

# Fostering Recursive Thinking in Combinatorics through the Use of Manipulatives and Computing Technology

Sergei Abramovich & Anne Pieper

Current trends in the revision of both mathematics curricula and traditional teaching strategies include the incorporation of discrete mathematics into the secondary school curriculum (Dossey, 1990). Serious consideration of this task has occurred as a result of the publication of the *NCTM 1991 Year Book* (Kenney, 1991), which accumulated perceptions, practical ideas and suggestions of many experienced professionals about discrete mathematics in general and combinatorics in particular. Also, Standard 12 of the *Curriculum and Evaluation Standards* for Grades 9-12 emphasizes the importance of this topic: “instruction should emphasize combinatorial reasoning as opposed to the application of analytic formulas for permutations and combinations” (NCTM, 1989, p.179).

Combinatorics is one of the oldest branches of discrete mathematics and goes back to the 16th century when games of chance played an important role in the life of a society (Vilenkin, 1971). The need for the theory of such games stimulated the creation of specific counting techniques and mathematical concepts related to the new real-life situations. Further efforts of French scholars Pascal and Fermat in the pursuit of theoretical studies of combinatorial problems laid a foundation for the theory of probability and provided approaches to the development of enumerative combinatorics as the study of methods of counting various combinations of elements of a finite set.

The importance of a recursive approach to enumerative combinatorics through using difference equations (recurrence relations) is a common theme (Anderson, 1974; Kenney & Bezuska, 1993; Olson, 1989). Many counting problems deal with arrangements that involve permutations and combinations. While the study of these concepts provides opportunities for using recursive reasoning as an effective problem-solving means, student learning of combinatorics has often been limited to plugging numbers into formulas for permutations and combi-

nations without developing any conceptual understanding of combinatorial ideas or using recursive reasoning as a means for approaching these ideas.

Mathematical visualization provided by computing technology and the use of concrete materials allow for significant improvement in the teaching and learning of basic combinatorial notions. This article describes how employing manipulatives for solving simple combinatorial problems leads to the discovery of recurrence relations for permutations and combinations, so that, in turn, numerical evidence and visual imagery generated by the computer spreadsheet through modeling these relations can enable all students to experience the ease and power of combinatorial reasoning, recursion, and mathematical induction proof. In addition to these teaching ideas, the interplay among geometry, combinatorics, and number theory is discussed.

According to Ahlfors et al. (1962), instruction should be a process of “extracting the appropriate concept from a concrete situation.” Combinatorics is an appropriate context for accommodating such instructional philosophy. For instance, what kind of combinatorial reasoning is behind whether one is sorting or selecting books from a shelf? How does one distinguish between combinatorics of a bookshelf and combinatorics of a combination lock or a touch-tone telephone? While these are real-world situations with which students are familiar, they need help in making sense of the mathematics that is behind the reality so that they can extract from it the appropriate concept.

Just as the use of dominoes in classroom activities helps to furnish the conceptual basis of combinatorics (Johnson, 1991), colored manipulatives are excellent tools that encourage and support combinatorial reasoning (Spangler, 1991). Because of this, we will formulate real-world situations in terms of colored disks (discussed in this paper as patterned disks) which were found to be extremely appropriate for individual or small group explorations and subsequent student demonstrations using an overhead projector. The paper reflects work done in a lab setting with preservice and inservice secondary teachers enrolled in a contemporary general mathematics course.

## ***Recursive Approach to Permutations***

*In how many ways can three colored disks be placed in different orders? Many real-world situations such as this*

---

*Sergei Abramovich has been on the Mathematics Education faculty at the University of Georgia since 1993 teaching a variety of undergraduate and graduate courses. His research interests include the study of the joint use of manipulative and computing activities in mathematics teaching and learning.*

*Anne Pieper currently teaches high school geometry at The Westminster Schools in Atlanta, Georgia. She recently graduated from The University of Georgia with an M.A. in Mathematics Education.*

deal with the concept of permutations—all the possible arrangements of a collection of elements, each of which contains every element once, with two such arrangements differing only in the order of their elements. Often, the emphasis is on the *number* of permutations rather than the permutations, or orderings, themselves.  $P(n)$  denotes the number of permutations of  $n$  objects; the number of orderings of three disks, then, would be denoted by  $P(3)$ .

To help students grasp the idea of permutations and make sense of how objects can be listed, have them consider a collection of three different patterned disks as shown in Figure 1.

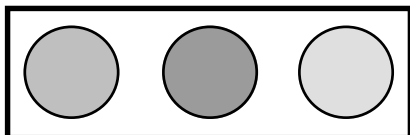


Figure 1. Three disks to be placed in different orders.

Having students actually manipulate patterned disks may further help to illustrate the concept. In order to determine the number of ways these three disks can be ordered, consider first the permutations generated by one and then two disks. Clearly, one disk has one ordering. Two disks can be ordered in two ways by fixing one and then placing the other to its left or to its right (see Figure 2). To consider the orderings of three disks, add the third disk to the possible orderings for two disks. For each ordering of two disks, the third disk can be placed in any one of three positions created by the other two disks: it can be placed to the left of the two, between the two, or to the right of the two (see Figure 3). Thus, there exist six different orderings for the three disks; that is,  $P(3) = 6$ .

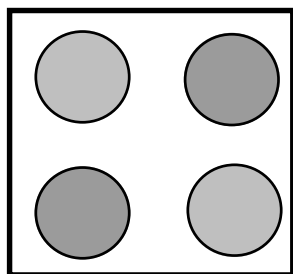


Figure 2. Permutations of two disks.

