A Four-Stage Model of Mathematical Learning

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Introduction

Research in education and applied psychology has produced a number of insights into how students think and learn, but all too often the resulting impact on actual classroom instruction is uneven and unpredictable (Sabelli & Dede, in press; Schoenfeld, 1999). In response, many in higher education are translating research in education into models of learning specific to their own disciplines (Buriak, McNurlen, & Harper, 1995; Felder, Woods, Stice, & Rugarcia, 2000; Jensen & Wood, 2000). These models in turn are used to reform teaching methods, to transform existing courses, and even to suggest new courses.

Research in mathematics education has been no less productive (Schoenfeld, 2000). This article[i] is in the spirit of those mentioned above, in that I combine personal observations and my interpretation of educational research into a model of mathematical learning. The result of this approach can be used to address issues such as the effective role of a teacher and appropriate uses of technology. That is, the model can be viewed as a tool that teachers can use to guide the development of curricular and instructional reform.

Before presenting this model, however, let me offer this qualifier. In my opinion, good teaching begins with a genuine concern for students and an enthusiasm for the subject. Any benefits derived from this model are in addition to that concern and enthusiasm, for I believe that nothing can ever or should ever replace the invaluable and mutually beneficial teacher-student relationship.

Related Literature

Decades of research in education suggest that students utilize individual learning styles (Bloom, 1956; Felder, 1996; Gardner & Hatch, 1989) and instruction should therefore be multifaceted to accommodate a variety of learning styles (e.g., Bodi, 1990; Dunn & Dunn, 1993; Felder, 1993; Liu & Reed, 1994). Moreover, strategic choices and metacognition are also important in research in mathematics education (Schoenfeld, 2000). Research in applied psychology suggests that problem solving is best accomplished with a strategy-building approach. Studies of individual differences in skill acquisition suggest that the fastest learners are those who develop strategies for concept formation (Eyring, Johnson, & Francis, 1993). Thus, a model of mathematical learning should include strategy building as a learning style.

Some mathematics students employ a common method of learning that might be characterized as the “memorize and associate” method. *Heuristic reasoning* is a thought process in which a set of patterns and their associated actions are memorized, so that when a new concept is introduced, the closest pattern determines the action taken (Pearl, 1984). Unfortunately, the criteria used to determine closeness are often inappropriate and frequently lead to incorrect results. For example, if a student incorrectly reduces the

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expression $\sqrt{x^4 + 4x^2}$ to the expression $x^2 + 2x$, then that student likely used visual criteria to determine that the closest pattern was the root of a given power. In mathematics, heuristic reasoning may be a sign of knowledge with little conceptual understanding, a short circuit in learning that often prevents critical thinking. Using heuristic reasoning repeatedly is not likely to build a strong foundation for making sense of mathematics.

I believe that the learning model most applicable to learning mathematics is *Kolb's model of experiential learning* or *Kolb's model,* for short (Evans, Forney, & Guido-DiBrito, 1998). This belief grows out of my experience teaching mathematics, but Kolb's model has also been used extensively to evaluate and enhance teaching in engineering (Jensen & Wood, 2000; Pavan, 1998; Stice, 1987; Terry & Harb, 1993).

In Kolb's model, a student's learning style is determined by two factors—whether the student prefers the concrete to the abstract, and whether the student prefers active experimentation to reflective observation. These preferences result in a classification scheme with four learning styles (Felder, 1993; Hartman, 1995):

- Concrete, reflective: Those who build on previous experience.
- Concrete, active: Those who learn by trial and error.
- Abstract, reflective: Those who learn from detailed explanations.
- Abstract, active: Those who learn by developing individual strategies.

These learning styles are not absolute, and all learners, regardless of preference, can function in all four styles when necessary (Kolb, 1984; Sharp, 1998). Indeed, in the *Kolb learning cycle,* each style is considered a stage of learning and students learn by cycling through each of the four stages (Harb, Durrant, & Terry, 1993; Kolb, 1984; Pavan, 1998). For example, the cycle begins with the student's personal involvement through concrete experience; next, the student reflects on this experience, looking for meaning; then the student applies this meaning to form a logical conclusion; finally, the student experiments with similar problems, which result in new concrete experiences. From here, the learning cycle begins again (Hartman, 1995).

