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Guest Editorial…

Learning to Frame Research in Mathematics Education

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Learning how to frame research is arguably the most important capacity for new researchers to develop. This is true both for doctoral students and for junior faculty as they establish research programs at new institutions and begin to help students conduct their own research. I take framing research to mean the construction of connections among research questions, the literature(s) within which those questions are positioned, theoretical perspectives or frameworks, and methods that generate warrants sufficient for making convincing claims. Making strong connections among all of these components is essential for building coherent arguments at the center of high quality research in mathematics education (Lesh, Lovitts, & Kelly, 2000; Simon, 2004). Examples of important contributions to our field that have relied on innovative combinations of questions, theoretical perspective, and methods include Schoenfeld's (1985) work on problem solving and Steffe's (e.g., Steffe, 2001; Steffe, von Glasersfeld, Richards, & Cobb, 1983) work on children's construction of whole and rational numbers.

Some recent publications have suggested that doctoral programs in education may not always be producing graduates with the necessary capacities for framing research. Boote and Beile (2005) sampled 30 dissertations from three state-funded colleges of education in the United States, found a wide range in the quality of the literature reviews, and reported that some of the literature reviews were little more than "disjointed summaries of a haphazard collection of literature" (Boote & Beile, 2005, p. 9). Moreover, weak literature reviews may be just part of a larger challenge for new researchers: Schoenfeld (1999) asserted that beginning researchers often do not learn what it means to make and justify claims about educational phenomena or how to frame workable research problems. Part of the challenge is that framing research is a creative process for which there are no formulae.

I struggled in graduate school to understand how authors came to pose the questions with which they began published articles and wondered if I would ever be able to articulate such questions. Perhaps others have had similar experiences. The purposes of the present article are to help doctoral students construct initial images of processes involved in framing research and to serve as a catalyst for reflection among junior faculty who, like me, are learning to support students conducting their own research. I will do this by describing how I framed my own dissertation study at a level of detail often omitted or left tacit in published articles. My dissertation was based on detailed analyses of videotaped interviews, but I hope my general points will help others to embark on other kinds of research projects as well.

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An Example of Framing Research

For my dissertation, conducted under Alan Schoenfeld at the University of California, Berkeley, I constructed a theoretical frame for explaining how pairs of eighth-grade students constructed knowledge structures for modeling with algebra a physical device called a *winch*. The winch (see Figure 1) exemplifies situations that can be modeled by pairs of simultaneous linear functions, a core topic in introductory algebra courses.

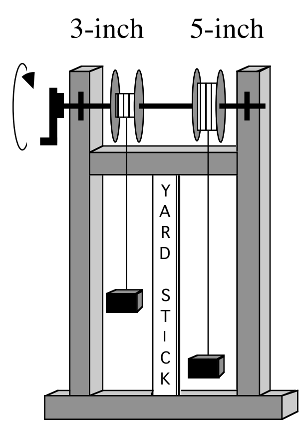

Figure 1. The winch. From "Inscribing the Winch: Mechanisms by Which Students Develop Knowledge Structures for Representing the Physical World with Algebra," by A. Izsák, 2000, *The Journal of the Learning Sciences, 9*, p. 33. Copyright 2000 by Lawrence Erlbaum Associates, Inc. Reprinted with permission.

The device stands 4 feet tall and at the top has a rod with a handle for turning two spools, one 3 and one 5 inches in circumference. Fishing line attaches one weight to each spool. I will refer to these as the 3-inch weight and 5-inch weight, respectively. Turning the handle moves the weights up and down a yardstick, allowing measurements of heights, displacements, and distances between the two weights. The theoretical frame highlights coordination and reorganization of knowledge for generating, using, and evaluating algebraic representations (Izsák, 2003, 2004).

Getting Ideas for Feasible, Significant Contributions

I entered my doctoral program interested in how students made sense of external representations of functions and began reading relevant literature. I was fortunate because by the early 1990s there were a handful of excellent recently published literature reviews (Kieran, 1992; Leinhardt, Zaslavsky, & Stein, 1990) and edited books (Janvier, 1987; Romberg, Fennema, & Carpenter, 1993) that addressed students' understandings of functions and the roles of external representations in mathematical thinking. These sources provided an overview of existing findings and extensive bibliographies that suggested the main researchers whose work related to my emerging interests. At the same time, my interests were also being shaped by numerous conversations with faculty and students, courses, and participation in research groups that, among other things, allowed me to observe more advanced doctoral students frame their dissertation projects.

