

Technology-Enabled Pedagogy as an Informal Link Between Finite and Infinite Concepts in Secondary Mathematics

Sergei Abramovich
Anderson Norton

This paper is a reflection on activities designed for computer-enhanced in-service training of high school mathematics teachers. The goal of these activities is two-fold: to promote advanced mathematical thinking, and to introduce new uses of existing technology tools. The authors suggest using jointly a computer-based graphing calculator, a dynamic geometry program, and a spreadsheet program in exploring linear algebraic equations to bridge finite and infinite mathematics structures. A linear algebraic equation may be introduced in the context of mathematically modeling a uniform movement. In turn, in the technology-rich environment, solving a linear algebraic equation can be introduced through the method of iterations that ultimately leads to the discussion of infinite processes. This opens a window on the complexity of infinite structures, which includes the convergent, divergent and cyclic behavior of iterative sequences. Computer-enhanced representations of infinite processes include bisector-bounded staircases and cobweb diagrams, animated pencils of straight lines, and iterations of sequences both in numeric and graphic notations. Finally, by exploring a piece-wise linear recursion, one can arrive at the frontiers of mathematical knowledge, and, in developing mature concepts of convergence, divergence and cycles, experience how chaos—a remarkable phenomena of modern mathematics—can arise in dynamic systems of a surprisingly simple form.

Sergei Abramovich is an associate professor in the Department of Teacher Education, State University of New York at Potsdam, where he teaches undergraduate and graduate mathematics education courses. His research interests include constructing computational learning environments for K-12 mathematics education and exploring effective ways of using computers as tools for conceptual development. His email address is abramovs@potsdam.edu

Anderson Norton is a Doctoral Student in the Department of Mathematics Education at the University of Georgia. He is interested in the nature of students' mathematical conjectures and creative uses of technology. You can find some of his work on his web page: <http://jwilson.coe.uga.edu/EMT668/EMT668.folders.F97/Norton/Anderson.html>. His email address is anorton@coe.uga.edu

Introduction

Modern technology tools offer a dynamic environment in which to visualize a complexity of infinite processes. A first experience with such processes in high school may emerge from the study of simple iterative sequences in connection with the concept of limit. This study can be grounded in using instructional computing to extend students' knowledge of properties of functions and algebraic equations. One computational approach to the concept of the limit of a sequence is to use jointly a graphing calculator such as GC2.2 [GC] (Robbins & Apple Computer, Inc., 1991), a dynamic geometry program such as Geometer's Sketchpad [GSP] (Jackiw, 1998), and a spreadsheet such as Microsoft Excel (Microsoft Excel 98, 1998). Using these computer-based tools can offer a dynamic way of exploring finite processes and enhance the development of advanced mathematical thinking dealing with infinite processes (Tall, 1992). Such an informal, explorative approach to the concept of the limit of a sequence, from algebraic, geometric, graphical and numerical perspectives, is a method long advocated by the National Council of Teachers of Mathematics (NCTM, 1989, 1991, 2000).

Solving algebraic equations, particularly linear equations, is an example of a finite process with which high school students are familiar. Contemporary mathematics pedagogy strives to engage learners in activities conducive to making connections between new ideas and previous experience. With such a focus on connections in mind, solving linear equations can be introduced to high school students through the method of iterations, something that ultimately involves the discussion of infinite processes. This paper demonstrates how the joint use of various technology tools can promote mature concepts of infinite processes and, through exploring a word problem on uniform movement, open a window on limits, cycles, and chaos. An alternative discussion of the topic and its extension to non-linear models through a tool kit approach can be

found elsewhere (Abramovich & Brantlinger, 1998; Abramovich, Brantlinger, & Norton, 1999).

The Situation: A Car and a Bicyclist

To begin, consider the following problematic situation, which is relevant to a typical high school algebra curriculum:

From a specific moment, the driver of a car will catch up with a bicyclist at rest in 4.5 hours. However, if the bicyclist is moving at that moment, at a speed that is one quarter that of the car, when will the car catch up with the bicyclist?