Kolb Learning in a Mathematical Context

Kolb's learning styles can be interpreted as mathematical learning styles. For example, "concrete, reflective" learners may well be those students who tend to use previous knowledge to construct allegories[ii] of new ideas. In mathematics courses, these learners may approach problems by trying to mimic an example in the textbook. Based on several years of observation, experimentation, and student interaction, I have interpreted Kolb's other three learning styles in a mathematical context:

- *Allegorizers:* These students consider new ideas to be reformulations of known ideas. They address problems by attempting to apply known techniques in an ad-hoc fashion.
- *Integrators:* These students rely heavily on comparisons of new ideas to known ideas. They address problems by relying on their "common sense" insights—i.e., by comparing the problem to problems they can solve.
- *Analyzers:* These students desire logical explanations and algorithms. They solve problems with a logical, step-by-step progression that begins with the initial assumptions and concludes with the solution.

- *Synthesizers:* These students see concepts as tools for constructing new ideas and approaches. They solve problems by developing individual strategies and new allegories.

The table in Figure 1 shows the correspondence between Kolb's learning styles and my interpretation in a mathematical context:

KOLB'S LEARNING STYLES	EQUIVALENT MATHEMATICAL STYLE
Concrete, Reflexive	Allegorizer
Concrete, Active	Integrator
Abstract, Reflective	Analyzer
Abstract, Active	Synthesizer

Figure 1. Kolb's Learning Styles in a Mathematical Context.

Moreover, I have not only observed that students are capable of functioning in all four styles, but that the preferred learning style of a student may vary from topic to topic. For example, students with a preference for synthesizing with respect to one topic may change to a preference of integration for another topic, and vice versa. In addition, when a student's learning style does not facilitate successful problem solving, I have observed that the student often resorts to heuristic reasoning.

These observations have led me to hypothesize that a student's preferred learning style for a given concept may indicate how well that student understands that concept. That is, a student's learning style preference may be a function both of the content and the level of understanding of the material. The existence of at least four different *styles* of learning may be indicative of at least four different *stages* of understanding of a mathematical concept, which again is in agreement with one of Kolb's original observations (Smith & Kolb, 1986). I believe this relationship can be used to improve instruction—i.e., a teacher's identification of how well students understand a topic can be used to design instruction so that it best addresses students at that level of understanding.

Stages of Mathematical Learning

There are models with more than four learning styles, and there may be models with more than four stages of mathematical learning. Furthermore, a given student may prefer a learning style for some reason other than level of understanding. However, I believe that if a large number of students in a given classroom prefer a particular learning style for a given concept, then that may indicate how well that group of students understands that topic (Felder, 1989, 1990, 1996). In fact, my experience in teaching mathematics suggests that it is useful to view each learner as progressing through the following four distinct stages of learning when acquiring a new concept.

- *Allegorization:* A new concept is described figuratively in a familiar context in terms of known concepts. At this stage, learners are not yet able to distinguish the new concept from known concepts.
- *Integration:* Comparison, measurement, and exploration are used to distinguish the new concept from known concepts. At this stage, learners realize a concept is new, but do not know how it relates to what is already known.

- *Analysis:* The new concept becomes part of the existing knowledge base. At this stage, learners can relate the new concept to known concepts, but they lack the information needed to establish the concept's unique character.
- *Synthesis:* The new concept acquires its own unique identity and thus becomes a tool for strategy development and further allegorization. At this stage, learners have mastered the new concept and can use it to solve problems, develop strategies (i.e., new theory), and create allegories.

