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Uncovering Algebra: Sense Making and Property Noticing

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This paper articulates a perspective on learning to discuss ways in which students develop personal sense and negotiate meaning in a middle school algebra context. Building on a sociocultural perspective that incorporates mental objects, learning is described as a mutually dependent process involving personal sense making and the public negotiation of meaning. Analysis of student problem solving is focused on the development of taken-as-shared meaning through an individual and collaborative analysis of the properties of various conceptual entities. The results suggest that functional properties inherent in linear relationships were more supportive in eliciting meaning making exchanges than were algebraic properties associated with generalized arithmetic, although the contextual nature of the linear tasks may have also supported the meaning making activity.

What is the difference between sense and meaning? Drawing on the philosophies of Vygotsky and Leont'ev, Wertsch (1991) distinguishes between sense and meaning by focusing on the personal and public aspects of activity. Lave, Murtaugh, and de la Rocha (1984) concur: "sense designates personal intent, as opposed to meaning, which is public, explicit, and literal (p. 73)." The personal act of reflection and the public act of communication relate in a manner that allows one's personal reflections to mediate and be mediated by one's interactions with the environment (Bauersfeld, 1992; Hiebert, 1992).

From this perspective, sense is based on one's individual reflections, whereas meaning has both a personal and public dimension. Throughout the paper, we will take the perspective that *sense* refers to the current status of cognitive acts constructed within an individual's mental plane, and that an individual's *meaning* is the inferred result of the intention or process of making one's sense knowable within a social environment. In other words, one's meaning is an intent to articulate one's sense, but meaning is also a public, "taken-as-shared" (Cobb, Yackel, & Wood, 1992) construct developed in a social context.

Because this paper focuses on learning and knowledge construction, we turn our attention explicitly to the differences between sense making and meaning making, with the latter explored in a social dimension. The philosophical complexities in the analysis of collective meaning making, as opposed to an individual's isolated meaning making, are greatly expanded due to the interactions between the personal and added social dynamics at play.

For example, a radical constructivist perspective emphasizes the interpretations of one's experiences in framing realties, so the distinctions between sense making and meaning making become quite complex. As Lerman (1996) states:

> Rejecting a picture theory of mind, that mental representations of reality are exact replicas of the real world, leads, for the radical constructivist, to the conclusion that one can only argue that all representations are constructed by the individual, and hence meanings are ultimately those in the individual's mental plane. (p. 137)

The cognizing individual constructs her or his own world (i.e., makes sense) out of the articulated meanings put forth by others in social interactions. But the meaning making of others is filtered by the organization of an individual's own experiences and sense-making processes, and any attempt to convey one's meaning is, in turn, filtered by the sense-making processes of others (von Glaserfeld, 1995). Thus, sense and meaning are quite blurred, as the result of attempts at collective meaning making are always unknown across the individuals' own mental planes. One's own meaning, as well as the perceived meaning of others, must remain in the internal realm of sense making.

Others, however, draw more separation between the notions of sense and meaning. For example, many researchers have articulated a view of learning in which "mental objects" are constructed through reflections on actions and activity. This mental

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abstraction has been called reification (Sfard and Linchevski, 1994), encapsulation (Dubinsky, 2000), and verb-noun status (Davis, 1984). For these researchers, sense making leads to distinct acts of meaning making where individuals construct and publicly share clear descriptions of these mental objects.

Positioned perhaps between the above two approaches, Cobb et al. (1992) describe a sociocultural perspective of learning in which students make use of experiences and interactions, mediated by developmental interventions of the instructor, in order to construct understandings that are either principally generated by one's sense-making activities or that approximate and expand on the taken-as-shared meanings of a community or society. This approach goes beyond the acquisition and participation metaphors of learning and includes aspects of knowledge creation (Paavola, Lipponen, and Hakkarainen, 2004). Here, it is possible that the approximation of others' meaning in an individual's sense-making activity can be thought to be negligible.