Much of the work on students' understandings of representations of functions documented difficulties (often termed misconceptions) that students exhibited when solving equations, solving word problems, and interpreting graphical representations. Taken together, these studies made clear that students struggled with standard or normative representations. The literature also contained several theoretical accounts of how students might learn in this domain. One family of closely related, highly visible accounts described process understandings of functions being reified or encapsulated into object understandings (e.g., Kieran, 1992; Sfard, 1991, 1992; Dubinsky & Harel, 1992). Researchers often relied on cross sectional data as evidence for process-object accounts of learning and none presented empirical evidence of reification or encapsulation processes occurring in actual students. Clearly, studies that captured how students might learn to represent functions and solve problems would contribute to the field, but was capturing such phenomena feasible? Perhaps this problem was too difficult for a dissertation project.

A number of faculty and doctoral students at the University of California, Berkeley shared my interest in the role of external representations in mathematical thinking and pointed me to a small number of then recent studies (diSessa, Hammer, Sherin, & Kolpakowski, 1991; Hall, 1990; Hall, Kibler, Wenger, & Truxaw, 1989; Meira, 1995, 1998) that demonstrated students' capacities to construct their own perhaps non-standard graphic, algebraic, and tabular representations of functions in the course of solving problems. These studies suggested both that past research had overlooked students' latent capacities for constructing representations and that capturing examples of students learning to represent functions and solve problems might be feasible. In particular, when analyzing interview data in which one pair of students worked with a device similar to the winch, Meira (1995) found a complex interplay between the students' understanding of the device and of a table that they were developing to represent that device. These data and the analysis were unlike any I had seen in the literature summarized in the previous paragraph. Subsequently, I discovered that other researchers (Greeno, 1993, 1995; Moore, 1993; Piaget, Grize, Szeminska, & Bang, 1968/1977) had also gained detailed access to young children's, middle school students', and high school students' implicit and explicit understandings of linear functions. Something about the winch was engaging to students and perhaps, with the right combination of tasks and students, I could capture data in which students learned to represent functions and solve problems. Furthermore, examination of previous studies suggested three categories of questions that I could pose to students with various initial winch set-ups:

(1) Predict the distance between the weights after an arbitrary number of cranks.

(2) Determine whether and, if so, when one weight would ever be twice as high as the other.

(3) Determine whether and, if so, when the weights would meet at the same height.

Finally, I needed a provisional theoretical lens or lenses to help me identify instances of learning. The existing literature suggested at least two options. One was to look for instances of reification or encapsulation as described in the process-object perspective mentioned above. Another was to look for the genesis of knowledge structures similar to those described in some recent research on representations of functions (e.g., Schoenfeld, Smith, & Arcavi, 1993). The latter studies documented instances where complex coordination of multiple pieces of prior knowledge was a dominant feature of learning. I did not adopt any one perspective a priori but rather began with a theoretical tool kit that might, or might not, suffice to explain any captured instances of learning. At this point, I was asking how might students learn to represent functions and solve problems, and I had some theoretical tools and a strategy for gathering data that were good enough to get me started.

Successive Approximations

I interviewed 14 pairs of students, 6 in spring 1996 and 8 in spring 1997. In most cases, I conducted four or five hour-long interviews with each pair. Because I was interested in capturing instances of learning, I needed students for whom the winch tasks were neither too easy nor too hard. Thus, I began in spring 1996 with a range of students. I interviewed 2 pairs of eighth-grade students from one school taking a pre-algebra course, 2 more pairs of eighth-grade students from the same school taking an Algebra I course, and 2 pairs of tenth-grade students from a different district who had completed an algebra course. At the beginning of the first interview with each pair, I read instructions that simply pointed out the handle, weights, and yard stick on the winch, a set of questions about the device, and scratch paper should the students need any. I made no mention of linear functions or of any representations. As the students worked, I intervened on occasion to clarify my instructions, to ask for further explanation of some comment, or to discuss possible strategies for making progress when students seemed stuck. Students could repeat actions with the winch as often as they liked and worked on problems until they reported satisfaction with their answer. Occasionally I moved students on to the next question when they appeared bogged down to the point of frustration.