That is, a student may prefer allegorization as a learning style only until he realizes that the idea they have been exposed to is a new one, after which that same student may prefer the comparisons and explorations that characterize integration. Similarly, once a student understands how the new concept compares to known concept, then she may desire to know all there is to know about the concept, and having done so, she may ultimately desire the mastery of the topic implied by a preference for synthesis. Thus experiencing different learning stages may impact the learning style of the student

The Importance of Allegories

Given that a student's preferred style may be due in part to a student's current level of understanding of a concept, the four stage model described in the previous section suggests that learning new concepts may fruitfully begin with allegory development. That is, a figurative description of a new concept in a familiar context may be a useful intuitive introduction to a new idea and should precede any attempts to compare and contrast the new idea to known ideas. Indeed, a student with no allegorical description of a concept may resort to a "memorize and associate" style of learning.

To illustrate the importance of allegory development, let us consider what might transpire if I were to teach a group of students the game of chess without the use of allegories. I would begin by presenting an 8 by 8 grid in which players 1 and 2 receive tokens labeled A, B, C, D, E, and F arranged as shown in Figure 2.

B1	C1	D1	E1	F1	D1	C1	B1
A1	A1	A1	A1	A1	A1	A1	A1
A2	A2	A2	A2	A2	A2	A2	A2
B2	C2	D2	E2	F2	D2	C2	B2

Figure 2. Chess without Allegories.

I would then explain that each type of token has a variety of acceptable moves—e.g., the "B" tokens can move vertically or horizontally but must stop when encountering another token, whereas "C" tokens have four possible L-shaped moves and need not stop if other tokens are in those paths. I would conclude my explanations by stating that the goal of the game is to immobilize the other player's "F" token. In response, students would likely memorize valid moves for each token, and then would memorize when to make those moves—a way of "playing" that does not seem like much fun.

In contrast, I believe people learn and enjoy chess *because* the game pieces themselves are allegories within the context of medieval military figures. For example, pawns are numerous but have limited abilities, knights can "leap over objects," and queens have unlimited power. Capturing the king is the allegory for winning the game. In fact, a vast array of video and board games owe their popularity to their allegories of real-life people, places, and events.

Thus, when I teach a course such as calculus or statistics, I try to develop an allegorical introduction to each major concept. To do so, I begin by identifying a context that is appropriate for a given class at a given time. For example, most of my students enter calculus with decent arithmetic skills; a limited background in algebra; and a mostly underdeveloped understanding of geometry, trigonometry, and functions. Correspondingly, I usually introduce calculus by using algebra and arithmetic to explore tangent lines to polynomial curves. In contrast, many calculus textbooks begin with limits of functions, including transcendental functions. I would argue that such an introduction does not lend itself to allegorical description and that the result is that calculus students are well entrenched in heuristic reasoning by the time they take the first test.

As another example, consider that when students hear the word "probability," they most likely think of rolling dice and flipping coins. If so, then random walks constitute a natural allegory for introducing nearly all of the primary ideas in statistics and probability. However, a course in probability and statistics often introduces normal distributions, statistical tests, expected value, and standard deviation as if they are intuitively obvious. My experience is that even when students make high grades in a statistics course, statistical concepts remain mysterious to them.

Components of Integration

Once a concept has been introduced allegorically, it can be integrated into the existing knowledge base. I believe that this process of integration begins with a *definition*, since a definition assigns a label to a new concept and places it within a mathematical setting. Once defined, the concept can be compared and contrasted with known concepts.

Visualization, experimentation, and exploration can play key roles in integration. Indeed, visual comparisons can be very powerful, and explorations and experiments are ways of comparing new phenomena to well-studied, well-understood phenomena. As a result, the use of technology is often desirable at this point as a visualization tool

For example, suppose that a certain class of students has a good grasp of linear functions and suppose that exponential growth has been allegorized and defined. It is at this point that students may best be served by comparisons of the new phenomenon of exponential growth to the known phenomenon of linear growth. Indeed, suppose that students are told that there are two options for receiving a monetary prize—either \$1000 a month for 60 months or the total that results from an investment of \$100 at 20% interest each month for 60 months. Visual comparison of these options reveals the differences and similarities between exponential and linear growth (see Figure 3). In particular, exponential growth appears to be almost linear to begin with, and thus for the first few months Option 1 will have a greater value. However, as time passes the exponential overtakes and grows increasingly faster than the linear option, so that after 60 months, Option 1 is worth $60,000 while Option 2 is worth $4,695,626.