Slavit (1997) has built on the above perspectives to articulate a theory involving the development of meaning in the context of algebraic ideas through an awareness of the properties of publicly negotiated and taken-as-shared mental objects. The ability to make sense of the properties that are embedded within and help define a situation, activity, symbol system, or idea can lead to the development of richer forms of sense- and meaning-making activity. These developments commonly occur in the cognitive act of formalization (Kieren, 1994), which allows an individual to make sense of mathematical constructs. For example, students over time might recognize that linear functions grow at a constant rate (from a numeric or graphic perspective), are continuous, and have exactly one x- and y-intercept (excepting vertical and horizontal lines). These and other properties lead to an understanding of linear functions borne from the sense-making process. Further, these understandings can then be weighed against one's perceptions of the taken-as-shared meanings of society as a whole, including negotiation with one's peers; these understandings could also be initially generated out of interactions with one's peers. In this perspective, the learner develops mental constructs associated with established mathematical objects and ideas by focusing on his or her interpretation and awareness of the properties that define these objects and ideas.

Therefore, constructing sense and meaning in algebra can be approached by focusing on the properties associated with the objects and ideas that help define this specific area of mathematics. Although algebra is a multi-faceted mathematical area, this study is primarily concerned with the two areas defined as "generalizing and formalizing patterns and constraints" and "study[ing] of functions, relations, and joint variation" (Kaput, 1995). Patterns in arithmetic computations can be identified by noting properties that these computations share, which can then be generalized to formal algebraic *rules*.[1] These generalizations form the basis of understanding algebra as generalized arithmetic and illustrate the development of sense and meaning in this area of algebra through a property-noticing process. Similarly, one can experience growth relationships between two varying quantities in a variety of situations to support the development of more abstract notions of functions, including the concept of covariation (a patterned change in one variable due to a patterned change in another; Kieran & Sfard, 1999; Slavit, 1997).

Method

The purpose of the study is to examine the sense-making and meaning-making activities of pairs of students engaged in problem-solving episodes related to the algebraic topics of generalized arithmetic and function. While not attempting to prove that collaborative learning environments are more effective than individual settings, the study investigates the manner in which pairs of students use specific cognitive and social processes in algebraic problem-solving environments. Data consist of 15 videotaped interviews involving problem-solving episodes with fourteen 7th grade and sixteen 8th grade students from two middle school classrooms in a rural, medium-socioeconomic status school in the northwestern United States. The students worked in pairs on two tasks (detailed in Figure 1) for approximately 20–30 minutes. Approximately half of the students in each of the two classrooms were randomly selected to participate, and all interviews were transcribed. The eighth grade students had a limited amount of formal exposure to algebra prior to the study, consisting mainly of equation solving, whereas the seventh grade students had no previous formal algebra instruction. Although whole-class discussion of problem solving strategies and solutions were common in both classrooms, the students had limited prior exposure to working in collaborative pairs.

Task 1) Two carnivals are coming to town. You and your friend decide to go to different carnivals. The carnival that you attend charges $10 to get in and an additional $2 for each ride. The carnival your friend attends charges $6 to get in, but each additional ride costs $3. If the two of you spent the same amount of money, how many rides could each of you have ridden?

Task 2) What digit is in the one's place of the number: $2^9 \cdot 3^4 \cdot 5^6$

Figure 1. Problem-solving tasks used to measure students' sense of functional algebra and generalized arithmetic, respectively.

The interviews were conducted in a small area of a quiet room, and the students were under no time constraints to complete the tasks. The students were given one copy of the first task and were told to "work the problem anyway you wish, but you may wish to work together." Only pencil and paper were provided. When it seemed that the students were at the end of their solution attempt, they were questioned on the manner in which they approached the task, and asked to think about other ways to solve the problem. This looking back stage was intended to promote critical reflection by the students on their solution strategy. These procedures were repeated for the second task.

Analysis was conducted on the videotaped segments as well as the students' written work. The analysis centered on the kinds of algebraic understandings (Kaput, 1995) the students seemed to bring into the problem-solving situation, and the kinds of algebraic understandings that the pairs utilized and constructed in their solution. Hence, the social, external, and mathematical constraints inherent in problem solving were present, but the analysis centered on the construction of understandings. Particular attention was given to the kinds of properties that the students attached to the algebraic ideas and mental objects that helped support their investigation.

