Mathematicians, Mathematics Teachers and Mathematical Discourse

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Discourses, History, Philosophy and Mathematics Education

Otte and Seeger (1994) describe the positivist-formalist conceptions and how they function in the use of motivation in the mathematics classroom. In particular, they consider the use of history and the effect on teaching. The discourse of motivation—typical in mathematics classes (similar to the discourse of applications; see Brousseau (1995, course 2, p. 18)—works as the "lesson's formality." The student recognizes its role as a ritual introduction; he or she knows that mathematics (what he or she is expected to learn) comes afterwards. The teacher will ask no questions related to the motivation section, unless there is a breach of the didactical contract (Brousseau, 1997, pp. 69-72). The mathematical notions in the motivation discourse behave analogously to the paramathematical notions (even though there is a different teaching contract at work with this discourse). It might be thought of as a weakly didactic "information contract", using Brousseau's (1996) categories (p. 18). It is worth remarking that in mathematics teachers' education, history of mathematics courses work in a similar way: they tend to be superficial and marginal.

It is not surprising that mathematicians regard, sympathetically and sometimes as valuable, activities carried out by non-experts such as the writing of textbooks. This is considered an acceptable mathematical activity because the product admits a mathematical evaluation. What is usually valued is the communication of the textualized knowledge—the adequate or simply acceptable transposition—especially when the texts do not question or venture into questions that are not up for debate (using Noss' phrase, see Noss 1994, p. 4).

We believe that a comparison of the different relations between mathematics on the one hand and philosophy, history of mathematics, and mathematics education on the other, is fruitful; the three of them analyze products of the mathematical activity. Past and current philosophies can question the essential foundations of mathematics, drawing limits to axiomatic systems. As is well known, there is only a relative impact of those findings on the mathematics activity. The community acknowledges the philosophical analyses, but they both continue treading their related but separate tracks. This bears description as a reasonable division of intellectual labor[1].

History works in very specific ways strongly based on its well-defined identity as a discipline. Instead, things are different with education/didactics; these are areas of research which draw from various other disciplines, including philosophy and history, which construct their own theories to analyze, reflect and elaborate frameworks on mathematical activity and productions (especially about their textualization[2] and communication). Thus, very close links are established with mathematics because, to analyze the teaching and learning of mathematics, it is necessary to analyze the mathematical objects to be taught and learnt.

History shows that mathematical rigor is built up successively, as if by layers, and this influences the impact of the analyses. There is a buffer between current rigor and previous "rigors;" some point inbetween can be accepted because one of the aims is successful teaching and learning. But the communication of the accepted knowledge, or rather the acceptable communication of accepted knowledge, is based on a consensus, and it is evaluated against the resulting standards. Although historically constructed, the consensus is uniquely dominant; it is resistant to challenges. We argue that education/didactics function in that critical zone, where the historicity of the consensus is emphasized and the communication of the accepted knowledge are looked upon critically. This is a far cry from the innocuous, at most reassuringly conformist, traditional didactics. Mathematics Education and Didactics of Mathematics are now much more critical disciplines than merely useful, prescriptive ones. As they move away from a practical, utilitarian conception, they become more conflictive. They are as such for both practicing mathematicians and practicing teachers. The studies now comprising the field (however well defined they may be) are exploring areas that were traditionally out of bounds. We propose a way to understand this.

The transference of the text of knowledge is, strictly speaking, the teaching of mathematics. The communication of knowledge is the concern of the mathematics community and it is the way in which the social project mentioned by Brousseau (1997) is interpreted (p. 24). The possibility of the transfer is out of the question. We contend that this absence of questioning is of interest and leads us to pose the analysis of the usual mathematical discourse as a probable obstacle to any reform. The current debates are showing the emergence of these preoccupations, and it is important to look at the possibility of transfer closely[3].

The fact that textualized knowledge is not permanent is not apparent from the discursive presentation, which, we argue, induces a belief—an image of the immutability of the text. As Noss (1994) points out, "there is an overwhelming temptation to

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view the subject matter as given, inevitable, natural" (p. 2). This seems to be a critical place to study the problems of the communication of textualized knowledge in general and of the teaching of mathematics in particular. We suggest that the immutable knowledge and the temporal text create a specific tension in mathematics education[4].

Chevallard (1991) in his discussion of didactical transposition describes one of the roles of a mathematics educator (didactician) as the study of the "change of object" that takes place between the object to be taught and the object actually taught. By asking the systematic question "is this the object whose teaching is intended?" we produce an epistemological crisis that allows the questioning of the transparency of the world of teaching "experienced before, either as a teacher or as a student" (p. 43). We suggest that the mathematician does not question the transparency either of mathematics or of the communication of the discourse and understanding.