This is a standard problem on a uniform movement and its solution can be found using a well-known relation that links distance, time, and velocity. Let x be the time sought. There are at least two different ways to reduce the problem to a linear algebraic equation in variable x . While each such equation would have the same solution, the reasoning that leads to the solution is very important for students developing ideas about infinite processes.

First, if one concentrates on the original distance between the vehicles, this distance can be represented as both $V \cdot x - 0.25V \cdot x$ and $V \cdot 4.5$, where V denotes the car's velocity. Therefore, one may derive the equality, $V \cdot x - 0.25V \cdot x = V \cdot 4.5$. This yields $3x = 18$, whence $x = 6$. So, it takes the car 6 hours to catch up with the bicyclist.

Second, if one concentrates on the total distance traveled, this distance can be combined as the sum of $4.5V$ and $0.25x \cdot V$. This is due to the fact that the car requires 4.5 hours to reach the starting point of the bicyclist and the additional riding time, x , of the biker to cover the remaining distance. Note that the distance is also given by the product, $x \cdot V$, of the total time and the car's velocity. This yields $0.25xV + 4.5V = xV$ or, factoring out the non-zero velocity,

$$0.25x + 4.5 = x \quad (1)$$

as another model for the problem. Of course, the solution to equation (1) is $x = 6$.

Exploring the Situation with the Graphing Calculator

One can use equation (1) as more than another algebraic way of solving the problem. It also allows for a graphical solution based on the construction of the

graphs of both sides of the equation. A graphical solution makes it possible to establish links between one's experience with functions and graphs and the development of an infinite mode in mathematical thinking. The use of the GC makes it possible to illustrate a graphical approach.

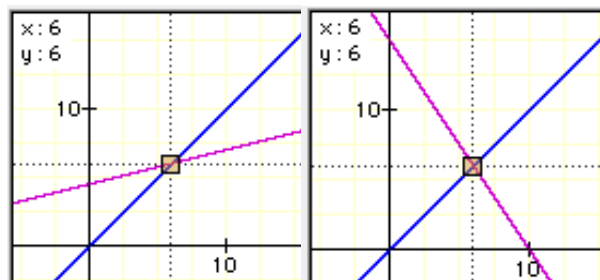


Figure 1. Graphical solutions of equations (1) and (3).

The left-hand side of Figure 1 displays the graphs of the left and right-sides of equation (1) set equal to y . The second of these two graphs is the bisector of the first quadrant angle. The inclusion of this graph is necessary for a geometric exploration of bisector-bounded staircases. But, first, consider the implications and limitations of using graphical and geometric representations of algebraic solutions.

The straight lines in Figure 1 meet at the point whose x -value is a root of equation (1). Although the visual offered by computer graphics (as well as computer coordinates) indicate that this root is $x = 6$, the apparent point of concurrency of the straight lines can only be determined within a certain degree of accuracy determined by the software's set up. Indeed, one could hardly point at $x = 6$ exactly, for the number 6, like any other number, is an abstraction; its geometric image (a point) is just a representation of the number rather than the number itself. Therefore the point that one recognizes as a common point for the two straight lines is an approximate geometric representation for the solution of equation (1). The idea is quite subtle for the present example which has a whole number solution, but similar graphical examples with irrational solutions may be used for emphasis. In any case, the use of the zoom-in feature of the GC enables one to refine any point chosen as an approximation to the solution. Continuing this technique of "zooming-in" on the refined point allows the user to approach the exact solution. Though the exact solution can never be

reached, the potentially infinite process of refinements sets the tone for further exploration.

In addition, a capability of the GC to graph functions depending on a varying parameter makes it possible to visualize dynamically how straight lines with different slopes, yet described by equivalent equations, intercept the bisector at the point (6,6). This can be done by animating slope m in the pencil of straight lines described by the equation

$$y = m(x - 6) + 6 \quad (2)$$

in the range from -3 to 3. For example, the right-hand side of Figure 1 shows a graphic solution to the equation

$$x = -1.5(x - 6) + 6 \quad (3)$$

Once the equivalence of the graphs of (1) and (3) is established empirically in a GC environment, the equivalence of their equations can be demonstrated algebraically. Equation (3) will be used later in this paper.