Next, in order to help students generalize for  $n$  disks, have them predict the number of permutations of four disks,  $P(4)$ , and explain their reasoning to the class. Ask students to describe the relationship between the orderings of four disks to that of three. Since the fourth disk can be placed in any one of four positions created by the other

three disks, four disks generate twenty-four permutations. Symbolically, this is represented as  $P(4) = 4 \cdot P(3) = 24$ . Have students again look at  $P(3)$  and determine whether a similar relationship exists. They should note that the number of permutations of three disks is equal to that of two disks multiplied by three, or the number of positions that the third disk can assume in relation to the other two.

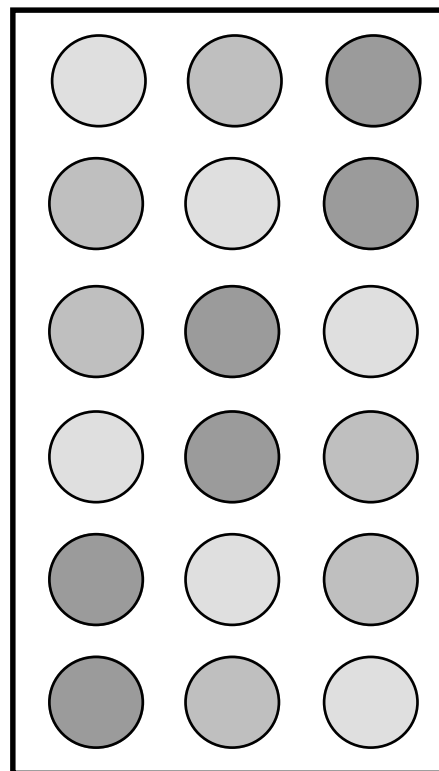


Figure 3. Permutations of two disks.

Such exploration of the permutations of three and four disks nicely leads to students generalizing for  $n$  disks; that is, the number of ways in which  $n$  disks can be ordered is  $n$  multiplied by the number of orderings of  $(n-1)$  disks. Formally, this is denoted as

$$P(n) = n \cdot P(n-1). \quad (1)$$

Point out to students that this is an example of a difference equation obtained by applying recursive reasoning; that is, the problem of counting permutations of  $n$  objects is reduced to an analogous problem involving a smaller number of objects. Kenney and Bezuska (1993) emphasize the power of recursive reasoning in solving discrete problems, and they argue that “the most challenging part of solving problems by recursion is making the connection of the solution to the given problem with the solution to the subproblems” (p. 677).

One may note that relation (1) is incomplete as a definition of permutations, however, since  $P(0)$  is not

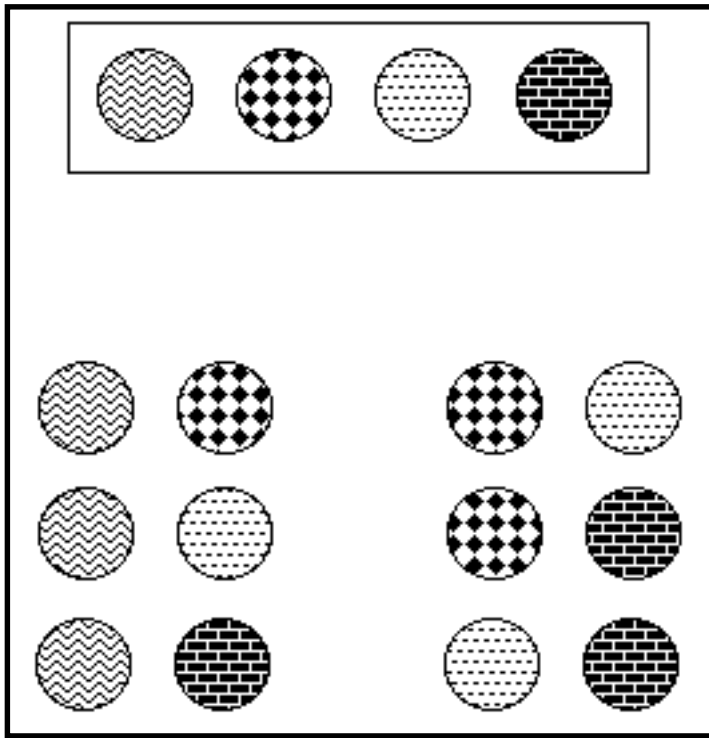


Figure 4. 2-combinations of four disks:  
 $C(4,2) = C(3,2) + C(3,1)$

defined. To explore this with students, ask them to consider what happens if  $n = 1$  or if  $n = 0$  in the above generalization. As they will discover, when  $n = 1$ , then due

