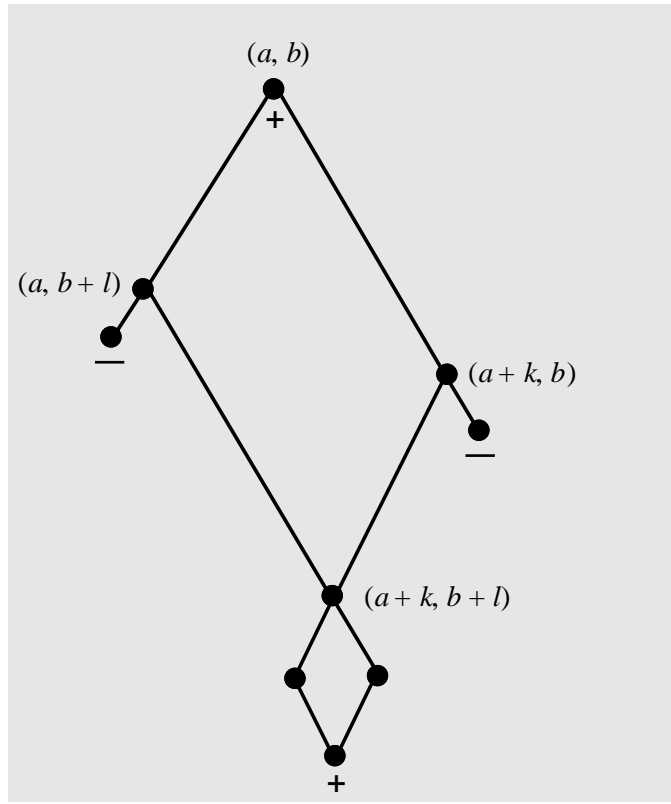


Figure 8: The coordinates of the four binomial coefficients.

Usually in mathematical research, one has a *conjecture* (a guess) to work on that comes from early experimenting. We certainly did some *experimentation* (we constructed a few examples such as those in Figures 1 and 2), but doing more than the cases $k = 1, 2,$ and 3 seemed not to be worthwhile. It would have taken a great deal of time, and we would have had no guarantee that we had listed all possibilities because there are so many towers for $k \geq 4$; it was unlikely that we would have seen a pattern from so few cases.

However, our initial experimentation did eventually suggest the two approaches to the problem that we have given in Sections 2 and 3. So we proceeded without a conjecture and came up with the answer and (at least) two *proofs*. We were lucky here that the straightforward approaches gave us the answer pretty well the first time, something which happens rarely. However, for mathematicians to be satisfied with the answer, it has to be accompanied by a proof. This is probably the only thing that separates mathematicians from physicists and other scientists. We justify our results by proof. Scientists are often in the position where they can only justify their results by repeating the experiment and getting the same answer.

In obtaining our proofs, we kept an eye on *past experience*. The difference method we used in our original but unreported method of finding an expression for t_k is a

technique that is commonly used. Our decision to look at the 0-1 sequences, reported in Section 3, was the result of previous experience with solutions of linear equations. To become good problem-solvers, students also need to be exposed to a range of mathematical techniques. It is not possible for them to invent from scratch every method that they need just when they need it.

When we had the answer, by methods which we have not explicitly described in this paper, we still did not feel satisfied, for two reasons. First, the nature of the answer suggested that there should be other approaches, which were *simpler* than those methods, and explained the simple binomial coefficient expression. The nose of a mathematician is always trying to smell out a better, nicer, way of doing things. This sense of smell led us into the more elegant methods via Pascal's Triangle and 0-1 sequences. Our final method (see Section 3) not only gave us an explanation of the presence of the binomial coefficient in the formula for t_k , but also, by setting up an explicit one-to-one correspondence, explained the presence of -1 in the formula.

The second reason for our lack of satisfaction was the basic instinct of the mathematician to *generalize* and *extend* problems. We have indicated our progress on these fronts in Section 4. This desire to generalize is coupled with another instinct and that is to *get something for nothing*. Once we had the answer for equal numbers of cans of the two types, it was easy to produce the answer for unequal numbers. Exactly the same proof worked. All we had to do was to change the characters slightly. It often turns out that a proof, if thought about in the right way, yields a proof of something more general.

Finally, when the result was recorded for others to read, we downplayed the historical development in favor of a

simpler presentation. This hides the sweat and tears involved in producing the result, but makes it easier for the reader to follow. Unfortunately, this style of writing, adopted in current books and journals, also serves to make mathematics appear easy to produce, and their subsequent disillusionment gives students the impression that they are not able to do mathematics themselves.

However, *all* the mathematical steps that we have outlined above are accessible to students in classroom problem-solving situations. We would advocate the use of open-ended problems both to allow students the joy of discovery and to encourage them to get a feel for what mathematics really is.

1. Remember that we have adopted the mathematical viewpoint that the (empty) 0-tower is a tower. Otherwise

we would have had to omit the binomial coefficient $\binom{0}{0,0}$.

2. This was called the Christmas Stocking Theorem by Dave Logothetti. Just look at the enclosed portion of Figure 4 and you'll see why. So far as we know, the theorem was first described by Chu-Shih Chieh in 1303.

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