Constructing a Bisector-Bounded Staircase

The "zooming-in" explorations in the computer graphics environment can prompt a geometric method of approaching the exact solution $x = 6$ no matter how far the starting point is from the exact solution. Take any starting point on the x -axis and mark it as x_1 . Draw a vertical line from x_1 to $(x_1, 0.25x_1 + 4.5) = (x_1, x_2)$, a point which belongs to the straight line

$$y = 0.25x + 4.5 \quad (4)$$

Draw a horizontal line from the point (x_1, x_2) to the point (x_2, x_2) on the bisector $y = x$. Draw a vertical line from the point (x_2, x_2) to the point $(x_2, 0.25x_2 + 4.5) = (x_2, x_3)$ on straight line (4). Draw a horizontal line from the point (x_2, x_3) to the point (x_3, x_3) on the bisector $y = x$ and so on. The resulting construction, made possible with the use of GSP, shows a kind of a staircase bounded by the bisector of the first quadrant angle and the graph of equation (4) (Figure 2). Indeed, this staircase becomes trapped between the bisector $y = x$ and straight line (4), so that the point $x = 6$ becomes its point of attraction. In other words, starting from an arbitrary point x_1 , the iterations of the sequence

$$x_n = 0.25x_{n-1} + 4.5 \quad (5)$$

approach the root of equation (1).

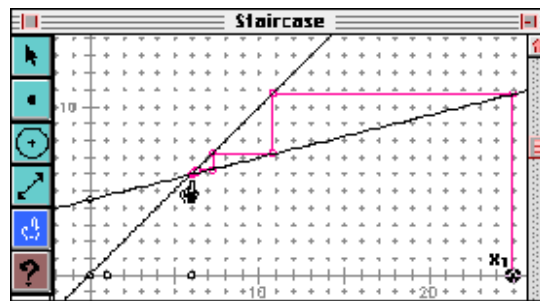


Figure 2. A GSP depiction of the bisector-bounded staircase.

The construction of such a bisector-bounded staircase can be done by playing a specially-created GSP script on the following five givens: a starting point, the x -axis, a straight line, the y -axis, and the bisector. The "dynamic" attribute of GSP—the possibility of altering various components of the givens and, by playing a script, observing a resulting construction—may also be useful in determining the behavior of iterations in various situations. Particularly, by replacing equation (4) with several equivalent forms with different slopes, one can examine the effects of altering the slope of the straight line. A summary of the results of computer explorations will be provided in the next section. One case, however, will be examined in detail.

Exploring Sequences Through the Variation of a Slope

Linear equation (4) gives birth to an iterative sequence (5) which, in turn, allows one to observe the case of monotone convergence. Each iteration of this sequence, or, in other words, each step of the staircase yields a value that approaches the solution to the original problem from one direction (depending on an initial value x_1). In fact, through explorations in a GSP environment, it can be determined that for any slope m such that $0 < m < 1$ the bisector-bounded staircase corresponding to equation (2) represents geometrically a monotone convergence of an iterative sequence. Such sequences can then be explored by using a spreadsheet. As described elsewhere (Abramovich & Levin, 1994), a spreadsheet environment is particularly amenable to the numerical study of the epsilon-N definition of the limit of a sequence.

Various other intervals of m in equation (2) reveal qualitatively different forms of convergence and divergence. These forms include oscillating convergence

($-1 < m < 0$), 2-cycles ($m = -1$), monotone divergence ($m > 1$) and oscillating divergence ($m < -1$). Note the omission of the uninteresting cases of a 1-cycle ($m = 0, m = 1$). Once again, a better understanding of each of these forms may be attained through examining numeric and graphic representations of corresponding sequences offered by a spreadsheet. As an example, the case of oscillating divergence is examined below in the multiple-application environment.