Numerous researchers have extended a sociocultural view of learning to research on teaching, grounded in participant's actions and perspectives (Cobb et al., 2003; Simon and Tzur, 1999). Researcher participation, and the methodology itself, can be jointly negotiated with the participants. Kieren et al. (1995) took a similar approach to research on learning, describing an *enactive learning environment* that attempts to balance the overall aspects of the learning situation with the cognitive and social backgrounds of the participants as they engage in mathematical activity. Kieren et al. used a group interview format where the mathematical activity and research focus are mediated by researchers, participants, and setting. Kieren and his colleagues prefer to balance the role of one's sense-making activity and one's ability to negotiate and construct meaning. As they state, instead of *situated* cognition or situated *cognition*, research should focus on *situated cognition*. Hence, analysis of student mathematical activity should be concerned with understanding the students' individual cognitive processes in the context of the entire setting, including the genesis and nature of interactions that lead to knowledge construction. Likewise, a discussion of the setting should be framed by the activity that occurs within. This perspective was the lens through which the interviews in this study were both conducted and analyzed. Discussion of results will focus on overall trends in the data, followed by a microanalysis of problem-solving interactions.

Results

Task 1 was designed to allow students to approach the problem either arithmetically or from a more algebraic perspective. Students can solve the problem by simply adding the cost of a ride or rides to each admission and finding the amount of money where

Person A	Person B	Number of Rides
10	6	0
12	9	1
14	12	2
16	15	3
18	18	4
20	21	5
22	24	6
24	27	7
26	30	8
28	33	9
30	36	10

Figure 2: Amounts of money spent by carnival attendees for a given number of rides.

these totals are the same (see Figure 2). However, students could also extend this strategy and find many other combinations where the amount of money spent is the same. Specifically, the students could express their answer in a manner that describes a variable relationship between the amount of rides and money spent. Students could discuss the general relationships in the numeric solution pairs that were obtained and then set the expressions $10 + 2a$ and $6 + 3b$ equal to each other, or they could draw the graph of this linear relationship. These strategies would suggest a more formal algebraic approach to the task than the computational method described above because of the greater degrees of abstraction present in the representation and solution process.

Task 2 could be worked in a purely computational way, where the value of $2^9 \cdot 3^4 \cdot 5^6$ is computed and the value of the one's place identified. However, students could also identify a factor of 10 present in the product, either through initial computation or by an examination of the factors, and recognize that the value of the product must end in zero.

Task 1

Overall analysis revealed that sense-making and property-noticing activities of some students occurred in this setting, which may not have been constructed if working alone. However, slightly less than half of the pairs of students worked almost independently of one another, and very little knowledge was shared. Students who worked collaboratively constructed solution strategies that involved meanings introduced by both students, and the congruencies and incongruencies in the individual strategies seemed to advance the collective approach in most, but not all, working pairs (see Table 1). The solution of Tim and Molly[2], discussed later, provides a clear example of a pair of students whose collective solution strategy yielded results that may not have evolved if working alone.

Every group began the problem by listing dollar amounts for a given number of rides, and every group but one found either the solution (one ride for Person A, two rides for Person B, \$12) or the solution (4, 4, \$18). Overall, more than half of the student pairs attended to covariance properties in regard to the number of rides or money spent (see Table 1). This analysis either involved the relationship between the increase in the number of rides of each carnival attendee for every new solution (three additional rides for Person A and two additional rides for Person B), or an increase in the amount of money spent for every new solution (one new solution for every \$6 increase).

A greater number of eighth grade pairs than seventh grade pairs performed an analysis of the rate at which new solutions were found in regard to money spent (see Table 1). This result was due to the fact that the seventh grade student pairs were more likely to focus only on situations in which both riders went on the same number of rides. Finding the amount of money each person would spend for the same number of rides does not include an analysis of the times when each spends the same amount of money but enjoys a different number of rides. Students who only focused on a uniform increase in the number of rides across riders found the solution (4, 4, \$18), but failed to find solutions such as (1, 2, \$12).

