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Guest Editorial…

Problem Solving *by* Analogy / Problem Solving *as* Analogy

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> Everyone talks about how important it is for a young quarterback to sit on the bench and watch the game. But instead of learning how to play, all they learn is how to sit and watch. (A paraphrase of Mike Ditka on ESPN's Sunday NFL Countdown, November 18, 2007)

If current frameworks, standards, and assessments are any indication, there is international consensus that students should be able to solve new (to them) mathematical problems ("real-world" and otherwise) in addition to knowing specific facts and performing basic calculations. Problem solving as a part of the mathematics curriculum has gone in and out of favor for several decades, perhaps due to the range of ways it has been approached in textbooks and classrooms. Too often, problem solving is taught very algorithmically and, as mentioned in another article in this issue, is seen as independent of mathematical content. In fact, I believe that mathematics is learned *through* problem solving, so when taught well, mathematical content and problem solving can't really be separated.

A number of problem solving "habits of mind" are taught explicitly in mathematics courses at all levels. Many of these ways of thinking can be traced to suggestions from *How to Solve It* and other publications by the father of modern problem solving, George Pòlya (1945, 1954). (In fact, in Volume I of his *Mathematics and Plausible Reasoning,* he wrote extensively about analogy in mathematics.) I won't restate these suggestions here since most have become part of the present day mathematical lexicon, but I *would* like to present some methods and ideas that I have found promising in helping my students become more successful problem solvers.

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Analogy *in* Problem Solving

A common trait of expert problem solvers is their ability to recognize connections between two or more problem situations and their solution methods. That is, by re-posing a problem in another context, the problem is often made more tractable.

> **The Handshake Problem:** *The twenty members of the math club met last Tuesday to plan next month's annual banquet. A tradition of the club is to start each meeting by having the members shake hands with each other. How many handshakes will occur?*
>
> There are two common strategies students (and others) use to approach this problem. The first is motivated by the observation that we could arrange the club members in some order (from first to twentieth). The counting of the handshakes usually starts something like this:
>
> *The first member shakes hands with each of the other 19 members, while the second member shakes hands with 18 others, the third members shakes hands with 17 people, and so on, until the 19^{th} person shakes one person's hands and the 20^{th} person doesn't shake any. Therefore, the total number of handshakes is*
>
> $$19 + 18 + 17 + \ldots + 2 + 1 = 190.$$
>
> (Of course, the 2^{nd} member is involved in a total of 19 handshakes, like all the others, but the handshake with the first member has already been counted, so it would be more correct to say that the 2^{nd} member was involved in 18 *more* handshakes, the 3^{rd} member (who had already shaken hands with members 1 and 2) was involved in 17 *more* handshakes, and so on.)
>
> The second strategy usually goes something like the following: *Each of the 20 math club members would have shaken the hands of the other 19 members, for a total of $20 \cdot 19 = 380$ hands being shaken. But each handshake requires two hands (or each handshake gets counted twice), so there are $380/2 = 190$ handshakes in all.*

Later, when asked to determine a closed form solution for 1 + 2 + 3 + … + n (determining the nth triangular number, for example), many students recognize that this sum corresponds to using the first method of solving the handshake problem for a group of n + 1 people. Since the second solution method involves the calculation $\frac{(n+1)\cdot n}{2}$, the students arrive at a closed form solution to the sum by taking advantage of this connection.

Of course, there are several alternative strategies that would give rise to the solution, but the example serves as an illustration of the point that the ability to make connections between apparently dissimilar problems is something we, as educators, hope to promote in our classes. In each class I teach, I try to convince my students that solving mathematical problems is an exercise in *reasoning by analogy*. In fact, we could argue that since each new problem is solved by somehow connecting it to our existing knowledge, this reasoning by analogy is *always* happening in one way or another.

Another (possibly less familiar) example came up recently in my Number Theory course for inservice elementary and middle school teachers.