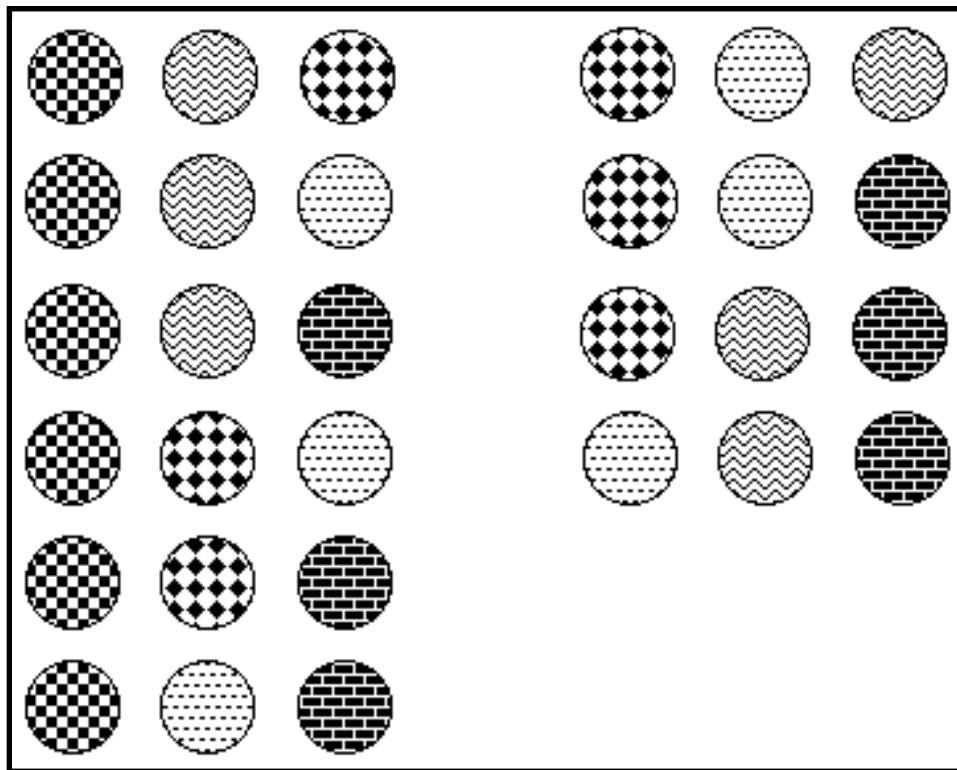


Figure 5. 3-combinations of five disks:  $C(5,3) = C(4,2) + C(4,3)$ .

to relation (1),  $P(1) = 1 \cdot P(0)$ . As discussed earlier with regard to disks, it is apparent that there is only one way to order one disk. Hence  $P(1) = 1$ . However,  $P(0)$ , the number of ways in which zero disks can be ordered, is a more difficult concept to understand. To define  $P(0)$ , one must again consider that

$$P(1) = 1 \cdot P(0).$$

Since  
 $P(1) = 1$   
 then  
 $1 = 1 \cdot P(0)$ .  
 Hence  
 $P(0) = 1$ .

Explain to students that this convention for  $P(0)$  is the starting point, or initial value, of  $P(n)$ . This initial value is necessary for formulating a complete recursive definition of permutations. An illustration of the applicability of the initial value can be found in computing compound interest where the initial deposit serves as the initial value. Such an example may help students better grasp the role of the initial value.

Keeping in mind the initial value, the formal recursive definition of permutations is thus

$$P(n) = n \cdot P(n-1), n = 1, 2, \dots ; P(0) = 1. \quad (2)$$

The recursive approach to permutations is described in Olson (1989), and a BASIC program is employed to numerically model the recursive definition given in (2). We will later suggest using a spreadsheet as an alternative to BASIC programming.

### *Recursive Approach to Combinations*

Modeling of many real-world situations deals with the concept of combination—the selection of a certain number of elements from a given set without regard to order. Whereas the concept of permutation focuses on the orderings of a certain number of objects (sorting books on a bookshelf), the concept of combination focuses on the selection of a subset of a certain

number of objects (selecting books from a bookshelf). An example of this in terms of the manipulatives is the following: *In how many ways can three disks be selected from five different disks?*

In general, problems involving the concept of combination can be expressed in the following form:

*Suppose there is a set of  $n$  given objects. Any choice of  $r$  objects from the set of  $n$  is considered as an  $r$ -combination of  $n$  elements. Two such combinations are different only if they differ in the elements they contain, not the order in which they are organized. How many different  $r$ -combinations can be selected?*

Note that the number of ways of choosing  $r$  objects from among a set of  $n$  given objects is a function of  $n$  and  $r$  and may be denoted  $C(n,r)$ . Traditionally, the example problem posed above has been solved using the formula

$$C(n,r) = \frac{n!}{r!(n-r)!} \quad (3)$$

The *Standards*, however, accentuate the importance of developing students' ability to solve problems of this type through combinatorial reasoning, as opposed to the application of formula (3). As with permutations, the concept of combinations can be introduced through a problem solving activity involving the use of manipulatives where recursive reasoning arises naturally.

Returning to the earlier example, suppose you have five patterned disks and you want to determine the number of ways three disks can be selected from the five. As Freudenthal (1973) noted, simple combinatorial techniques can be acquired "by having the student disentangle complex situations in which such combinatorics play an essential part" (p. 597). In this vein, first ask students to solve a simpler problem of finding the number of combinations that arise from selecting two disks from four disks. In carrying out this task, the teacher may discover that students act at random in creating the combinations. However, when asked to be systematic, students often unconsciously apply recursive reasoning. Indeed, as observed in an actual mathematics classroom, a student may fix one disk, say the one with zig-zag pattern, and exhaust all 2-combinations of four with it (see Figure 4). Then the "zig-zag" disk is put aside and not used at all, thus resulting in creating 2-combinations of the remaining three disks.

Observing Figure 4, one may note that the right-hand side represents  $C(3,2)$  because this is a construction of all 2-combinations of three elements. But what is on the left-hand side? In order to create all 2-combinations with the zig-zag disk, we in fact should select one element from

three and this can be done in  $C(3,1)$  ways. Therefore,

$$C(4,2) = C(3,2) + C(3,1) \quad (4)$$

In much the same way, the original problem can be solved through recursive reasoning, that is, by splitting  $C(5,3)$  into the sum of two combinations. Figure 5 represents the partitioning of  $C(5,3)$  into  $C(4,2)$  on the left (compare to Figure 4) and  $C(4,3)$  on the right. In other words,

$$C(5,3) = C(4,2) + C(4,3) \quad (5)$$

Relations (4) and (5) are recursive formulations of  $C(4,2)$  and  $C(5,3)$ , respectively. It is clear that  $C(5,3) = 6 + 4 = 10$ .