Students investigating Task 1 in ways that did not involve a uniform increase in the number of rides had to perform a complex analysis of the covariance properties present in the situation. The students first had to recognize that the two quantities relating to the amounts of money spent by each attendee increase at different rates as more rides are taken. Then the students had to realize that this difference impacts another form of covariance—each person's number of rides is different for each solution. The latter difference is due to a 3:2 ratio between the number of rides of the two attendees from one equal spending value to the next (see Figure 2). Therefore, to make these kinds of generalizations, the students had to simultaneously negotiate two different covariance situations and relate them to their analysis.

The eighth grade students were more likely to discuss a general solution to the task by finding a pattern in the solutions displayed in a numeric table of values they created. An example of this kind of analysis is provided below. Many of these groups took a more formal approach by making explicit notice of the manner of covariation in the number of rides and money spent, as just described. These kinds of observations are pivotal when attempting to generalize a solution for an arbitrary number of rides.

Two other groups made explicit mention of covariation that was inappropriate to the problem context, and four groups made no explicit mention of any notion of covariance. One pair of students explored these notions using algebraic symbols, but the rest of the students did not go beyond the numeric table and verbal descriptions.

Table 1

Algebraic understandings of covariance and solution possibilities exhibited by students on Task 1

	No. groups	
Group's algebraic understanding	7th grade (7 groups)	8th grade (8 groups)
Made explicit notice of the differences in covariation in the number of rides and money spent between the two carnival attendees (3:2 increase in number of rides per additional solution)	3	3
Made explicit notice of covariation in regard to amount spent (a new solution for every 6 dollar increase)	1	4
Made no explicit mention of any notion of covariance	3	1
Stated or suggested that they realized that their were a theoretically infinite number of solutions to the task	3	4
Stated or suggested that they realized that there were "a lot" of solutions to the task	2	4
Found only one solution and did not seem to think others could be found	2	0

Analyzing the property of covariance (i.e., slope) from multiple situational perspectives eluded many of the student pairs, most of whom were in the seventh grade (see Table 1). Overall, some general differences existed in the level of algebraic reasoning and awareness of algebraic properties between the students in the two grade levels. This may have been due to the eighth grade students' prior exposure to algebraic ideas, including notions of variable and equation solving.

Task 2

Collaborative problem-solving behaviors by the student pairs on Task 2 occurred less frequently than on Task 1. All but two of the student pairs either worked in isolation, with each student computing the value of the product individually, or with no real collaborative problem-solving activity. The latter involved negotiation of individual computational duties, such as one person calculating 2^9 and the other person finding the values of the other two factors. All students who worked in isolation compared their answers to the calculations of their partner. Overall, there was very little interaction between the students on this task.

The solution strategies on Task 2 were also more uniform. Only one seventh-grade pair and one eighth-grade pair who successfully answered the question did not calculate the value of the expression. These two student pairs noted the presence of the factors two and five in the product, recalled the property of a factor of 10 producing a zero in the one's place, and then utilized this property to construct a solution. However, unlike Task 1, simple probing questions, such as "Is that the only way you could have done it?", elicited generalizations in many of the student pairings who had already solved the problem through direct calculation. In particular, the realization of the presence of a factor of 10 led to the development of strategies in three of the seventh grade pairs and three of the eighth grade pairs similar to that provided by the two pairs of students mentioned above. Therefore, this task did not initially elicit algebraic solution strategies, but these behaviors did arise when prompted by the researcher.

Enactive Learning: A Closer View

This section will focus on the problem-solving strategies of Tim and Molly, two eighth graders, as they worked together on Task 1. It will explore the contextual, social, and cognitive factors that may have played a role in their problem-solving processes. The transcription of the solution strategy developed by Tim and Molly illustrates how they initially constructed an understanding of the problem based on the properties of the two functions that represent the amounts of money spent by the two carnival attendees. These properties involved the initial amount (admission) and rate of increase (cost per ride). Analysis centered on their individual sense-making and collective meaning-making activities as they engaged in a solution attempt. This interview was chosen because of the degree of collaboration and collective sense making that occurred. Although shared meaning developed between Tim and Molly, the previously discussed data indicates that such activity was not found in all of the interviews.