This particularly interesting case can be introduced through constructing a staircase for equation (3), already known from the GC setting. This can be done by manually changing a slope of a straight line in a GSP environment until such alterations create a graph for the equation (at least approximately). One can then iterate the sequence

$$x_{n+1} = -1.5x_n + 15 \quad (6)$$

by using a spreadsheet. Finally, sequence (6) can be explored with the GC.

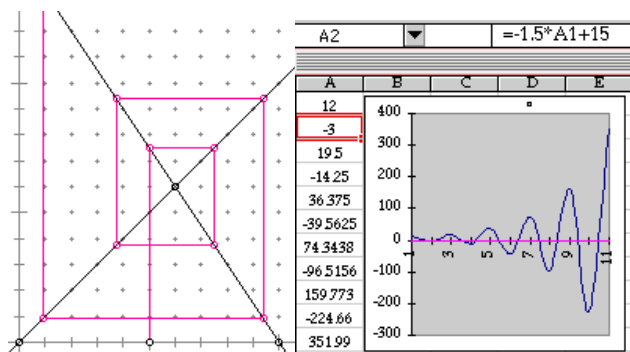


Figure 3. Sequence (6) yields bi-directional growth (left) and divergent oscillations (right).

Explorations of slope caused by click-and-drag alterations in a GSP environment suggest that any slope $m < -1$ brings the case of oscillating divergence. More specifically, in building a new staircase, one may plot two points, say $(0,15)$ and $(6,6)$, to construct the straight line described by the equation $y = -1.5x + 15$. One then constructs the bisector, chooses a starting point, and plays the GSP script mentioned above on five geometric objects. Figure 3 shows a spiral staircase (a cobweb) that seems to run away from the point $(6,6)$. This diagram demonstrates the bi-directional growth of iterative sequence (6).

The use of a spreadsheet (Figure 3) allows for a numeric representation of sequence (6) by first selecting

an initial value x_1 and placing it in the top cell of column A. The entry of the next cell, displayed in the formula bar, is determined by sequence (6). This is copied down column A yielding consecutive iterations of sequence (6). The modeling data exhibits the case of oscillating divergence. Spreadsheet charts provide a graphical illustration of this phenomenon.

Cycles: A Special Case of Oscillations

When the slope of a line for a given equation is negative and less than 1 in absolute value, one should expect a phenomenon similar to the last case (oscillations). This time, however, a staircase or, better, a cobweb is being squeezed in toward the point of intersection of the new line and the bisector (oscillating convergence). The question arises, then, as to what kind of a cobweb would be generated by a straight line with a slope that separates these two intervals—the intervals $m < -1$ and $-1 < m < 0$ (namely, $m = -1$). A plausible conjecture is that such a cobweb neither approaches nor escapes the point of intersection. One can examine such a case with yet another equivalent form of equation (1), $x = -x + 12$. In fact, the resulting sequence $x_{n+1} = -x_n + 12$ yields a cycle as displayed in Figure 4.

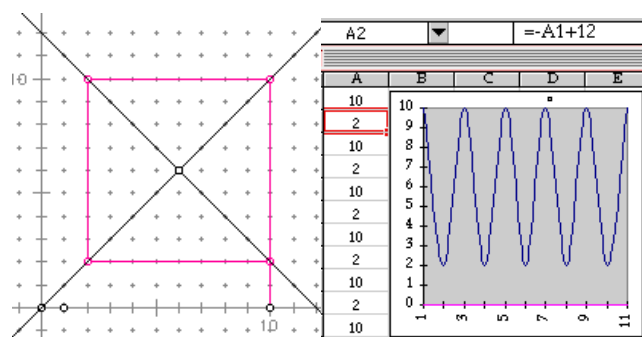


Figure 4. The new sequence, with $m = -1$, yields a cycle (left) whose periodic behavior is graphically illustrated (right).