To enhance students' understanding of recursion formulas, the teacher can use a visual imagery for combina-

| $r \backslash n$ | 0        | 1 | 2 | 3 | 4        | 5        |
|------------------|----------|---|---|---|----------|----------|
| 0                | $C(0,0)$ |   |   |   |          | $C(5,0)$ |
| 1                | $C(0,1)$ |   |   |   |          |          |
| 2                |          |   |   |   | $C(4,2)$ |          |
| 3                |          |   |   |   | $C(4,3)$ | $C(5,3)$ |

Figure 6. A grid chart for combinations.

tions. Indeed, combinations may be organized in a grid chart and located in cells as shown in Figure 6. Looking at the cell of  $C(5,3)$  in the chart and using the equality  $C(5,3) = C(4,3) + C(4,2)$ , students can develop an L-shaped geometric representation for such a relation. They can then check whether  $C(4,2)$  and  $C(4,3)$  can be shown using a similar L-shaped relation. Students can then express any combination using such an L-shaped geometric interpretation. That is, if the right cell has coordinates  $C(n, r)$  then the two to the left will have coordinates  $C(n-1, r)$  and  $C(n-1, r-1)$  (see Figure 7).

But what happens when you reach the cell containing  $C(1,0)$  in Figure 6? Students will recognize that there is no upper cell to use in determining  $C(1,0)$ . This brings up an important concept explored earlier with permutations—that of an initial (boundary) condition. All cells along the top row involve the boundary condition at  $r = 0$ , that is,

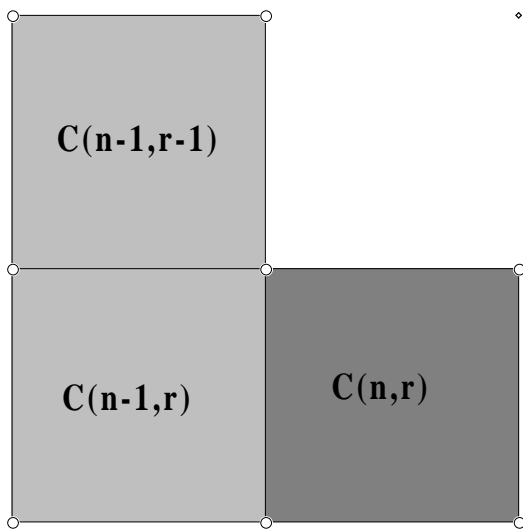


Figure 7. Visual imagery of combinations.

they correspond to combinations of  $n$  objects taken zero at a time. Using the L-shaped representation, students can determine that since  $C(2,1) = C(1,0) + C(1,1)$ , it follows that  $C(1,0) = C(2,1) - C(1,1)$ . Returning to the patterned disks, recall that  $C(2,1)$  refers to the number of ways in which one disk can be chosen from two disks, namely, two. Similarly,  $C(1,1)$  refers to the number of ways in which one disk can be chosen from one, and hence  $C(1,1) = 1$ . Thus  $C(1,0) = C(2,1) - C(1,1) = 2 - 1 = 1$ . From similar reasoning it follows that  $C(3,0) = C(2,0) \dots = C(n,0) = 1$ . Students should note that this is also the case for  $C(0,0)$ : using the L-shaped rule,  $C(1,1) - C(0,1) = 1 - 0 = 1$ .

Questions also arise with regard to the cells in the far left column, namely, where  $n = 0$ . Suppose for instance, you consider  $C(0,3)$ , the number of ways to choose three disks from zero disks. There are zero ways to carry this out, as you are choosing more than zero disks. Students can experiment with different cells in the far left column to determine that this is true for any  $r \geq 1$ .

By exploring the boundaries of the grid, students thus determine the boundary, or initial, condition for combinations in the following form:

$$C(0,r) = 0, r \geq 1; C(n,0) = 1, n \geq 0. \quad (6)$$

The grid enables students to recognize the need for such a condition and furthermore allows them to specifically determine what the condition is. Finally, translating the visual image of Figure 7 into symbolic form results in the general recursive formula for combinations:

$$C(n,r) = C(n-1,r) + C(n-1,r-1). \quad (7)$$

which along with relation (6) completes the recursive definition for  $r$ -combinations of  $n$  elements.

### Using Spreadsheets

According to the *Standards*, one way in which mathematical connections need to be fostered is by developing students' abilities to translate among different representations. The effective use of spreadsheets to introduce permutations through simple recursion (difference equation) is generally known (Cornell & Siegfried, 1991; Kenney & Bezuska, 1993). What is less known is that a spreadsheet can handle the modeling of the recursive definition for combinations; that is, it can generate Pascal's triangle in the form of a rectangular array (Abramovich & Levin, 1994). In this way, both for permutations and combinations, the spreadsheet serves as a numerical representation with which students can make mathematical connections. Also, it enables students to focus on the patterns that emerge from the permutations and combinations by decreasing the emphasis on hand calculations. Students can use the numbers generated by spreadsheets to make conjectures, use numerical evidence as a visual interpretation of inductive proof, and then subsequently develop a proof by translating visual imagery into a symbolic form. Microsoft Excel and AppleWorks can be used for producing spreadsheet templates for classroom use. The spreadsheet used in this article is Microsoft Excel 4.0 for the Macintosh.

### Exploring Permutations Using Spreadsheets

As we have noted earlier, spreadsheets can be used effectively for modeling the recursive definition for permutations. The spreadsheet template shown in Figure 8, which the teacher can help the students construct, represents the results of modeling. In column **A** (beginning with cell **A2**) non-negative integral values of  $n$  are listed. The initial value  $P(0)$ , that is, 1, is defined in cell **B2**. The spreadsheet function  $=B2*A3$  is defined in cell **B3** and computes the value of  $P(1)$  as  $P(0) \cdot 1$ . This function is copied all the way down to cell **B15**; the cells referred to are updated accordingly. For example, then, the function in cell **B4** is  $=B3*A4$ .

