Metacognition and Mathematical Problem Solving: Helping Students to Ask The Right Questions

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The acquisition of problem solving, reasoning and critical thinking skills has been identified by the National Council of Teachers of Mathematics (NCTM, 1989) as a critical goal. Lester (1985) defines this goal as helping students to think within a mathematical context:

> The primary purpose of mathematical problem solving instruction is not to equip students with a collection of skills and processes, but rather to enable them to think for themselves. The value of skills and processes instruction should be judged by the extent to which the skills and processes actually enhance flexible, independent thinking (p. 66).

Students' abilities to think flexibly can be developed and enhanced by teachers modeling their own thinking processes, giving students opportunities to problem solve, and helping students become aware of their own thought processes as they solve mathematical problems. This process of analyzing our own thought processes is called metacognition and includes thinking about how we are approaching a problem, the strategies we choose to use to find a solution, and the questions we ask ourselves about the problem are all part of metacognition.

Schoenfeld (1985) has characterized metacognitive skills as "aspects of mathematical 'understanding' that extend beyond the mastery of routine facts and procedures" (p. 361) and noted that these skills do not usually develop in mathematics instruction because of the focus on factual and procedural knowledge. Campione, Brown and Connell (1989) state that "successful learners can reflect on their own problem solving activities, have available powerful strategies for dealing with novel problems, and oversee and regulate those strategies efficiently and effectively" (p. 94). They also indicate that assessing this type of learning requires dynamic rather than static measures.

While static measures test knowledge and process, dynamic measures depend on determining how a student uses knowledge and skills to progress beyond a starting point. Dynamic measures are better predictors of gains in performance and are significantly more diagnostic than learning scores from static tests.

In light of these considerations, this study examined how students used their thinking skills to complete a problem solving task and how those thinking skills change given practice. Additionally, the study sought to determine if guidance in the form of hints given to subjects would make a difference in the manner in which they attended to the task.

Subjects

College students enrolled in a mathematics content course required of all prospective elementary teachers served as subjects for this study. This group was chosen with the assumption that older subjects would be more likely to be able to reflect on their own thought processes and analyze their own performances. All students in this class were required to have taken an introductory algebra course or tested out of it, and the present course would be the terminal college mathematics course for most subjects, except for a few mathematics minors. Subjects were randomly assigned to one of two groups, either a treatment or control group. Each group contained seventeen members with the treatment group having 2 men and 15 women and the control group 3 men and 14 women. Examination of the two groups after group assignment showed them to be comparable in range of abilities, age, mathematical background, and success in previous math courses.

The Task

A public domain computer program by Arlene Cram entitled "PHANTNUM" (Apple translation by Preston Marsh) served as the problem solving task. A computer game was used for four reasons. First, subjects can work in a computer lab with minimal supervision. Second, this game provides instant responses to student clues and allows for individuals to work at their own pace. Third, the game provides feedback as to incorrect guesses without

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telling the correct answer. Fourth, the computer records all entries and responses made by subjects so a comparison between self-report and actual responses can be made.

In this game, ten letters, A through J, appear on the screen, each of which represents one of the ten digits (0 to 9). The purpose of the game is to identify the digit represented by each letter by performing addition, subtraction, or multiplication. The object of the game is to use critical thinking about mathematical relationships to guide asking for clues and making guesses in order to retain the most points possible. Each player begins play with 500 points and can proceed by making guess as to the value of a letter or by asking for a clue using an operation with the letter. Each wrong guess costs 50 points while each clue asked costs only 5 points. For example a subject may type in the clue A * A =? and the computer will respond with the letter which represents the last digit of the answer. So with a print out of A * A = A, A could be 0,1, 5, or 6.

This game presents a problem solving situation (or task) because the answer is not readily apparent, and subjects need to determine a strategy for gathering information before a possible solution can be tested. After gathering data, decisions based on responses to the clues solutions.

Procedure

The two groups were assigned to different computer labs in different buildings on campus and were asked to complete the task during a designated class period. Both groups were told to play the game three times. Some subjects got caught up in the game and did more, however, only the first three trials were used for this study. The to use a strategy or strategies to attend to the problem.

When all three trials were completed, subjects were asked to reflect on their experience and answer the following questions:

1. What mathematical skills, concepts or relationships did you need to know to do this task?
2. What operation was most helpful at the beginning of the task and at the end?
3. What digit was the easiest to find? Why?
4. What process or strategy did you use?
5. At what point did you stop using clues and start solving for digits or when did you “get it”?

Results

Several statistics were calculated from the results of this study. Average mean scores were obtained for the first and third trial for both the treatment and control group. Also, the average number of clues used and the average number of wrong guesses was determined for each group. A one way ANOVA was calculated for each category. The results are shown in Figure 2.

