Children's Number Sequences: An Explanation of Steffe's Constructs and an Extrapolation to Rational Numbers of Arithmetic

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In regarding young children as mathematicians we need to pay close attention to how children develop their mathematical thinking. It would be misleading to assume that young children think mathematically in the same way as adult mathematicians. Children have to develop their mathematical structures and their ways and means of operating mathematically. These structures and operations have to be constructed from the child's own activities. They cannot be given to the child "ready made." One of the most fundamental mathematical structures that a child develops early on in life is that of a *number sequence.* The basic activity that leads to the construction of a number sequence is that of *counting.* The activity of counting, however, does not occur all at once for a child. Steffe, von Glasersfeld,. Richards and Cobb (1983) indicated in their research of young children's counting activities that early counting progresses through five distinct types of activity, from the counting of perceptual unit items to the counting of abstract unit items. They liken this progression to Piaget's concept of "object permanence" (p.117).

A child's number sequence, however, is not a static structure. It also progresses through several developmental changes that are brought about through adaptations in the children's counting activities as they encounter more complex numerical situations. Following their work on children's counting types, Steffe and Cobb (1988) further developed the notion of children's abstract number sequences from their teaching experiments with first and second grade children. They described the development of three successive number sequences: the Initial Number Sequence (INS), the Tacitly Nested Number Sequence (TNS) and the Explicitly Nested Number Sequence (ENS). What follows are my attempts to clarify my own thinking about these hypothetical number sequences and an extrapolation from these sequences to a Generalized Number Sequence (GNS) and subsequently to the Rational Numbers of Arithmetic (RNA).

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Key Psychological Aspects of Number Sequences

While it is beyond the scope of this paper to attempt a thorough psychological discussion of learning, the following key psychological aspects of number sequences are important for making sense out of the developmental sequence presented in this paper. The following brief definitions are provided merely as an orientation for the reader rather than as a thorough explanation of terms. It is hoped that their meanings will become clearer through the examples given in the descriptions of the number sequences.

Schemes

Number Sequences are mental constructs-schemes. According to von Glasersfeld (1980) a scheme consists of three parts: (1) an assimilatory structure or recognition template through which a child recognizes a situation as relevant to this particular scheme, (2) an action or operation associated with the situation, (3) a result of the action or operation. All three parts of the scheme are related to an overriding goal of the scheme.

Interiorized Activity

The possible operations associated with a number sequence emerge from the *interiorization of activities* that children engage in through applications of their prior number sequence. That is to say that children may do things in action first what they are not yet able to do mentally. The interiorization of activity is a process of reflective abstraction (von Glasersfeld, 1995). The activity is first internalized through mental imagery; the child can mentally re-present the activity. This mental re-presentation still carries with it contextual details of the activity. The activity becomes interiorized through further abstraction of these internalized re-presentations whereby they are stripped of their contextual details. For instance, a child could first internalize the act of counting six cubes by forming mental records of the counting acts that would include the child's pointing actions and images of the cubes. If the cubes were then covered and the child asked to tell how many cubes were covered the child might re-enact his counting acts over the cover, mentally projecting images of the cubes that he is counting from the records of the prior activity. This *reprocessing* of his records of prior experience can lead to a further abstraction, whereby the pointing act alone can now stand-in for counting any objects. Eventually the *result* of such figurative counting will, itself, stand-in for the counting acts. It is at this point that we would say that the child's counting acts have been interiorized.

Re-interiorization

The higher-level number sequences are constructed through a process of *re-interiorization* of the lower level number sequence. Re-interiorization is achieved through *recursively* applying the operations of a scheme to the results of a scheme. For instance, when a child can take the result of counting a set of objects as something to be counted (a necessary action in order to establish the numerocity of a set of sets).

Iterable and Composite Units

The construction of an *iterable unit* is key to the development of multiplicative schemes, and is the result of *reversible* operations (One iterated five times produces one five, that can be partitioned into five ones). *Reversible operations* are established through *recursive* applications of the activity of a scheme to the results of the scheme. Iterable units are the building blocks for *composite units*—that is a unit composed of unit items. For instance five one's can be taken as one five.

Folding Back

The operations of lower order number sequences reappear in higher order sequences but are applied to the more complex unit items. This may be similar to Kieren and Pirie's (1991) notion of *folding back* in their *recursive* model of mathematical thinking.

With these psychological aspects of number sequences in mind, I now present brief descriptions of the number sequences developed by Steffe et al., starting with their description of the counting schemes of pre-numerical children and working through to the construction of the Generalized Number Sequence.