Ask students to compare the spreadsheet's calculations with those done in class for the permutations generated by two, three, and four disks. Have them investigate, using the spreadsheet, the relationship between the permutations of  $n$  disks and  $n-1$  disks for different values of  $n$ . Have them make conjectures about such a relationship. Too, students can explore the role of the initial value by entering different values for  $P(0)$  into cell **B2** and observing the change in different cells' values.

|    | A  | B          |
|----|----|------------|
| 1  | n  | P(n)       |
| 2  | 0  | 1          |
| 3  | 1  | 1          |
| 4  | 2  | 2          |
| 5  | 3  | 6          |
| 6  | 4  | 24         |
| 7  | 5  | 120        |
| 8  | 6  | 720        |
| 9  | 7  | 5040       |
| 10 | 8  | 40320      |
| 11 | 9  | 362880     |
| 12 | 10 | 3628800    |
| 13 | 11 | 39916800   |
| 14 | 12 | 479001600  |
| 15 | 13 | 6227020800 |

Figure 8. Spreadsheet modeling of permutations

The famous Pólya's maxim states that most discoveries start with recognizing a pattern (Pólya, Tarjan, & Woods, 1983). In looking at the numbers produced by the spreadsheet, students may first discover regularities within each triple of cells shaped as shown in Figure 9. This provides a geometric interpretation of the recursive defini-

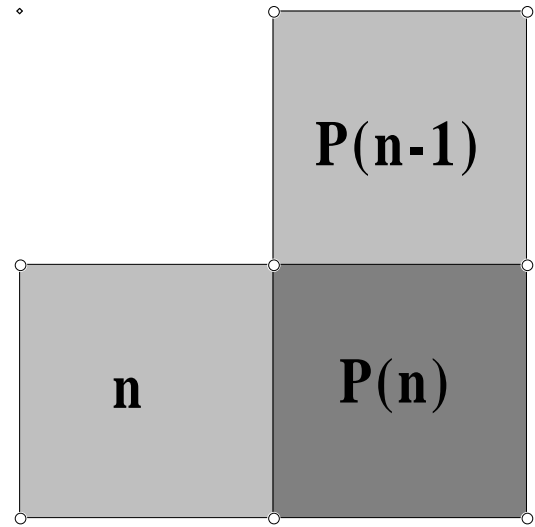


Figure 9. A geometric interpretation of the recursive definition of permutations.

tion of permutations: the product of two numbers in the cells shaded light equals the number in the third cell. Applying this rule to itself several times, i.e. replacing the top cell in Figure 9 with two diagonally adjacent cells, results in the picture shown in Figure 10. The final aggregation of cells (far right of Figure 10) provides a geometric interpretation of the closed form formula for permutations:

$$P(n) = 1 \cdot 2 \cdot 3 \dots n = n! \quad (8)$$

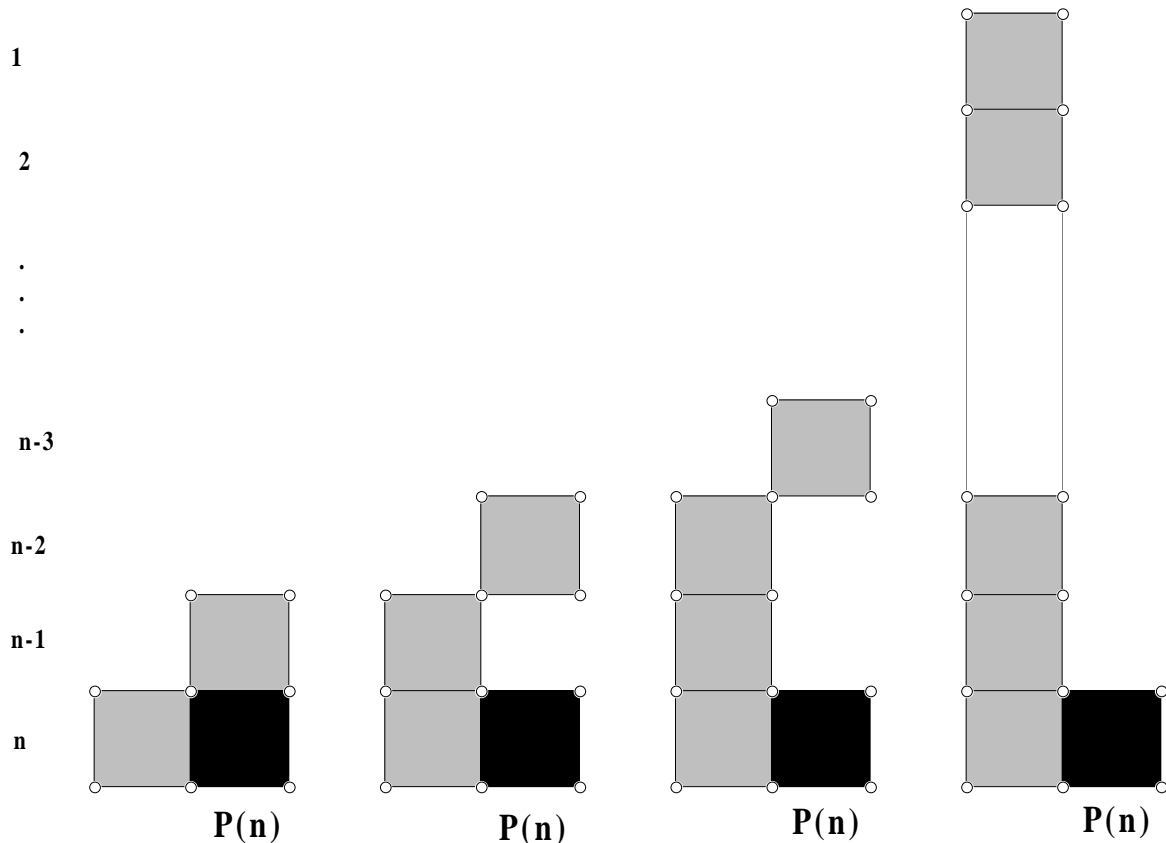


Figure 10. Visual discovery of relation (8).