After Tim found an initial solution of one ride and two rides (1, 2, $12), Molly conjectured that more solutions would be possible. They discussed the covarying properties of the two dependent variables, and this led Tim to the solution (4, 4, $18). At this point, the two participants realized that infinitely many solutions were possible, but they had no means of describing what these solutions would be. Therefore, the students approached the problem from a computational perspective by making use of the functional notion of covariance to find alternate solutions, but they were unable to generalize to the arbitrary case. Hence, their solution showed aspects of algebraic thought, but they failed to develop functional properties associated with arbitrary quantities. Instead, the idea of covariance was contemplated from an arithmetical perspective. Further, the students' ability to articulate their sense-making processes led to additional meaning making *between* the pair, leading to the acquisition of higher degrees of understanding of the mathematical situation.

A closer look at specific portions of the interview reveals particular instances of sense making and instances where the students noticed properties that led to their solution. The following segment occurred at the onset of their solution attempt on Task 1:

T & M: (read problem, mumbling)

Molly: I don't know.

Tim: Wow (exasperated). (pause) Do you have a calculator?

DS: No.

Tim: OK, so right now we know that each person paid at least 10 and 6 dollars.

Molly: And then they could also go on two times, but it depends how many rides he'd want to go to.

Tim: How many rides can each—

Molly: Actually, it depends on how many rides he'd want to go on.

Tim: No actually, um, actually it's pretty much asking what, like, sort of asking, it's almost like asking lowest common multiple, almost, or something like that, but anyway.

Molly: You actually learned something (gibingly).

Tim: That wasn't funny (good-humoredly).

Molly: One of my family's jokes.

The beginning portion of the interview illustrates several important aspects present throughout the interview. First, the two students felt comfortable with one another and did not appear to be nervous or affected by aspects of the interview setting (e.g., the video camera, presence of researcher, pressure of solving the problem). Second, the students were individually engaged in the task and actively sought an understanding of the context and solution strategy.

The next few lines of the interview illustrate that the two were beginning to make use of each other's sense-making activities.

Molly: This actually depends on how many rides you went on, to go on, if you wanted to go on like two rides, you could spend

Tim: It depends, no, OK

Molly: for each

Tim: OK, if I wanted, if I was here and you there, right, if I wanted to go on two rides that would be a total of 14 dollars, for me, if you wanted to go on two rides then it would only be 12 dollars for you, so it would end up costing

Molly: Oh, I was mixed up, OK (laughs) well, one of them, one person didn't have to ride at all to get 10 dollars, no

Tim: OK, so what we are trying to figure out is, they spent the same amount

Molly: Yeah I know, OK, and six dollars, what, what adds up to being, lets see, 10, 20, (long pause), OK

Molly began to explore the problem by advancing on her initial sense of the situation, which involved an understanding of the need to consider "how many rides." She made use of the cost of admission and ride price, situational properties that correspond to the linear functional properties of y-intercept and slope. They were beginning to construct and make use of shared meanings of the situation.

Tim utilized Molly's remarks regarding the need to consider the case of going on two rides to begin his numerical analysis. After these computations, Molly recognized that this would not be a desired solution and said, "I was mixed up." In the last two comments, they returned to their individual sense-making activities. Tim then made the following key insight:

Tim: I could go on one and you could go on two and then we'd each spend 12 dollars (very confidently). That's the answer (chuckles). And 12 dollars is one answer, or, 'cause

Molly: Yeah that's true.

Tim: I'd go once,

Molly: You'd go once

Tim: You'd go twice (writes) 12

Molly: So you'd

Tim: So

Molly: one person would go on two rides and the other person would go on one.

Tim: No, one person would go. Yeah (put pencil down) I get it, there's that one.

Using the covariance properties previously explored, Tim decided to vary the number of rides for each person, changing the (2, 2) case to (1, 2). This produced a solution that Molly was able to immediately verbalize.

Tim's personal solution became a shared and meaningful one, although Tim was clearly the initiator of nearly all of the public meaning. But this immediately changed when I became part of the interaction after assuming from Tim's last comment that the problem-solving activity had ended:

DS: You spent 12? OK, so the one was two and the other was one ride?

Molly: Well you could spend more time and

Tim: You could spend a lot more money, and then

Molly: Yeah

Tim: You could

Molly: One person could go on two rides and the other person could go on four rides and you'd still get the same thing.