Since subjects were asked to keep a complete list of all clues asked, guesses made, and the running score, this list could be examined for (1) the presence of some strategy to

HINTS FOR THE PHANTNUM GAME

IN GENERAL:

1) Go slowly. Think about your answers.
2) Consider all the options carefully before making a choice. Remember, you can get 10 clues for one wrong guess.
3) Figure out a logical approach. Have a PLAN, don’t just guess randomly.
4) Look for patterns. For instance, if a letter comes up in two different clues, try to figure out what the relationship is.
5) If you’re not sure about an answer, ask another clue.

SPECIFICALLY:

1) Remember that each digit (0-9) is used only once.
2) Remember that the computer only responds with the ones digit even if the answer contains two digits.
3) Ask yourself questions like the examples below:
   Questions #1 If A*A = A and A + A = A, what is the value of A?
   Questions #2 If A*A = A and A*B = B, what is the value of A?
4) Once you have determined the value for two or more letters, go back over your clues and “plug” in digits to help solve the equations.

Figure 1: Instruction Sheet for the Treatment Group.

Category	Control	Treatment	F	D
first trial mean	213.90	280.60	2.02	.165
third trial mean	336.30	391.40	2.81	.104
first trial clues	16.59	15.24	.80	.337
third trial clues	13.88	13.24	.40	.532
first trial wrong guesses	4.00	2.53	23.09	.000
third trial wrong guesses	1.71	1.00	1.79	.

Figure 2: Comparison of Control and Treatment Groups

guide their problem solving, (2) the identification of types of strategies used, and (3) the change in strategies from one trial to the next. The results are shown in Figure 3.

Finally, usbjects were asked to answer the open-ended questions following the trials. These questions were designed to force verbalization of their thinking processes as they reflected on the task. The variety and frequency of responses is found in Figure 4. In general, most participant responses to the exercise itself were positive. "[This activity] challenges you to be systematic, to be able to realize/create patterns and to better understand mathematical operations."

Discussion

The only significant difference between the control group and the treatment groups was in the number of wrong guesses made on the first trial although the third trial scores approach significance (p = .104). At the beginning, the control group subjects were more likely to start randomly guessing at possible values with very few clues. It seemed they did not think about the loss of 50 points being significant until they ran out of points. The added hint for the treatment group that you could ask ten clues for the same amount of points as one wrong guess apparently made them more cautious and more thoughtful about the guesses they made. The treatment group asked for more clues in the beginning until they were sure of one or two values. Then they asked fewer clues to develop the rest of the values. On the other hand, many of the subjects in the control group would ask for a few clues, make a guess, then ask for more clues until they got a right answer. Then they would repeat the process for another value. Frequently the only connection they made among the steps was the elimination of possible values rather than using two known values to find a third. So although the total clues used were nearly the same, the treatment group seemed to make better use of the clues than the control group did.

Even though there was no statistically significant difference between the groups in scores, all trials showed the treatment group achieving higher scores. Had the sample size been bigger it is possible that the difference would have been significant. Both groups, however, showed a significant increase ($p<.001$) in gain score over the trials indicating an improvement in the use of strategies to complete the task.

As illustrated in Figure 3, only two subjects out of the eight from the control group persisted in gathering information from random clues. All other subjects eventually found a strategy or were using one from the first trial It is important to note that all subjects in the treatment group used some sort strategy from the beginning. On the other hand, half of the control group was unable to generate a strategy immediately. In all subjects, those using at least one strategy at the beginning either changed to a more efficient strategy or used a more appropriate strategy in later trials.

In examining the strategies applied to the task, several patterns emerged. Those starting with a random pattern of asking clues generally changed to asking clues to find a single digit. In subsequent trials, these subjects would cluster various operations around one particular letter and then form an hypothesis about its value. Those subjects starting with this repeated operation strategy learned to identify which operations gave the most information and changed to clues using squares (A*A, B*B, etc.) to generate specific patterns.

As subjects worked through the task, regardless of the strategy used, the pattern they recognized led most often to

Strategy	First Trial		Third Trial	
	C	T	C	T
Random (no strategy)	8	0	2	0
Use of strategy	9	17	15	17
Identity	3	6	7	8
Looking back	5	7	2	4
Squares	1	2	2	10
Repeated operations with the same number	5	15	7	8
Change in strategy from first trial to third	N/A	N/A	10	11

Note: Subjects could give more than one response.

Figure 3: Comparison of Strategies Used

the identity elements (0 and 1) first. In answering the questions, "Which digit was the easiest to find?", almost everyone in both groups indicated these digits even if they were not able to use the term "identity". Zero was recognized as the easiest to find most often because it could be identified not only as an additive inverse statement (A - A = ?), but also by the zero property of multiplication. "The 0 was easiest to find. This was the case in the problem A * B = A. In this situation, either A = 0 or B = 1." Interestingly, though, only two subjects were able to identify zero in one step by using the additive inverse property.

Not only were the responses to the above question similar between the two groups, but in fact, almost all responses were nearly identical for both groups. In spite of the fact that subjects were able to use numerous number relationships and logical comparisons in doing the task and were able to report which ones they used when asked, the majority only perceived that they used basic addition, subtraction and multiplication facts. This may indicate that they have synthesized different relationships under the general headings of number operations or they didn't perceive of "thinking skills" as mathematical concepts. Again, this seems to support the idea that the groups were similar in abilities and backgrounds regarding mathematics.