Pre-Numerical Counting Schemes

Pre-numerical children may have a "number-word sequence" that they use in conjunction with pointing acts in their attempts to "count" the items of a collection; however, the result of their counting acts does not signify the cardinality of the collection—it may only signify closure of their attempts to make definite the indefiniteness of a plurality (the "manyness" of the collection). For instance, these children may "count" a collection using the number-word sequence "one, two ... five" and then when asked to count again the same collection will use the number-word sequence "one, two ... six." The children may have pointed to an object twice in this second sequence or may not have coordinated their pointing acts with their utterances. These children may experience no perturbation in arriving at different number words as the result of "counting."

Perceptual and Figurative Counters

Pre-numerical children can learn to coordinate their pointing acts with their number-word sequence. When they see this coordination as a *necessity* for finding "how many" objects they have, the result of counting is an *extensive* meaning for their number words: counting a collection "one, two ... five" results in the child having "five" things "out there." Such children are regarded as *perceptual* counters. The objects need to remain in their perceptual field in order for them to be able to count them. If children who can count perceptually end on a different number word when counting the same collection a second time, they might experience perturbation. This perturbation may stimulate them to organize their counting activity to make sure that they only count each object once (e.g. aligning the objects before counting, or moving each object away from the remaining objects with each count).

Perceptual counters can learn to count things that *stand in for* the objects to be counted. For instance, having just counted a collection of objects, and asked to tell how many objects they have after the counted objects have been covered, some children will point to the cover and count *imagined* objects, or they may develop finger patterns for two, three, four or five objects and count these finger patterns. They may even draw pictures of the objects on paper, or make tally marks for each object as they count. Such children have *internalized* their countable items and have become *figurative counters.*

For both perceptual and figurative counters the result of counting is still "out there;" it is not yet *interiorized.* They cannot take that result of counting and use it as input for a second count. For instance, having counted a collection of five objects, when more objects are added to their counted collection, these children will *count all* of the collection to find how many they now have. This necessity to *count all* distinguishes the pre-numerical child from the numerical child.

An Initial Number Sequence (INS)

Children who need to *count all* can be encouraged to mentally re-present their counting acts by covering the prior counted collection when more objects are added. These children will often attempt to visualize the objects under the cover, pointing at the cover while recounting the covered objects. Such activities will help them to at first *internalize* (make mental representations of) their counting acts, and eventually *interiorize* the results of those counting acts: the result of counting a collection is not only "out there" but is also symbolized mentally by the *interiorized* number word, that now carries with it the records of the experience of counting. The child has constructed an abstract number sequence in the sense that each element of the child's number word sequence can now stand for the sub-sequence of counting acts that results in that number word. For instance, the number "five" stands for the counting sequence "one, two ... five." Children who have made this accommodation in their number word sequence can now *count-on* when more objects are added to a previously counted collection. Thus, having just counted out five objects, when given three more, such children will say "five ... six, seven, eight," pointing to the new objects in turn. The "five" now stands for having counted the first five objects. The number five is now a *numerical composite:* it can stand for a collection of five items. The *numerosity* of a collection is finally established.

With an INS the interiorized records can only be used to symbolize the *results* of counting acts; they cannot yet be used as *input* for counting acts. Children with an INS may even be able to use numerical composites as counting items, thus generating "counting by" actions. For example, children may count a large collection by twos (because it is quicker) but will not be able to tell you how many times they counted by twos. The question "How many twos?" doesn't make sense yet because the "two" stands for two things in a collection (a numerical composite), not one countable item (a composite unit item). Similarly, children may learn to count by fives or tens but not be able to keep track of their "counting-by" acts.

A Tacitly-Nested Number Sequence (TNS)