How many ways can you spell NUMBERS?
Starting at the top and always moving to adjacent letters in the row directly below, how many paths can you take to spell **NUMBERS**?

N
U U
M M M
B B B B
E E E E E
R R R R R R
S S S S S S S

As anticipated, many students observed that, starting with the **N** in the first row, there are 2 choices (right or left) when you move down to the second row. Similarly, there are 2 choices when moving from each subsequent letter, so there are 2^6 = 64 different paths that spell **NUMBERS**. However, several students used a strategy I didn't expect; however, in retrospect, I should have foreseen it. These students kept track of the number of ways to get to each individual letter using a method we had used earlier to count taxicab paths between lattice points in the coordinate plane, labeling each letter as shown below.

$\mathbf{N}^1$
$\mathbf{U}^1$ $\mathbf{U}^1$
$\mathbf{M}^1$ $\mathbf{M}^2$ $\mathbf{M}^1$
$\mathbf{B}^1$ $\mathbf{B}^3$ $\mathbf{B}^3$ $\mathbf{B}^1$
$\mathbf{E}^1$ $\mathbf{E}^4$ $\mathbf{E}^6$ $\mathbf{E}^4$ $\mathbf{E}^1$
$\mathbf{R}^1$ $\mathbf{R}^5$ $\mathbf{R}^{10}$ $\mathbf{R}^{10}$ $\mathbf{R}^5$ $\mathbf{R}^1$
$\mathbf{S}^1$ $\mathbf{S}^6$ $\mathbf{S}^{15}$ $\mathbf{S}^{20}$ $\mathbf{S}^{15}$ $\mathbf{S}^6$ $\mathbf{S}^1$

The superscript for each letter represents the number of paths that start at the top **N** and lead to that letter.

Adding up the number of ways to get to each letter on the bottom row, they also arrived at the total of 64 paths that spell **NUMBERS.** One student spoke up, "Hey, that gives us a way to prove that conjecture we made earlier—that the sum of the entries in a row of Pascal's Triangle is a power of 2!" Of course, I was *very* happy that she made this connection, but I was even more pleased since I knew it had occurred naturally; I hadn't "telegraphed" the strategy for them since it hadn't occurred to me that anyone would solve the problem that way.

The surprising connections continued when I asked my students to consider the same question with the following configuration. My goal was to give the other students an opportunity to use the student's method we had just discussed.

N
U U
M M M
B B B B
E E E
R R
S

I had assumed that they would recognize that there were 20 ways to get to the S in the middle of the 7th row, as determined in the previous problem. Most did as I expected, but one student looked at it another way. She noticed that she could label the first 4 rows as before.

$\mathbf{N}^1$
$\mathbf{U}^1$ $\mathbf{U}^1$
$\mathbf{M}^1$ $\mathbf{M}^2$ $\mathbf{M}^1$
$\mathbf{B}^1$ $\mathbf{B}^3$ $\mathbf{B}^3$ $\mathbf{B}^1$
E E E
R R
S

But then she started from the **S** at the bottom and worked up, noticing there were the same number of ways to go *down* to each **B** as there were ways to get *up* to them.

$$
\begin{array}{c}
\mathbf{N}^{1} \\
\mathbf{U}^{1}\quad \mathbf{U}^{1} \\
\mathbf{M}^{1}\ \mathbf{M}^{2}\ \mathbf{M}^{1} \\
\mathbf{B}^{1,1}\quad \mathbf{B}^{3,3}\quad \mathbf{B}^{3,3}\quad \mathbf{B}^{1,1} \\
\mathbf{E}^{1}\quad \mathbf{E}^{2}\quad \mathbf{E}^{1} \\
\mathbf{R}^{1}\quad \mathbf{R}^{1} \\
\mathbf{S}^{1}
\end{array}
$$

She then computed the sum 1•1 + 3•3 + 3•3 + 1•1 = 20, using the following explanation. There is 1 way (1•1=1) to spell **NUMBERS** by going through the first (leftmost) **B**. That is, there is 1 way to get to that **B** from the top and only 1 path *from* that **B** to the **S** at the bottom. There are also 3•3=9 ways to go through the second **B**, i.e., 3 paths down to the **B** and 3 paths from there to **S.** Identically, there are 3•3=9 ways to go through the third B. Finally, only 1•1=1 way exists by going through the last **B**. This strategy yields a total of 1 + 9 + 9 + 1 = 20 paths that spell **NUMBERS**.