Molly's current sense of the solution allowed her to expand the situation by linearly increasing their solutions, with some minor prompting from me. However, Molly's generalization was inaccurate, as it did not include a proper analysis of the rates of increase in the number of rides between solutions.

While Molly was verbalizing the meanings she constructed that led to this conjecture, Tim makes sense of Molly's remarks and challenges the meaning put forth by Molly after conducting a few computations:

DS: OK, what would they spend in that case?

Molly: Well, if you had

Tim: (writes) 6 times 30

DS: Or what makes you say that?

Tim: and if the other person goes on four, or wait a minute, this person goes on two.

Molly: It would be the same amount just as the first time.

DS: By the first time you mean when the one person rides one and the other person rides two?

Molly: OK, one person

Tim: No it wouldn't (confidently). OK, wait, so you're saying, so you're saying

Molly: OK, you go

Tim: one person goes four times, goes on four rides

T & M: and the other person goes on two

Tim: that's 12, 14, this one's 9, 12, 15, 18, so, no, that's not exactly true.

Molly: What (contentiously)?

Tim: You said one person goes on two and another person goes on four.

Molly: So, but wait.

Tim: 'cause this person has to pay six just to get in and three for each ride

Molly: yeah

Tim: that's four rides, that's a total of 12 right there, plus another six is 18, so that's not necessarily true.

Molly: Well, um, if you take, it's 10 and six, and then the first time, one person goes on one once

Tim: that's two dollars right there

Molly: yes, that's two dollars, and then another person goes on another time, that's two times, and then you go on it again, six, and another two, I guess that doesn't work. It's worked in the past for me.

Using the meanings put forth by Tim, Molly recognized the faults in her sense of the situation and altered her belief in her own solution of (2, 4). The pair constructed a collective sense of the precise nature of the covariation in the situation and used this to further their understanding and solution of the problem, including a greater awareness of the property of covariance. Eventually, their meaning making exchanges collectively led to the (4, 4) solution, but either participant made no further generalizations. However, Tim does suggest that he believes more solutions are possible:

Tim: You could find other answers if you'd keep going but, if you had all the time in the world I'm sure you could find a lot more answers.

Molly: If you were an old fogey I bet you could, because you'd have a lot of time.

Despite failing to advance the solution, the above dialogue illustrates how the articulation of sense and joint construction of meaning are interdependent processes that can lead to knowledge acquisition.

Molly and Tim successfully completed Task 2, but their analysis was not as thorough and did not employ any algebraic methods. Their solution strategy involved performing the entire multiplicative computation stated in the problem, and then examining the digit in the one's place in their final answer. The pair shared the computation and writing duties throughout, but did not construct a solution in a truly collaborative manner.

Molly: It's two to the ninth, so one more (multiplication of two).

Tim: OK, that's 512. Here, you do this for a while. (hands Molly the paper)

Molly: OK.

After the pair completed the computation, I initiated a discussion that did not generate a more algebraic approach, with Tim concluding by stating, "I think there's a shorter way but we don't know." The use of the term "we" suggests that Tim was viewing the pair as a fully functioning team throughout this task as well, even though no real collaboration occurred. However, unlike Task 1, the pair was unable to utilize algebraic methods in this solution because the ability to identify and make use of the effect of multiplication by 10 was not apparent. As discussed before, this result was typical across all but two of the groups without prompting from the researcher.

Analysis of the work of another eighth-grade pair provided a different perspective on the nature of collaboration and the use of algebraic properties. Like Tim and Molly, Bill and Gloria collaboratively explored Task 1 and arrived at multiple solutions and a detailed analysis of the 3:2 ratio that existed between the increase in rides. On Task 2, both students initially asked for calculators, and then divided up the computation.

Bill: I'll do two to the ninth and you do three to the fourth.

Gloria: OK.

The pair continued computing silently for a very long time, with both eventually working on 5^6 together. However, before this computation was completed, Gloria stated, "Wouldn't we get zero?" Bill either ignored or dismissed this comment and continued his computation. After several more minutes, the pair arrived at an answer of zero.

Gloria: Zero will be there, and if you multiply anything by zero you get zero.

DS: Can you look at the problem and see why?