By adding further constraints to children's counting situations (such as covering the objects that were added, as well as the ones already counted) children with an INS will need to develop ways of keeping track of their counting-on (and counting-by) actions. Many children will use fingers or pointing acts to represent the covered items. This act of representing what is to be *counted-on* is a first step in using items of their INS as input for further counting acts. At first they may use finger patterns to stand-in for the amount to be counted-on and simply use each finger in the pattern as a *counted item* as they continue counting using their INS. When the amount to be counted-on is too large (e.g. greater than ten) for the child to use a finger pattern, the child needs to develop some other way of keeping track of their counting acts as they count-on. For instance, having counted a pile of 12 counters into a cup and 15 counters into another cup, in trying to find how many counters in both cups together, the child may start to count on from 12, raising a finger for each count: "13 (one finger), 14 (two fingers), 15 (three fingers) ... 22 (ten fingers raised)." At this point the child may lower all raised fingers and continue counting "23 (one finger), 24 (two fingers)…." If the child stops after all five fingers of one hand are raised then the child probably used a pattern of "all ten fingers plus one hand" for 15. However, many children will not have developed such a pattern with only the INS and will not know when to stop counting-on. This causes a perturbation that the child needs to neutralize. They have to develop a means of keeping track of counting-on 15 more from 12. This forces a reprocessing of their INS that results in some form of double-counting. The abstract unit items of their INS become *countable items.* In effect they use their number sequence from 1 to 15 in a novel way–to mark off a segment of their number sequence from 12 onwards. They may say something like the following while raising their fingers: "13 is one, 14 is two … 27 is 15." In reprocessing a segment of their number sequence in this way they have made a tacit nesting of the sequence from 1 to 15 within the sequence from 1 to 27. This constitutes a re-interiorization of their INS that Steffe calls the Tacitly Nested Number Sequence (TNS). The elements of the INS were interiorized counting acts. The elements of the TNS are now *countable abstract unit items.* The act of reprocessing one through 15 as countable items also creates a composite structure. The result (the nested interval from 13 through 27) has a numerocity of 15. There is an awareness of "15" now as a composite whole, not just the interiorized result of counting from one through 15. In essence, the child can now *unite* the results of counting. Thus a collection of five items, when counted, can now be taken as *one thing: a composite unit item.*

For instance, when counting-by a composite unit (rather than a numerical composite) children with a TNS can keep track of how many times they have counted by two or by five. The question "How many twos?" now makes sense because the two can be taken as one thing and can be used to *segment* a sequence into an unknown number of intervals of two items. Thus children who have constructed a TNS may be able to find how many groups of two they could make out of twelve covered items by counting 1, 2; 3, 4; 5, 6; ... 11, 12 while keeping track of the number of paired counting acts. Alternatively, they may count by "twos" up to twelve (2, 4, 6, 8, 10, 12), keeping track of how many times they counted. This "keeping track" involves coordination of unit items at the two levels of their TNS. This coordination, however, can only take place in activity because the nested intervals have to be produced (they are not yet explicit). Thus *enactive* multiplicative schemes become possible constructs for children with a TNS but are not possible constructs for children with only an INS.

An Explicitly-Nested Number Sequence (ENS)

Although children with a TNS can begin to construct multiplicative structures during their activities, they are limited by the *tacit* nature of their nested number sequence. By this I mean that the nested intervals can be used to *segment* the number sequence in the act of counting. The segmented sequence (e.g. the "six twos" in the number sequence from 1 to 12) is a *result* of applying the operations of a TNS. It cannot yet be taken as a given. However, to establish abstract multiplicative schemes (multiplicative reasoning) children need to be able to *start* with "six twos" as an assimilated structure. Being able to assimilate such structures requires yet another re-interiorization of the child's number sequence. *Each element* of a number sequence needs to be *an abstract composite unit item* nested within the number sequence. What I mean by this is that an element of a number sequence, e.g. the number six, contains the records of counting from 1 to 6 as abstract unit items. The number six can be regarded as "1" six times as well as the result of counting the first six elements of a number sequence. One way to think of this is that each element of the number sequence is a bar of a frequency bar graph, containing its own number of units. (I do not mean to imply that this is how children imagine the elements of their ENS!). A critical step in this re-interiorization process is the establishment of an abstract unit item "one" as an *iterable* unit. The iterable one is a product of repeatedly applying the "one more item" operation when double counting. Each time the child counts-on they create one more item in the sub-sequence (or nested interval). It is the re-interiorization of this "one-more item" that results in an awareness that, having counted-on six more items, these six items are six ones.

With an iterable unit item a child can engage in part-to-whole reasoning. The number six can be thought of as containing six ones, any number of which can be disembedded from (pulled out of) the composite unit of six *without destroying the unit six*. Three is contained in six and can be compared to the unit six. An iterable one is the product of *reversible operations.* Not only does "six" consist of six "ones" but any one of those six "ones" can be iterated to recreate the composite unit of six.

As an example of how children with an ENS might act differently from children with only a TNS, consider the following problem: 1+1+1+1+1. Children with only a TNS will probably solve this problem by successive additions: 1+1 is 2, 2+1 is 3, 3+1 is 4, 4+1 is 5. They have to *generate* the nested sums, whereas for children with an ENS the nested sums are implicit and they would simply see the problem as five ones, and know that five ones are the same as one five. Their number concept for five is reversible.