Subjects were eventually able to identify which clues led them to the most limited possibilities, which operations were the most valuable at which time, and what patterns to look for in the clues. The subjects were easily able to verbalize how they went about the process of solving the task, illustrating the presence of metacognition processes. "[I] put combinations of letters together and after about 5 clues, I would poll my information and make an educated guess." What they didn't seem to realize was when they had sufficient data to simply start solving for values rather than using more clues from which to make guesses. Only three subjects from the treatment group and two subjects from the control group were able

Question: What mathematical skills, concepts or relationships did you need to do this task?

Skills, concepts, relationships.	Control	Treatment
Addition	11	12
Subtraction	9	9
Multiplication	12	14
Identity	2	5
Last digit	1	2
Square	0	1
Other multiplication	1	1
Trial and error	5	3
Order of operations	0	1
Associative property	2	0
Deduction	5	2
Solving equations	2	0

Question: What operation was most helpful at the beginning and the end of the task?

	Control		Treatment	
Operation	Start	End	Start	End
Addition	7	13	8	10
Subtraction	1	9	0	5
Multiplication	15	3	17	8

Question: What digit was easiest to find? Why?

	Control	Treatment
Digit		
zero	13	14
one	7	8
none	1	0
stated identity element	13	10
Why easiest		
fewest calculations	0	1
zero property	6	8
additive inverse	0	2

Question: What process (strategy) was used?

Strategy	Control	Treatment
Guess and check	6	5
Look for a pattern	6	6
Solve for identities first	8	9

Question: At what point did you stop using clues to guess at values and start solving for specific digits?

Solve for digits after	Control	Treatment
several digits were known	13	11
finding 0 and 1	2	3
after finding only 1	2	3

Figure 4: Responses to survey questions.

to reason out that once the digit 1 was found, every other digit could be determined in a single step by adding 1 to a known value. Most seemed to want validation by having several digits known before solving for the remaining digits. "First I found 1 and 0 using multiplication. Then I added 1 and other numbers [unknowns] until I got 0 which meant that the number had to be 9. Then I took 9 - 1 to get all the rest of the numbers."

Conclusions

Since the sample size was small, it is not possible to make sweeping conclusions about all students based on the results of this study. However, to the extent that results here support or duplicate other research, some conclusions can be made. First, students get better at problem solving when given practice. This comes as no surprise and supports NCTM in its emphasis to incorporate more critical thinking and problem solving into the mathematics curriculum at all levels. Additionally, students who are allowed time to work at problem solving situations do more than simply find a solution. Given multiple opportunities to practice, they learn to be more efficient in their choice and use of strategies, to generalize from one situation to another, and to discriminate relevant characteristics more quickly. "Doing this task several times allowed me to recognize which approaches were constructive and the logical sequence that was most helpful in solving the problem."

Second, students in this study were able to identify the thinking processes they used to accomplish a task. "At first I used a guess and check strategy. After I came up with a few numbers, I tried to look for a pattern." Also, some students reported that while they were writing their responses , they were able to see alternatives to the approach they used and would make those corrections if they did the task again. Again this supports studies that show the value of verbal expression in helping to clarify one's thinking such as in group work, cooperative learning, keeping logs or journals, and writing reports (Human, 1993; Stewart, 1993). However, it is not likely that younger children would be quite as adept at so thoroughly verbalizing their thinking as the adults in this study, but it still indicates the value of verbalization in building metacognitive skills.

Third, the subjects in this study were definitely aided in their problem solving by the hint sheet as indicated by the higher scores on all the trails and by the use of more efficient strategies by the treatment group. Although subjects seem to benefit from repeated trials, so that all subjects increased their scores over the three trials, there seems to be merit in students starting with a viable strategy and improving it rather than taking considerable time to focus on any strategy. Since classrooms continue to be inhibited by time constraints, it is not always possible to allow students extended time periods to deduce appropriate strategies. In the work environment, the time factor may be even more critical. Consequently when students are learning to problem solve, judicious guidance can give students a headstart. Helping students to ask themselves the right questions is more difficult than teaching a set procedure for a given circumstance, but it is worth the effort. The key is to help students structure their own thought processes so that they can generate their own questions and strategies appropriate to the task.

Problem solving skills are an important part of any mathematics program. Giving students ways in which they can monitor their own learning and thought processes can be effective in helping them become better problem solvers and ultimately better "thinkers" for any mathematical (or other) task. Some general conclusions can be drawn and suggestions for future study can be made as a result of this study. Although limited in scopewith a sample population, the indication is that students can be helped to mnitor their own cognitive procssing by using metacognitive techniques which shows potential for improving their problem solving skills. Continued research with different age groups, larger sample sizes, and/or additional problem solving activities is necessary to find ways that teachers can assist students to be more aware of their own thinking as they attempt to solve problems.

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