Bill: (long pause) Not really.

Gloria: We got zero here (at one point in the computation) and it stayed zero after that.

Although Bill and Gloria began working collaboratively on the computational aspect of the problem, they did not share in any meaning-making activity regarding the conceptual aspects embedded in this task. Gloria began to explore this, but did not progress either individually or collaboratively with Bill. As stated above, this lack of group meaning making was true of the majority of paired groupings on this task.

Conclusion

In many situations, learning is a collective process of privately constructed personal sense and publicly negotiated meaning. The ability to utilize one's sense to articulate meaning, as well as make sense of other's stated meanings, enriches the taken-as-shared network being constructed. Learning is a dynamic interplay between one's sense, one's stated meanings, and the sense one makes out of other's stated meanings. This study provides evidence where these three aspects of learning can combine to form a collective, rich understanding of a problem-solving situation, or a situation where sense- and meaning-making do not fully develop.

This study does not intend to make the case that students are more successful working together. Rather, it tries to articulate how specific conceptual processes are utilized when working on algebraic tasks in both an individual and collaborative environment, and what kinds of algebraic tasks might elicit individual and collaborative problem-solving behavior. The main mathematical object of analysis in Task 1 was the linear growth relationship that contained the property of constant rate of change and a covariance property between the two variables. In the case of Tim and Molly, the taken-as-shared meanings that developed were certainly a product of the two individual's sense-making activities, but the development of these meanings also effected future sense-making and meaning-making activities in the task. These students were both willing and able to participate in this personal and shared process, and their collective sense- and meaning-making activities were at the heart of their learning experiences.

Overall, the tasks and learning environment were able to elicit sense-making activities based upon the individual students' understandings that were then transferred into meaning-making activities shared by the student pairs. These activities led to solution strategies that involved the generalization of arithmetic constructs into more algebraic realms. This occurred in the context of interactions between students in Task 1, but interactions with the researcher also led to these occurrences in Task 2 for several student pairs. The most significant advancements made by the student pairs appeared to have occurred in Task 1, where the students explored aspects of the covariance to advance their understandings of the task, the number of solutions found, and their ability to articulate a generalized solution. The fact that there was more interaction on Task 1 may have been due to the fact that the linear growth properties inherent in this task were more apparent to the students than the multiplicative or number theory properties inherent in Task 2. The use of a context involving two people may have also made Task 1 easier to model than Task 2. As a result, the students were able to participate in richer meaning making exchanges about Task 1, and develop more advanced solutions.

It appears that, for these middle school students, algebraic understandings of the property of covariance on Task 1 were readily available. These understandings allowed many of the student pairs to approach the task in algebraic realms, making use of the properties of covariance, correspondence, and slope to identify multiple solutions to the task. Understandings of appropriate arithmetic properties which may have led to a more generalized approach to Task 2 were not as apparent. Hence, in this study, the students were better able to utilize their sense-making activities to uncover properties related to aspects of functional algebra than with properties associated with algebra as generalized arithmetic. But the limited scope of this study does not begin to allow for generalizations of this result. Further research is needed to discuss differences in students' facility with various algebraic properties and the ability to work with these properties at various levels of generality (Kaput, 1995).

Moreover, students with procedural problem-solving tendencies (which many of the students appeared to possess) would not be expected to utilize algebraic understandings on Task 2 because a solution strategy requiring direct computation is immediately recognized. But, in Task 1, a student who begins to compute the number of rides associated with various amounts of money spent by the carnival attendees is generating information that can lead to a discovery of notions of slope and covariance, which happened frequently in these student pairs. Therefore, the problem-solving behaviors of the students may have led to differences in their ability to recognize properties of the mathematical situations and to solve problems with various degrees of generality across the two tasks. These data illustrate the complexities inherent in the personal and public interplay of students' knowledge construction processes that often go unnoticed in an interactive, problem-solving environment.

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[1] For example, $a^2 - b^2 = (a + b)(a - b)$ expresses the relationship between the difference of the squares of any two numbers and the product of their sum and difference. An additional example involves the discovery that the result of any product that contains a factor of 10 must have a zero in the one's place.

[2] All participant names are pseudonyms.