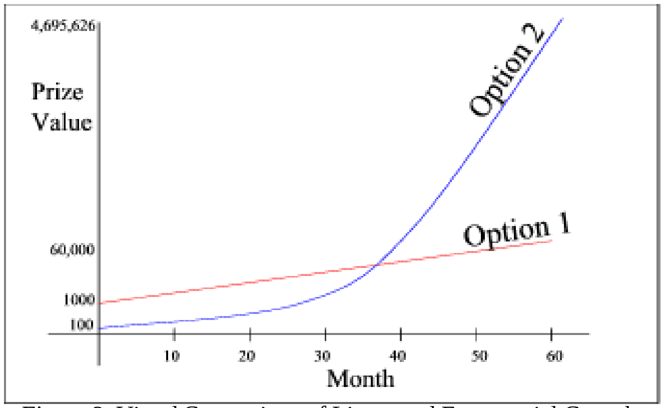

Figure 3. Visual Comparison of Linear and Exponential Growth.

Other comparisons that may be appropriate at this point include comparing an exponential to polynomials of increasing degree or comparing a sine wave to an exponentially damped sine wave. In my opinion, comparisons such as these are of no value before a student realizes that exponential growth is a new type of growth they have not yet imagined in their context of algebraic functions. Moreover, presenting a comparison such as Figure 3 to a group of students who have spent some time concentrating on the properties of the exponential may lead them to wish audibly that they had seen Figure 3 before studying all those unmotivated properties. That is, comparing new ideas to known ideas seems to me to be most natural and most beneficial in the second stage of learning.

Analysis

Once a student has experienced an allegorical introduction to a new concept and has compared the new concept to known concepts, he is ready to consider the new concept independent of other ideas. Indeed, at this stage, the new concept takes on its own character, and the student's desire is to learn as much as possible about that character. Learners in the analysis stage want to know the history of the concept, the techniques for using it, and the explanations of its different attributes. Furthermore, they want information about the relationship of the new concept to known concepts that goes beyond comparisons, such as the sphere of influence of the new concept within their existing knowledge base.

As a result, learners in the analysis stage desire a great deal of information in a short period of time. Thus it seems appropriate to lecture to a group of such learners. Unfortunately, many of us who teach mathematics too often assume that all of our students are at the analysis stage for every concept, which means that we deliver massive amounts of information to students who have not even realized that they are encountering a new idea. This phenomenon appears to occur for the limit concept in calculus. Studies have shown that few students complete a calculus course with any meaningful understanding of limits (Szydlik, 2000). Instead, most students resort to heuristics to survive the initial exposure to the limit process.

Synthesis

Finally, the synthesis stage involves mastery of the topic, in that the new concept becomes a tool the student can use to develop individual strategies for solving problems. For example, even though games often depend heavily on allegories, some would argue that the fun part of a game is analyzing it *and* developing new strategies for winning. Indeed, most people would like to reach the point in a game where they are in control—that is, the point where they are synthesizing their own strategies and then using those strategies to develop their own allegories of new concepts.

However, synthesis is a creative act, and not all students will be able to act as synthesizers with a given concept within the same period of time. The cycle of learning may break down at this point due to an inability to use the concept under study to generate allegorical descriptions of a subsequent concept. Consequently, learning mathematics may not be feasible for most students without the assistance of a teacher.

The Role of the Teacher

The four stages of mathematical learning cannot be reduced to an automated process with four regimented steps. Appropriate allegories should be based on a student's previous experiences, and consequently new allegories must be continually developed. Some concepts require more allegorization, integration, and analysis than others, and it may not be a judicious use of time to ask students to synthesize their own allegories for new ideas.