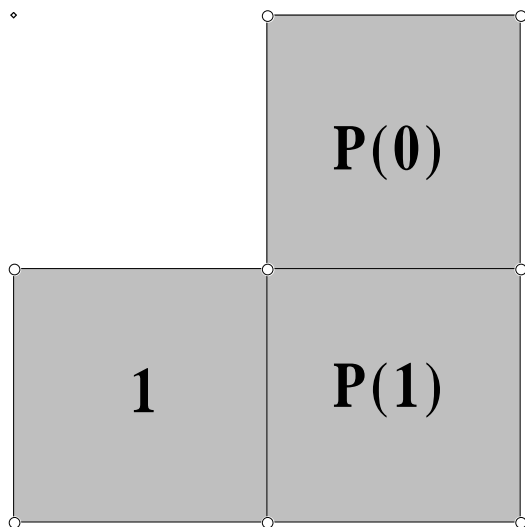


Figure 11. Visual imagery of induction proof of relation (8): The basis step.

Mathematical visualization can be further used to develop a geometric representation of a proof by mathematical induction. As Ralston (1990) pointed out,

Induction is often taught in what I like to call the “mathematics as magic” mode in which students are presented with the results ... and then shown how to prove the result correct by induction. A

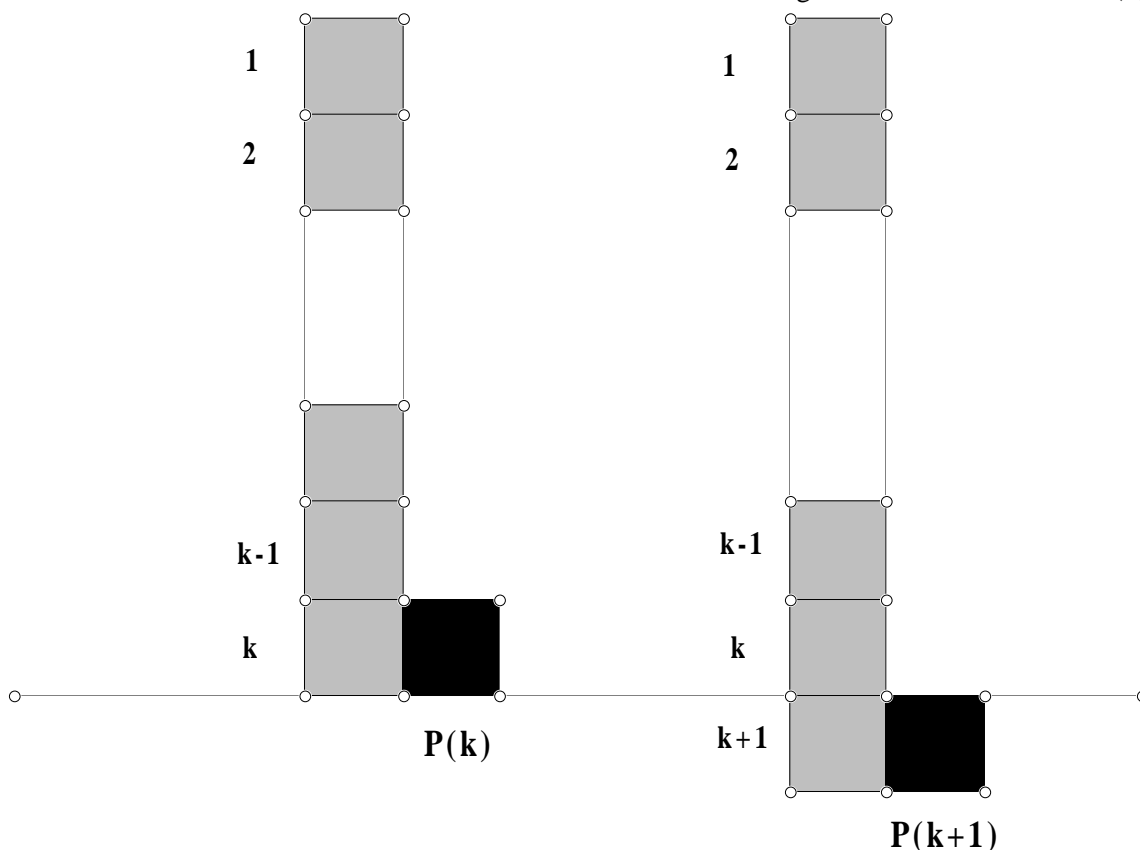


Figure 12. Visual imagery of induction proof of relation (8): The induction step.

much better approach is to use the *inductive paradigm* — compute, conjecture, prove — to infer the result and then prove it (p.80).

Indeed, the basis step for relation (8) holds true since for  $n = 1$ ,  $P(1) = 1 \cdot P(0) = 1$  (as Figure 11 shows). The inductive step is shown in Figure 12. By virtue of the rule of permutations (see Figure 9), the assumption that the product of the cells shaded light equals the cell shaded dark remains true when  $k$  is replaced by  $k + 1$ .