Several results from the first round of data collection allowed me to refine how I framed my research. First, the tasks seemed particularly engaging for the Algebra I students. These students often made connections between their algebra course and the interview activities after recognizing that turning the handle generated patterns in heights of weights and distances between weights. Especially promising, given my research question, were examples in which these students constructed apparently novel, yet sensible, equations after struggling to generate expressions and equations that represented height and distance patterns. The pre-algebra students and tenth-grade students evidenced less potential for learning when working on the winch tasks: The pre-algebra students tended to approach all the problems by making tables, and the tenth-grade students could set up and solve equations much more readily. Second, one pair of Algebra I students apparently reasoned about equations in ways consistent with both the process and the object perspective as described in the literature, yet struggled to coordinate their understandings of their representations and the winch. This suggested that the process-object account was insufficient for explaining how students might learn to represent situations that could be modeled by linear functions. Third, detailed analysis of this same pair suggested that they had constructed a new knowledge structure resembling one described by Sherin (2001).

Analyzing these data led me to frame my research with greater precision: I focused my attention on how Algebra I students could construct knowledge structures for modeling the winch with algebra and took initial steps at creating theory for explaining such phenomena. The theory proposed two learning mechanisms, notation variation and mapping variation, that described processes by which students refined and coordinated their algebraic expressions and the correspondences they established between parts of those expressions and features of the winch (see Izsák, 2000, for details). At this point, I needed further examples of students' constructions and my committee was intimating that I should attempt "deeper" theory that explained in greater detail how students drew on their existing knowledge.

In spring 1997 I interviewed 7 more pairs of Algebra I students and 1 more pair of pre-algebra students using essentially the same set of tasks. The students came from the middle school used in the first round of data collection. As I had hoped, the second round of data collection produced further examples in which students constructed apparently novel, yet sensible, equations after struggling to represent height and distance patterns on the winch with expressions and equations (Izsák, 2003, 2004). One of the most interesting episodes occurred when one pair of students examined the winch set up so that the 3-inch weight started by the 14-inch mark and the 5-inch weight started by the 0-inch mark. Both weights went up when the students turned the handle, and the question was: Can you predict how far apart the weights will be as you turn the crank? If so, how? The students treated all distances as positive and struggled to represent a pattern that first decreased to zero and then increased. Eventually, one of the students generated a pair of expressions $14 - 2n$ and $2n$ where n represented number of cranks. She used the first expression to calculate distances before the weights met and the second to calculate distances after. In so doing, she counted n from two different starting points. Her partner understood that computations made with these expressions matched distances measured on the winch, yet rejected the pair of expressions as a legitimate representation.

As the students continued to work, they evidenced a range of criteria for evaluating algebraic representations. Two of the several examples were *single equation*, the criterion that single equations are better than multiple ones, and *consistent interpretation*, the criterion that the number of cranks always be counted from the same starting point. The students continued to work until they generated $|(n - 7) * 2| = d$, an equation that correctly predicted distances between the weights and simultaneously satisfied all the criteria they had mentioned (Izsák, 2004). These data made clear that the students' knowledge for evaluating algebraic representations fundamentally shaped the direction of their work. Furthermore, analysis of subsequent interviews with the same pair of students suggested that they had constructed a new knowledge structure (Izsák, 2004). Thus, criteria apparently played a role in learning.

At this point, I was confident that my data were sufficient for making legitimate progress on my research question, but how I framed my study continued to evolve through further analyses of both data and literature. In the end, students appeared to draw on a range of knowledge about physical causality in the winch, for using algebraic equations (e.g., substituting values or solving), and for evaluating algebraic representations. This range of knowledge, and the ways students used it in the course of solving winch tasks, suggested a good match between my data and an epistemological perspective known as knowledge-in-pieces (diSessa, 1988, 1993). diSessa developed this perspective to explain emerging expertise in Newtonian mechanics. The perspective holds that knowledge elements are more diverse and smaller in grain size than those presented in textbooks. Growth and change consists of multiple, related processes including not only the construction of new knowledge elements but also the coordination of diverse knowledge elements and the extension or constriction of conditions under which particular elements may be applied productively. Thus, the final framing that has appeared in published reports of my dissertation (Izsák, 2000, 2003, 2004) evolved through continuous analysis of existing literature and over the course of two rounds of data collection and analysis.