As a result, there must be an intermediary—i.e., a teacher—who guides the development of allegories for the students, who determines how allegorization, integration, and analysis should be used in presenting a concept, and who prompts students to synthesize and think critically about each concept.

Indeed, it has been suggested that the ideal classroom would include each of the four processes in the Kolb cycle (Hartman, 1995; McCarthy, 1986). That is, full comprehension requires learning activities fitting each stage of learning (Jensen & Wood, 2000). McCarthy has identified four roles for the teacher based on the Kolb learning cycle—evaluating, motivating, teaching, and coaching.

Likewise, the four stages of mathematical learning described above imply at least four different roles for the teacher of mathematics.

1. *Allegorization:* Teacher is a storyteller.
2. *Integration:* Teacher is a guide and motivator.
3. *Analysis:* Teacher is a source of information.
4. *Synthesis:* Teacher is a coach.

In the stages of allegorization and analysis, the role of the instructor is one of active leadership, while in the stages of integration and synthesis, the instructor is a mentor, guide, and motivator who emphasizes active learning, exploration, and expressions of creativity.

I will explore each of these roles in turn. When a teacher first introduces a concept to a group of students, the teacher may act as a storyteller to meet the students' need for allegorization. That is, students need a teacher to provide intuitive introductions to new ideas in familiar contexts—historical, arithmetic, scientific, or otherwise. For example, even though I teach college students, I keep a set of measuring cups in my office as an allegory for arithmetic of fractions. Usually, using the measuring cups to demonstrate that $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ is more than sufficient to motivate the idea of a common denominator.

Students who have realized that a new idea is being considered need to compare and contrast that new idea to known ideas. Thus, a teacher may find it fruitful to define the new idea in a way that allows it to be differentiated from known ideas, and then may engage students in focused exploration that will reinforce and clarify the comparisons of the newly defined concept to previously defined ideas. Among these comparisons may be visualizations and numerical experiments with a predicted outcome that must be prepared in advance.

Students who understand the nature of a new concept are ready for someone to provide a great deal of information about the concept in a short period of time. Thus, students who are in the analysis stage may benefit from a teacher who knows the subject in great depth and detail. In addition, students in this stage may benefit from a teacher who provides a number of different sources for information about the idea.

Students who are at the stage of synthesis still need a teacher to advise and direct them. That is, a teacher in the role of a coach may foster the growth of these learners by helping them to develop discipline and structure in their creative activities. Personally, I believe that many of the students who feel bored or even stifled in our educational system are students with great potential who are waiting for someone to offer them a different direction. Thus, teachers need to foster in all students the realization that doing mathematics is a creative activity and that such creativity is both enjoyable and rewarding.

Conclusion

Educational research, applied psychology, and research in mathematics education have produced a great many insights and potential improvements to mathematical instruction. However, as has been realized in other fields, it is important that teachers translate the results of that research into a form appropriate for use in the classroom. Sabelli and Dede (in press) use the phrase “Scholarship of Practice” to describe this idea.

The four stages of mathematical learning presented in this article speak to this purpose. Educational researchers have demonstrated the importance of multiple learning styles. Applied psychologists have established the importance of strategy-building and stages of skill development. Mathematical researchers have identified many areas where mathematical instruction can and needs to be improved.

I have simply combined these ideas into a working model that describes what students may experience in mathematics courses. The model suggests that concepts need to be allegorized first, integrated next, analyzed third, and synthesized last. It also implies that teachers should play many different roles in the classroom to meet students’ needs in the different learning stages—for example, adopting the role of storyteller during the allegory stage and acting as a coach at the synthesis stage.

This model has become an invaluable tool in my own teaching. It allows me to diagnose student needs quickly and effectively; it helps me budget my time and my use of technology; and it increases my students’ confidence in my ability to lead them to success in the course. I hope it will be of equal value to my fellow educators in the mathematics and mathematics education professions.

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[ii] An allegory is a figurative description of an unknown idea in a familiar context.