Steffe (1994) provides the example of a child with an ENS, Johanna, who could operate with composite units in a way analogous to the way a child with a TNS could operate on abstract unit items. Johanna could solve counting-on problems with units of units of three, keeping track of the number of composite units of three as she was counting on with them. She had produced four rows of three from 12 blocks, and was asked to find out how many more rows of three she could make if she had 27 blocks altogether. She was able to apply her double counting coordinations to two different unit types, feeding the results of the first coordination (five rows of three from the complement of 12 in 27) back into her units-coordination scheme to count-on from the previously established unit of four rows of three, to arrive at a total of nine rows of three. This *recursive* application of her units coordinating scheme was possible because the elements of her TNS were re-interiorized, providing her with two levels of abstract units (unit items and composite units) as material for operating. The process of re-interiorization itself is a *recursive* process, and it is this recursive process that produces the reversible number concept.

Another example involves the construction of partitive and quotitive division operations. Children with a TNS, when asked how many strings of four beads they could make out of a box of 24 beads (quotitive division), would count by fours until they reach 24, keeping track of their fours to arrive at a result. However, when asked to find how many beads would be in each of six strings made from 24 beads (partitive division) they might have trouble conceptualizing the problem. At best they might make a guess as to how many beads might be in each string and then use their guess to *segment* the number sequence from 1 to 24, attempting to solve the problem as a quotitive division problem. Children with an ENS might solve the quotitive problem in much the same way as the child with only a TNS but the partitive problem could be solved by *partitioning* 24 into six equal parts. They can pose the problem for themselves as "Six of what will give me 24?" They may still have to choose a number and *iterate* it six times to see if the result is 24 but they start with a sense of the resulting structure and this makes a solution more probable. The child with an ENS, however, would not be able to solve the problem by finding how many sixes are in 24. This *commutative* solution requires yet another re-interiorization of the number sequence.

A Generalized Number Sequence (GNS)

Children with an ENS can establish multiplicative schemes that involve two levels of units. They can form composite unit items and can use these items as input for further operations (counting, combining, comparing, segmenting and partitioning). They can form a numerical composite of composite unit items as a result of these operations. For instance, the result of combining six groups of 5 is a collection of six items, each one of which is one five. They can unpack each five to arrive at a result of 30 unit items. These 30 unit items may even be taken as one composite unit of 30. What they cannot do is to take the six groups of five as one thing that they can use for further input. They can *produce* units of units of units but cannot yet symbolize them. As stated above, they can operate with composite units in much the same way as children with only a TNS operate with single items. A re-interiorization of the number sequence is required to take the results of these operations on composite units as material for further operations. This re-interiorization of an ENS is what constitutes a GNS.

This re-interiorization of the ENS results in *iterable composite units* in much the same way as the re-interiorization of the TNS resulted in iterable "ones." With an iterable composite unit of four, say, a child can conceive of "four" six times as being the same as having a composite unit of six with a *unit* of four items in each of the six unit items that constitute the six "ones." The "ones" in the composite unit of six have become place holders for any type of unit (singletons or composites). The example of Johanna above points the way in which this re-interiorization might take place–through recursive applications of a child's units-coordinating operations to the results of those operations: abstract composite units. Just as the iterable one was the abstraction of the repeated application of the "one more item" operation when double counting by ones, the *iterable composite unit* is the result of the abstraction of the repeated application of the "one more four" (say) when double counting by fours.

Children with a GNS can begin to build exponential structures. They can also coordinate at least three different levels of units. Whereas children with only an ENS will most likely have been working with place value notation in the hundreds (and possibly the thousands), they will experience difficulty in explicating the numerical relations involved in the place value system. Children with a GNS can explicitly demonstrate these place value relations and extrapolate them to higher and higher exponents of the base number.

An example of the strategic power that a GNS provides can be seen in Nathan's construction of a procedure for finding the lowest common multiple (LCM) of two numbers. This construction took place during Nathan's third grade year during his work with the Fractions Project[1]. Nathan did not have as his goal the generation of the LCM of two numbers. The LCM was never referred to in any of our teaching episodes with Nathan. I use it only as a short hand name for Nathan's procedure.