Discussion

What from this example might be of use to doctoral students and junior faculty who are beginning to help students conduct their own research? First, in my view the term literature review is misleading and might be better described as an analysis of prior work. This means much more than summarizing the results of a set or sets of studies. It means identifying and summarizing main questions that bodies of research have addressed, theoretical perspectives and methods used in different bodies of research, and main results that have emerged *across* studies. It also means analyzing the possibly tacit assumptions underlying bodies of research and possible limitations that cut across related studies. Various handbooks (e.g., Berliner & Calfee, 1996; Grouws, 1992; Richardson, 2001) contain reviews by acknowledged experts in a wide variety of areas and are good places to start.

As my example illustrates, analyzing prior work can help identify problems for which you have a reasonable chance of making progress. Some problems, though important, may simply be too hard for a new researcher working individually. For instance, studies that have come the closest to constructing links between professional development for teachers and the learning of their students have been ambitious collaborative efforts among several seasoned researchers. High quality studies in a given area based on data that an individual researcher might reasonably collect and analyze can suggest an "approximate size" for a dissertation study. I stress the word approximate because published articles often report only a portion of a larger study, even a dissertation study.

Second, figuring out how to get data that contain warrants sufficient for making convincing claims is hard work. Pilot studies can help gauge the promise of a particular research design and can suggest refinements. I interviewed a wider range of students in the first year and narrowed in on Algebra I students the second as I searched for a good match between students and winch tasks. If I had not found a good match between students and winch tasks during the pilot, I would have needed different students, different tasks, or both. Pilot studies also provide an opportunity to work out kinks in data collection before the data really count. For instance, technically strong videotaped data often requires experimentation with placements of cameras and configurations of microphones, especially in classrooms. Other aspects of research design can arise in studies using qualitative methods and the larger point applies to quantitative studies as well. For instance, a questionnaire or other instrument might need piloting and refinement.

Third, framing research is often a dynamic process that unfolds over the course of a study. This point is important because the process of framing research often differs from the logic authors present in published articles—the questions for the study, literature review, methods, analysis and results, and concluding discussion. By way of analogy, the final presentation of a proof in mathematics that works forward from givens does not necessarily reflect how the proof was constructed through initial efforts that led to dead ends, processes where the person worked forward from the givens and backwards from the conclusions hoping to connect the two strands of logic in the middle, and so on. The final elements in a coherent argument are often assembled through cycles of successive approximations that lead to increasingly refined and coordinated questions, theory, methods, and results. Thus, questions posed at the beginning of articles are not necessarily those with which studies were launched.

Finally, theoretical perspectives shape research questions, data collection, and analysis in ways that emphasize some aspects of phenomena under study while suppressing others. One way to understand more deeply how theories shape which aspects get emphasized and which get suppressed is to consider how the same phenomena might be investigated and explained using different theoretical perspectives. In my view, there are strong advantages to beginning studies with sets of alternative theoretical perspectives and to examining the extent to which each alternative does, or does not, help explain data. In my case, I started with two cognitive perspectives and then moved away from the process-object account when it clearly did not fit my data. Instead, the knowledge-in-pieces perspective was a good match because it allowed the forms and types of knowledge at play to be part of what was under investigation. To illustrate how a theoretical perspective can suppress some aspects of a phenomenon, had I begun my study committed to finding schemes and operations (e.g., Steffe, 2001), I might have overlooked comments that evidenced students' criteria for algebraic representations. Furthermore, all of the perspectives I have discussed leave social aspects of learning in the background. Because we are not always aware of ways in which theoretical perspectives shape what we attend to and "see" in data, considering different perspectives can help make the rationale for theoretical decisions more explicit and, in turn, facilitate stronger connections among all of the components essential to high quality research. Moments where all the components start to come together are exciting and make all the hard work very rewarding.

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