Now students may better understand the formal proof of relation (8) through mathematical induction. Actually, for  $n = 1$  the basis step has already been established. Assuming relation (8) to be true for  $n = k$  results in the following transformation of  $P(k + 1)$ :

$$P(k + 1) = (k + 1) \cdot P(k) = (k + 1) \cdot k! = (k + 1)!$$

### Exploring Combinations with Spreadsheets

As with permutations, spreadsheets can be used for modeling the recursive definition for combinations. The earlier activity involving the grid for combinations is aimed at bridging the gap between the traditional educational environment and spreadsheet use for quickly making the necessary replications and calculations. The spreadsheet shown in Figure 13 models numbers  $C(n,r)$  as the

|    |    | Figure 13 |   |   |   |   |    |    |    |    |     |     |     |     |      |      |      |       |       |       |
|----|----|-----------|---|---|---|---|----|----|----|----|-----|-----|-----|-----|------|------|------|-------|-------|-------|
|    | A  | B         | C | D | E | F | G  | H  | I  | J  | K   | L   | M   | N   | O    | P    | Q    | R     | S     | T     |
| 1  | r  | 0         | 1 | 2 | 3 | 4 | 5  | 6  | 7  | 8  | 9   | 10  | 11  | 12  | 13   | 14   | 15   | 16    | 17    | 18    |
| 2  | 0  | 1         | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1   | 1   | 1   | 1   | 1    | 1    | 1    | 1     | 1     | 1     |
| 3  | 1  | 0         | 1 | 2 | 3 | 4 | 5  | 6  | 7  | 8  | 9   | 10  | 11  | 12  | 13   | 14   | 15   | 16    | 17    | 18    |
| 4  | 2  | 0         | 0 | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36  | 45  | 55  | 66  | 78   | 91   | 105  | 120   | 136   | 153   |
| 5  | 3  | 0         | 0 | 0 | 1 | 4 | 10 | 20 | 35 | 56 | 84  | 120 | 165 | 220 | 286  | 364  | 455  | 560   | 680   | 816   |
| 6  | 4  | 0         | 0 | 0 | 0 | 1 | 5  | 15 | 35 | 70 | 126 | 210 | 330 | 495 | 715  | 1001 | 1365 | 1820  | 2380  | 3060  |
| 7  | 5  | 0         | 0 | 0 | 0 | 0 | 1  | 6  | 21 | 56 | 126 | 252 | 462 | 792 | 1287 | 2002 | 3003 | 4368  | 6188  | 8568  |
| 8  | 6  | 0         | 0 | 0 | 0 | 0 | 0  | 1  | 7  | 28 | 84  | 210 | 462 | 924 | 1716 | 3003 | 5005 | 8008  | 12076 | 18564 |
| 9  | 7  | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 1  | 8  | 36  | 120 | 330 | 792 | 1716 | 3432 | 6435 | 11440 | 19448 | 31824 |
| 10 | 8  | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 1  | 9   | 45  | 165 | 495 | 1287 | 3003 | 6435 | 12870 | 24310 | 43758 |
| 11 | 9  | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 1   | 10  | 55  | 220 | 715  | 2002 | 5005 | 11440 | 24310 | 48620 |
| 12 | 10 | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0   | 1   | 11  | 66  | 286  | 1001 | 3003 | 8008  | 19448 | 43758 |
| 13 | 11 | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0   | 0   | 1   | 12  | 78   | 364  | 1365 | 4368  | 12076 | 31824 |
| 14 | 12 | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 1   | 13   | 91   | 455  | 1820  | 6188  | 18564 |
| 15 | 13 | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 1    | 14   | 105  | 560   | 2380  | 8568  |
| 16 | 14 | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0    | 1    | 15   | 120   | 680   | 3060  |
| 17 | 15 | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0    | 0    | 1    | 16    | 136   | 816   |
| 18 | 16 | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 1     | 17    | 153   |
| 19 | 17 | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0     | 1     | 18    |
| 20 | 18 | 0         | 0 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0   | 0   | 0   | 0   | 0    | 0    | 0    | 0     | 0     | 1     |

Figure 13. The spreadsheet **COMB** (combinations).

solution to relation (7) through the following programming. In row 1 (beginning with cell B1) and in column A (beginning with cell A2) non-negative integral values of  $n$  and  $r$  are defined respectively. Entering the unity into row 2 (beginning with cell B2) and zero into column B (beginning with cell B3) results in relation (6) on the borders of a spreadsheet template. The spreadsheet function =B2+B3 is defined in cell C3 and computes the sum  $C(0,0)+C(0,1)$ . This function is replicated across all columns and down all rows to cell T20; hence, the template is immediately filled up with numbers corresponding to  $C(n,r)$ . The “screen snapshot” of part of the spreadsheet shown in Figure 14 displays in cell G5 and in the Formula Bar, respectively, the number 10 and the formula =F4+F5, which computes this number.

As with permutations, combinations can be viewed geometrically within the spreadsheet. As discussed earlier, relation (7) can be associated with its visual imagery, or L-shaped triple of cells (Figure 7), in which the sum of the two left cells equals that on the right. While previously shown within a grid, this rule can also be easily visualized by students in the context of the spreadsheet. Then, in much the same way as with permutations, through applying this rule to itself students can discover new patterns and regularities among numbers with combinatorial meaning. Furthermore, students can observe recurring numbers along each column of the spreadsheet (Figure 13). How does combinatorial reasoning explain this pattern? How can manipulatives be used to support this reasoning? Is it possible to discover Fibonacci numbers in the spreadsheet? And if so, how can the connection between Fi-

bonacci numbers and combinations be explained in the language of combinatorics? Answering these questions and discovering new patterns among combinations can serve as an extension of the basic activities. For more details see Abramovich & Levin (1994).