This situation can be described with the idea of control through epistemological vigilance.. Two types of epistemological vigilance are at work on mathematical objects—one mathematical, the other didactical. An example can be drawn from the interactions between mathematics educators and mathematicians, as seen in the reports of the American Mathematical Society Resource Group (1998):

> Another example is the confusion—at least in our reading of the [NTCM] Standards—between calling for more mathematics in the preparation of teachers (which all of us enthusiastically support) and calling for changes in the nature of the mathematics taught to teachers (which some of us support to some extent, but about which others have deep concerns. (p. 271)

A variation of this conflict can be observed in the discussions of the epistemological status of certain individual constructions, as in the "teaching experiments" carried out by radical constructivists such as Confrey (1991). The ongoing debates about proofs in general mathematics education are equally relevant and better known.

Epistemological vigilance focused upon vigilance of the discourse produces some remarkable effects. It is, we argue, too rigid a framework within which to construct texts of knowledge designed for non-mathematics students (like mathematics for economists, for engineers or for mathematics teachers). On the other hand, this emphasis in the correct presentation produces conflicts that tend to be resolved by means of "superficial texts" that presumably can be understood by people who "don't really need mathematics" but rather have only to use it. And all this happens without breaking the rules of discourse. The underlying assumption is that there is no access to mathematics except through a very specific discourse.

The special characteristics of the teachers' case add difficulties to its study. People who teach mathematics are, in effect, a particular kind of mathematicians—those who introduce other people to (a part of) the world of mathematics. No matter the depth of their knowledge, their relation with the mathematical community is institutionally conditioned, and they have the ultimate responsibility to manage the double vigilance described. The implications are complex since, for instance, "many elementary school teachers are convinced that theory, "official knowledge", is a discourse, a convention, of relative or doubtful efficiency, to which one can make personal adjustments, or for which one can substitute "parallel" knowledge" (Brousseau, 1997, p. 241).

Texts and Dissociations

From a different point of view, given a mathematical problem or in general any mathematical task, the question: "What else must be known to complete the task, to solve the problem?" brings up the issue of the so-called prerequisites. Chevallard (1991) remarks that prerequisites can never be exhaustively explicitly stated; they work as preconstructs. Chevallard's observations at the beginning of Bourbaki's treatise about the note "Mode d'emploi" are quite to the point (p. 59). Because Bourbaki may be seen as the most notable and recent enterprise to present the whole of mathematics as a written discourse, it seems pertinent to this discussion.

The cognitivist approach to the analysis of student's learning behavior, provides an interesting perspective to some aspects of the question of mathematical discourse and education. Polanyi in his book "The Study of Man" (1959) makes a distinction between explicit and tacit knowledge. Greeno (1987) quotes him: "Explicit knowledge is "what is usually described as [simply] knowledge, as presented in written words or maps, or mathematical formulas", whereas "tacit knowledge [is] such as we have of something we are in the act of doing" (p. 62).

Now we notice that the "bureaucratic communication of knowledge" (Verret's very apt expression quoted by Chevallard (1991, p. 57)) requires a delimitation of knowledge into parts—a taxonomy. On the other hand, mathematical activity requires the inverse process—the melting of the delimitation and the dismantling of the taxonomy. "Knowledge in action" has a logic that rejects the limits imposed by certain forms of communication. It is clear that mathematics teachers are acquainted with the taxonomy, that is with the classification of explicit knowledge; but it is not so with the perception of the change of dynamics in mathematical activity in general.

Polanyi's distinction, we suggest, bears a parallel with the result of the process of "becoming implicit" which is concurrent with the other process, "becoming explicit through a discursive presentation" (see Chevallard, 1991, p. 60)

If we assume, although it is a simplification, that the production of knowledge is previous to its communication, then it is easy to understand that the logic of science is quite different from the logic of its communication, particularly in its teaching.

In this respect, as a reference, we can describe this as the antinomy of problem solving in the classroom: "no problems" versus "problems as the focus". We argue that this problem is negotiated through discourse and that both terms converge in the discourse.

Once an object of knowledge is chosen to be taught, it can be referred to as the object to be taught. This object, if it is to function in a teaching process, needs a teacher and a learner. Because the teacher is assumed to accept the object to teach, the learner in turn must accept to be instructed; the learner's stake in the didactical contract leads to the consideration of the object to learn. This is the object of knowledge chosen as object to teach but conditioned by the fact that it makes sense insofar as "it must be learnt". The object to learn has a logic different from the object to teach. The time of teaching, for instance, differs from the time of learning (in Chevallard's sense, it is non-chronological time): "The order of learning is non-isomorphic to the order of the presentation of knowledge, the learning of knowledge is not a copy of the text of knowledge" (1991, p. 63). It is as if it were implicitly said, "I will teach this but you must learn something else," where "this" is explicitly stated and "something else" is the responsibility of the learner.