A cycle is a pattern that occurs with repetitions. The period (duration) of a cycle is the length of time (measured in iterations) between repetitions. The cycle in question has a period of 2, and it is worth noting that no cycles of duration greater than 2 occur through the iteration of linear functions. In fact, in exploring the cases for various slopes, one finds that cycles of any non-trivial kind (cycles with period greater than 1) occur only for $m = -1$. Since all cases of bisector-

bounded staircases with $m = -1$ are geometrically equivalent to one another, the claim about 2-cycles follows from a single case. In a general setting, cycles become more interesting and more important because of the patterns and predictability they offer. Cyclic sequences exhibit predictable behavior in that their values form a fixed and bounded pattern. In contrast, the case that follows deals with the bounded yet unpredictable behavior of a simple iterative sequence.

Moving Toward Mathematical Frontiers

The use of graphs of linear functions as a vehicle for generating knowledge about infinite processes also prompts the consideration of a piece-wise linear function whose graph resembles a tent. Through exploring such a function in GSP and spreadsheet settings, one comes across an extremely interesting type of sequential behavior called chaos. The existence of chaos in dynamic systems was already known to Poincaré in the 19th century; however, the systematic study of chaos began only about three decades ago. Mathematically, chaos is defined as an aperiodic bounded dynamic in a deterministic system with sensitive dependence on initial conditions (Bruce, Giblin, & Rippon, 1990).

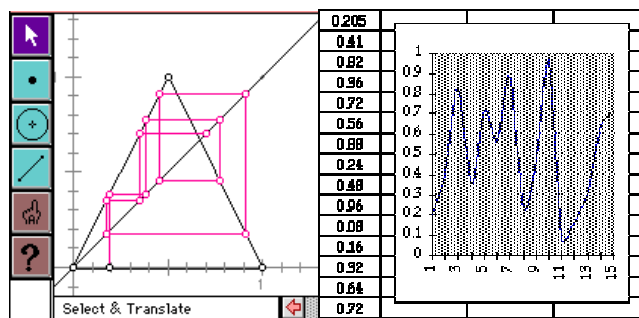


Figure 5. Sequence (7) yields a cobweb of chaos (left) and aperiodic bounded dynamic (right).

To understand this definition, consider the following iterative sequence, commonly referred to by professional mathematicians as the tent map:

$$x_{n+1} = 1 - |2x_n - 1| \quad (7)$$

Equation (7), considered simultaneously as a formula and sequence, can be used to exemplify each aspect of the definition of chaos. GSP and a spreadsheet can be used jointly to illustrate the point (Figure 5).

First, a deterministic system is one that allows for a definitive (rather than random) rule, like formula (7), that governs the behavior of a system. An aperiodic dynamic, which is exhibited by sequence (7), means that no value is ever repeated. In turn, bounded dynamic means that a staircase (or the “orbit” of any initial point) remains in a definite bounded region for arbitrarily large numbers of iterations, as implied by an inspection of Figure 5. Finally, as the spreadsheet template of Figure 6 shows, sensitive dependence on initial conditions means that two starting points, that are initially close to each other, will gradually drift apart as the number of iterations of sequence (7) grows large.

| A | B | C | D | E | F |
|---------|---------|---|---|---|---|
| 0.30512 | 0.30513 | | | | |
| 0.61024 | 0.61026 | | | | |
| 0.77952 | 0.77948 | | | | |
| 0.44096 | 0.44104 | | | | |
| 0.88192 | 0.88208 | | | | |
| 0.23616 | 0.23584 | | | | |

Figure 6. Sensitive dependence of sequence (7) on initial conditions.