### Making Mathematical Connections

As Olson (1989) pointed out, combinations can give “an alternative perspective from which to think about a familiar mathematical topic” (p. 571). For instance, because students are familiar with figurate numbers from middle school (Ben-Chaim, Lappan, & Houang 1989), they may observe that numbers in row 4 of the

spreadsheet are triangular numbers. At the same time, these are 2-combinations. The natural question arises: How can we connect triangular numbers to combinations, or, in more general terms, number theory to combinato-

|   | A | B | C | D | E | F | G  | H  | I |
|---|---|---|---|---|---|---|----|----|---|
| 1 | r | 0 | 1 | 2 | 3 | 4 | 5  | 6  |   |
| 2 | 0 | 1 | 1 | 1 | 1 | 1 | 1  | 1  |   |
| 3 | 1 | 0 | 1 | 2 | 3 | 4 | 5  | 6  |   |
| 4 | 2 | 0 | 0 | 1 | 3 | 6 | 10 | 15 |   |
| 5 | 3 | 0 | 0 | 0 | 1 | 4 | 10 | 20 |   |
| 6 | 4 | 0 | 0 | 0 | 0 | 1 | 5  | 15 |   |
| 7 | 5 | 0 | 0 | 0 | 0 | 0 | 1  | 6  |   |
| 8 |   |   |   |   |   |   |    |    |   |

Figure 14. Screen-snap of the spreadsheet **COMB**



rics? One way to respond to this question is to use a geometric situation from which triangular numbers can be generated. Let us take two straight lines — they can intersect at one point only. This corresponds to the first triangular number, namely, 1. Add a third line — it can be concurrent with the previous two lines at two new points. Thus the total number of points is represented by the sum  $1 + 2$ , which is the second triangular number, namely, 3. Then add a fourth line — 3 new points of intersection are obtained. In this instance the sum  $1 + 2 + 3$  occurs, which is the third triangular number, namely, 6. Continuing in this vein, when the  $(n + 1)$ th line is added,  $n$  new points are obtained, yielding the sum  $1 + 2 + 3 + \dots + n$  which is the  $n$ th triangular number.

Indeed, two lines can intersect at only one point, and this situation can be represented by  $C(2,2) = 1$ . Each time a new line is added, it is possible to intersect each of the old lines once. Thus, for three lines, there are  $C(3,2) = 1 + 2 = 3$  points of intersection. This generalizes to  $C(n+1,2) = 1 + 2 + 3 + \dots + n$ . An alternative discussion on the interplay between triangular numbers and combinations is provided by Abramovich (1995).

Making connections among geometry, number theory, and combinatorics in this way, students can experience mathematics as a whole. Similarly, by making connections among different representations — namely spreadsheets, manipulatives and visual imagery — students can develop a richer understanding of permutations and combinations. Research shows that “mathematics becomes difficult for students when it concerns topics for which there do not exist simple physical or visual representations” (Dubinsky, 1991, p. 201). Thus, teachers must strive to provide students with a variety of visual representations when teaching combinatorics.

## References

- Abramovich, S. (1995). Technology-motivated teaching of advanced topics in discrete mathematics. *Journal of Computers in Mathematics and Science Teaching*, 14 (3), 391-418.
- Abramovich, S., & Levin, I. (1994). Microcomputer-based discovering and testing of combinatorial identities. *Journal of Computers in Mathematics and Science Teaching*, 13 (2), 223-246.
- Ahlfors, L. V. et al. (1962). On the mathematics curriculum of the high school. [Memorandum]. *American Mathematical Monthly*, 69, 189-193.
- Anderson, I. (1974). *A first course in combinatorial mathematics*. Oxford: Clarendon.
- Ben-Chaim, D., Lappan, G., & Houang, R. T. (1989). The role of visualization in the middle school mathematics curriculum. *Focus on Learning Problems in Mathematics*, 11, 49-60.
- Cornell, R. H., & Siegfried, E. (1991). Incorporating recursion and functions in the secondary school mathematics curriculum. In M. J. Kenney (Ed.), *1991 NCTM Yearbook. Discrete mathematics across the curriculum, K-12*. (pp. 149-157). Reston, VA: The Council.
- Dossey, J. (1990). *Discrete mathematics and the secondary mathematics curriculum*. Reston, VA: The National Council of Teachers of Mathematics.
- Dubinsky, E. (1991). Constructive aspects of reflective abstraction in advanced mathematics. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 160-202). New York: Springer Verlag.
- Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, The Netherlands: Reidel.
- Johnson, J. (1991). Using dominoes to introduce combinatorial reasoning. In M. J. Kenney (Ed.), *1991 NCTM Yearbook. Discrete mathematics across the curriculum, K-12*. (pp. 128-136). Reston, VA: The Council.
- Kenney, M. J. (Ed.). (1991). *1991 NCTM Yearbook. Discrete mathematics across the curriculum, K-12*. Reston, VA: The Council.
- Kenney, M. J., & Bezuska, S. J. (1993). Implementing the discrete mathematics Standards: Focusing on recursion. *Mathematics Teacher*, 86, 676-680.
- National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: Author.
- Olson, A. T. (1989). Recursion mathematics. *Mathematics Teacher*, 82, 571-572, 576.
- Pólya, G., Tarjan, R. E., and Woods, D. R. (1983). *Notes on introductory combinatorics*. Boston: Birkhauser.
- Ralston, A. (1990). The effect of technology on teaching collegiate mathematics. In F. Demana and B. K. Waits (Eds.), *Proceedings of the first annual conference on technology in collegiate mathematics: The twilight of the pencil and paper* (pp. 78-82). Reading, MA: Addison Wesley.
- Spangler, D. A. (1991). The pigeonhole principle: A counting technique for the middle grades. In M. J. Kenney (Ed.), *1991 NCTM Yearbook. Discrete mathematics across the curriculum, K-12*. (pp. 55-58). Reston, VA: The Council.
- Vilenkin, N. Y. (1971). *Combinatorics*. New York: Academic Press.