Nathan's goal was to find a partition for a candy bar (in the Microworld TIMA: Bars, Olive & Steffe, 1994) that would allow him to pull out both one third of the bar and one fifth of the bar. His procedure was to count by threes and by fives until he found a common number in the two sequences. He would think to himself in the following way: 3, 6; 5, 10; 9, 12, 15; 15. It's 15. He would then put 15 parts in his bar, pull five out for one third and three out for one fifth. He was able to coordinate and compare his two sequences of multiples until a common multiple was found, but he also knew how many of each multiple he had used to get to this common multiple; he had kept track of each sequence. Nathan was able to carry out these coordinations because "three" and "five" were available to him as *iterable composite units.*

The following year Nathan was able to use his knowledge of factors to obtain a similar result in a more efficient way: In a more complicated task requiring the children to make a fraction of a stick starting from a different fraction (e.g. make a ninth of a unit stick using a twelfth of a unit stick), Nathan was eventually able to partition the 1/12 into three to make 36ths. He knew this would work because "both 9 and 12 add up to 36 ... four nines are 36 and three twelve's are 36, so four of these will be 1/9." As powerful as Nathan appeared in these examples, there were limitations in his operations on fractions that suggested another accommodation beyond the GNS as necessary in order to overcome these limitations.

Rational Numbers of Arithmetic

The example given in the last section of making a fraction using a different fraction as a starting point was not easily achieved by the two boys (Arthur and Nathan) who were given this task. Even though we were convinced by their prior actions that both children had constructed a Generalized Number Sequence, it took Nathan four teaching episodes over a period of a month to construct a scheme that he knew he could use to solve such problems. For Arthur, the struggle took much longer.

One stumbling block that they met was to name a fraction of a fraction as a new fraction. They could produce and anticipate a "fifth of a fifth" but did not know its value as a single fraction. In order to establish its value they had to iterate the result 25 times to reproduce the unit stick. They then knew that the result was 1/25. Their operations for making a fraction of a fraction, while recursive in nature, were not yet *reversible.* Eventually, they developed the ability to mentally project the partition into three equal parts of say one twelfth, into each of the 12 twelfths in the unit in order to establish the value of a third of a twelfth as 1/36. Would the symbolization of this procedure constitute a re-interiorization of their GNS *inwards* rather than the outward direction of their GNS that provides for exponential structures? That is, instead of producing units of units of units, could they produce units *within* units *within* a unit? And is this re-interiorization *inward* the necessary accommodation of a GNS that will generate **Rational Numbers of Arithmetic**? The answers to these important questions have been addressed in a published paper (Olive, 1999) and are still being developed in our ongoing research. Our current results indicate that the journey from the Generalized Number Sequence to the Rational Numbers of Arithmetic is both long and complex. It does involve a reinteriorization inwards that produces units within units within a unit, but on the way to this restructuring, children need to develop the following schemes and operations:

- A *Unit Fraction Composition Scheme* for establishing a unit fraction of a unit fraction as indicated above.
- A *Fraction Composition Scheme* for finding any fraction of any fraction (e.g. 2/3 of 3/5 of a unit)
- A scheme for establishing *fractions as measurement units* so that questions such as "How much of 3/4 is 1/8?" make sense and can be answered.
- A *Co-measurement unit for fractions* so that fractions can be simplified and any fraction can be generated from any other fraction.

When children have constructed fractions as measurement units they have the possibility of constructing any fraction from any other by finding a *co-measurement unit* for the two fractions. From a mathematician's point of view it is just such a possibility that generates closure on the field of rational numbers. By the end of their fifth grade year both Nathan and Arthur were able to make ratio comparisons among fractions and to create any fraction starting from any other fraction.

Implications for Practice

In my courses on children's mathematical thinking for pre-service and inservice elementary teachers I have found that even a brief introduction to the distinguishing aspects of these number sequences helps these students and teachers to make sense of the difficulties they perceive children having in learning school mathematics. The use of video examples from our research projects and hands-on experience with our computer tools has helped to bring these theoretical ideas to life. With only a rudimentary understanding of the differences in children's mathematical thinking represented in these hypothetical models, these students and teachers have been able to recognize different levels of thinking when working with young children, as suggested through the children's counting activities and, thus, plan more appropriate activities for these children.

Rather than insisting that children abandon their counting activities as soon as they start to learn their "basic addition facts" our research indicates that we should encourage children to build on these counting activities to develop their own sophisticated counting strategies that will lead to modifications in their number sequences. After all, if we are serious about regarding young children as mathematicians we should value their ways and means of doing mathematics, and use these ways and means to foster the growth of their mathematical thinking. By realizing that children's numerical activity is limited by their available number sequences, we will be better able to foster that growth.

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[1] Children's Construction of the Rational Numbers of Arithmetic, NSF Project No. RED-8954678, co-directed by Leslie P. Steffe and John Olive. 1991-1995.