Conclusion

Mathematics education research has shown that teaching the concept of the limit of a sequence through a limited set of examples could cause misconceptions (Tall, 1992). For example, students who have only seen sequences which converge monotonously, are likely to dismiss the convergence of a dampening sine function. Technology-enabled pedagogy, as suggested in this paper, makes it possible to demonstrate sequences with different behaviors using visualization and multiple representation of infinite processes. Sequences with qualitatively different behaviors can be generated, examined, and compared to each other using a multiple-application environment. Mathematical visualization provided by newer tools of technology can enhance the development of concepts of convergence/divergence and allow for clearer understanding of more advanced ideas in calculus and analysis.

The approach taken in this paper is consistent with an informal and explorative discourse on infinitesimal aspects of mathematics, as advocated by the NCTM Standards (NCTM 1989). In addition, the prescribed investigations offer mathematics teachers a means by which to introduce high school students to chaos—a

mathematical science which is at the frontier of knowledge about processes occurring in nature. The authors have demonstrated that the phenomenon of chaos could follow in a natural progression from the discussion of bisector-bounded staircases. Throughout such a discussion, there are occasions for students to gain insight on the complexity of the concept of infinity. Perhaps the most notable feature of this approach is that, through appropriate use of tools of technology, one can generate the entire spectrum of qualitatively different convergence/divergence behaviors for simple infinite processes, all proceeding from a linear algebraic problem on a uniform movement.

References

Abramovich, S., & Brantlinger, A. (1998). Tool kit approach to using spreadsheets in secondary mathematics teacher education. In S. McNeil, J.D. Price, S. Boger-Mehall, B. Robin, J. Willis (Eds.), *Technology and teacher education annual, 1998*. Charlottesville, VA: AACE.

Abramovich, S., Brantlinger, A., & Norton, A. (1999). Exploring quadratic-like sequences through a tool kit approach. In G. Goodell (Ed.), *Proceedings of the 11th Annual International Conference on Technology in Collegiate Mathematics*. Reading,

MA: Addison Wesley Longman.

Abramovich, S., & Levin, I. (1994). Spreadsheets in teaching and learning topics in calculus. *International Journal of Mathematical Education in Science and Technology*, 25, 263-275.

Bruce, J.W., Giblin, P.J., & Rippon, P.J. (1990). *Microcomputers and mathematics*. Cambridge: Cambridge University Press.

Jackiw, N. (1991). *Geometer's sketchpad* [Computer software]. Berkeley, CA: Key Curriculum Press.

Microsoft Excel 98 [Computer software]. (1998). Redmond, WA: Microsoft Corporation.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

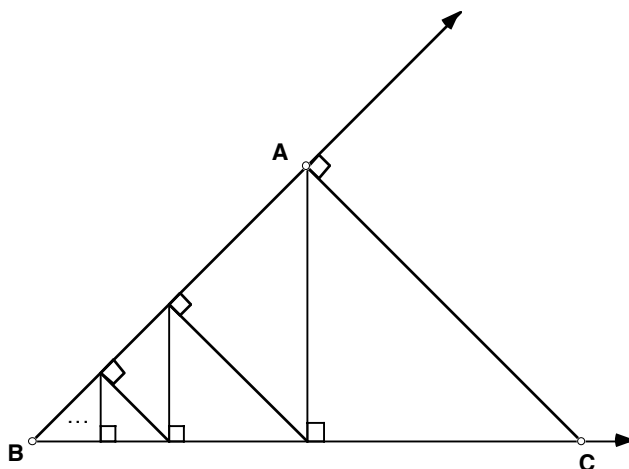
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Robbins, G.M., & Apple Computer, Inc. (1998). *Graphing calculator – 2.2* [Computer software]. Woodside, CA: Pacific Tech.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495-511). New York: Macmillan.

From the Archives...

A spider has woven a web beginning with segment CA and then zigzagging between rays BA and BC as shown in the diagram below. Suppose that $BC = 1$ and that $m\angle ABC = 45^\circ$. Calculate the length of the zigzag path from C to B .



Possible variation: Vary the measure of $\angle ABC$.

This problem first appeared in the first issue of TME (Summer 1990). It was provided by Dr. Jim Wilson and was part of an ongoing feature called From Dr. Wilson's Notebook...