Brousseau (1997) seems to refer to something closely related when he says:

> [...] we implicitly thought that learning situations were almost the only means by which knowledge is passed on to students. This idea arises from a rather suspect epistemological conception, as an empiricist idea of the construction of knowledge: the student placed in a well-chosen situation, should, on contact with a certain type of reality, construct for herself knowledge identical with the human knowledge of her time…. There is the idea that knowledge can be taught but that understanding is in the student's hand. (p. 237)

The negotiation through discourse takes place in the interplay between mathematical notions and paramathematical notions in conjunction with the role of the implicit prerequisites or protomathematical notions (Chevallard, 1991, p. 51). We can illustrate this when, in a course, the notion of function, introduced by the ostensible use of examples, is later defined as a particular relation (a set of ordered pairs). This is a shift where the paramathematical are no longer "functions" but "set theory". Similarly, in linear algebra courses it is sometimes considered that a relatively detailed, rigorous knowledge of polynomials is necessary to grasp of the notion of eigenvalue. Here the question of what works paramathematically and what mathematically arises again. The order of presentation of formal mathematics and the ingenuous, acritical belief in its use as the tool for communication has another easily identifiable effect: the excessive development of the prerequisites.

It is worth considering the common division of most mathematics university courses into two sections, one labeled "theoretical", the other "practical" (which includes problem solving and help sessions). There is a different contract in each case. In the theoretical case the discourse of presentation of mathematical notions and results is at work while in the practical an "empirical learning contract" seems to be established (Brousseau, 1996, p. 25). When solving problems formally related to the discourse of theory, students are supposed to engage in a process of straightforward acquisition of mathematical skills and problem solving strategies.

That students perceive this difference can be seen when they find it difficult to conceive both sections of the same domain as parts of a whole. The distinction between the formal presentation of mathematics on the one hand and mathematical activity, "doing mathematics" on the other, was pointed out by Polya. The serious difficulty for the learners to put the two discourses together is remarkable. Schoenfeld (1987) has described an example taken from geometry classes to illustrate the students' attitudes towards theorem and theorem proving (theory) and ruler and compass constructions (practice); from this he distinguishes between two behaviors that he called "mode of confirmation" and "mode of invention" (p. 196-197). These modes neatly describe two quite different perceptions of two different discourses.

Other examples can be easily observed. The presentation of integers as equivalence classes of ordered pairs of natural numbers is sometimes seen as proof of rigorous mathematical knowledge from a teacher's perspective; but clearly this presentation appears and works within a global perception of the formal mathematical discourse. Other processes of dissociation are usual with the Induction Principle. One is the difficulty to tell apart the principle as a fundamental property of the natural numbers and the application as method of proof. Another has to do with the confusion between the sum notation $\sum_{i=1}^{n}$ and the notion of induction. This is an example of "name substitution," and of a real "object substitution" underlying it.

It is from the perspective of communication of textualized knowledge that problems that concern mathematics education might be amenable to a sharper treatment, where the role of the usual discourse seems important. Both the discourses and the texts, because of their historicity, are relevant to "the visibility of meanings"[5]. Somehow it is around these questions that the entrenched belief of the importance of mathematics for a technological society might be confronted with the seemingly intractable teaching and learning problems, as well as with the equally entrenched and surprising belief that, somehow, some people will come up with the correct answer.

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Notes

[1] This assertion applies mainly to the "classical" philosophy of mathematics which is interested in the foundations of mathematics. But there are new studies which, not surprisingly, put an emphasis on the relationship between mathematics education and philosophy. See Ernest (1994) and Otte (1997).

[2] See Alagia (1998).

[3] It is interesting to consider observations such as those of the mathematician H. Wu that touch upon the discourse, *when talking to his fellow mathematicians*. See Notices Americal Mathematical Society, 44 (7), June-July 1997.

[4] This might be a source of differences between education in natural sciences and mathematics education. Also, if this is valid, it supports the critique about the conceptual fragility of more traditional versions of didactics, not centered around the object of knowledge.

[5] Visibility of mathematical meanings is a concept used by Noss (1994, 1996, 1998).

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awarded stipends, university credit, or SDU's.

Lastly, the PRIME project could provide additional access to materials and publications that support teacher development, such as the NCTM publications, and be an advocate for financial help toward NCTM membership and attendance of conferences. Access to Internet facilities, including personal e-mail accounts, use of distance learning facilities, and library membership to the University of Georgia could be eventually granted.

Schools participating in the project could enjoy benefits, such as graduate assistants to work with teachers with such efforts as computer-oriented lessons, tutorials, and any other activities involving mathematics teaching. Other possibilities are the use of university resources including software, class sets of graphing calculators, laptops, manipulative materials, and supplementary texts, and access to university personnel to assist in the teaching of certain aspects of the curriculum.

The 1998-1999 PRIME mentor teachers were required to attend a three hour workshop in the fall which included the following:

- highlights of teacher evaluation instrument (e. g. GTEP),
- highlights of the NCTM *Standards*,
- highlights of the Teacher Support Specialist programs,
- highlights of the UGA preservice teacher program,
- general strategies from successful practitioners,
- and a list of recommended publications and articles.

Three additional academic meetings were scheduled that provided an opportunity for PRIME mentor teachers and UGA faculty to discuss the project, the preservice teacher education program and other topics of the mentors' choice. Teachers who wish to be selected as PRIME mentors for 1999-2000 must hold a current Georgia Professional Certificate in secondary mathematics with two years of teaching experience in the field of mathematics, demonstrate performance as a superior teacher, be committed to professional growth, and exhibit excellent conferencing and supervisory skills.

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