Of course, the students aren't the only ones who are learning in a class that is based on problem solving. As the student was explaining her solution, it occurred to me that she had just outlined a very natural, elementary proof of the following theorem:

> The sum of the squares of the entries in row *n* of Pascal's Triangle is the middle term of row *2n*. That is, $\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$.

Following the student's lead, I was able to reason by analogy to see the connection with another problem and make a connection I had never seen before. When I gushed about how cool I thought her solution was, she was taken aback by my surprise, saying, "I thought this is what you *wanted* us to do! I was just trying to use what we did before."

This was gratifying because a common mantra I repeat in classes I teach is, "How is this like something I've done before?" By this, I do *not* mean to give the impression that problem solving is *remembering* how to solve a problem. Rather, being successful at problem solving requires the ability to sift through all the stuff you know and pick out those facts and techniques and strategies that might apply to the problem at hand.

Meta-Problem Solving

Discussion of the actual process of problem solving is missing from the vast majority of mathematics textbooks, which tend to focus on the *final* process of proof. How the author *figured out* the solution is often missing, thus leaving the impression that the author *just knew* how to solve the problem. (One reason for this could be the traditional textbook style, while another is the need of the publisher to reduce costs by limiting the amount of text and, therefore, paper.) I know that if I'm not careful, I can also give this impression in my classes, especially if I'm in a hurry. (When are we *not* in a hurry, though?) When leading a discussion of a problem, I attempt to "fill in the blanks" with explanations of my thinking: *What I was thinking here was…* or *I noticed that this problem felt similar to …* or *I wasn't sure what to do, so I decided to try …* I also elicit suggestions from the class so they won't think the whole discussion was "scripted." We often pursue strategies that I'm pretty sure (or I know) won't be successful, because much can be learned from "dead ends," as well. This seems to convince students that I *really am* figuring out the problem with them rather than remembering (or just knowing) how to do it. They see that it's okay if you don't *know* what to do when you encounter a problem and that they, too, can figure out the problems.

These discussions of meta-problem solving—the processes of trying and failing, trying again and making partial progress, and so on, that problem solving entails—inevitably involve statements like *Doing mathematics is like hiking through the woods; Problem solving is like driving a car; Problems are just really big Sudoku puzzles.*

Analogies *of* Problem Solving

The use of analogy in solving problems allows the solver to connect the familiar (a previously used method, strategy, or context) to the unfamiliar (a new problem). Therein lies the power of analogy. Even when I try to share my thinking during problem solving discussions, students have difficulty applying the methods and strategies they've previously used to solve a new problem. I had believed this was due to unfamiliarity with having to solve *problems*, rather than *exercises* designed to practice a particular skill or use a specific, easily identified, concept. As previously mentioned, I have always tried to model the problem solving process in classes, but realize that at the end of a problem discussion, even when insisting on student input for direction and strategy, I am often the one who solved the problem. I have repeatedly shared with students what I consider one of the most important facts about problem solving: you're not expected to *know* how to solve a problem; you're supposed to *figure out* how to solve it. But even after providing them with many problem solving experiences, often in more than one course, many still have trouble getting started, saying that they're not sure what the "right way" to solve the problem is. I believe that this is due to the fact that they don't have a sense for what actual problem solving *feels like*. By building explicit connections to other activities with which they might be familiar, I hope my students are able to feel more comfortable with the processes (and inherent uncertainty) of problem solving. None of these analogies are perfect, and it is easy to stretch them to their breaking points, but I have found that each of them serves as a touchstone for the problem solving process. I'll explain one in detail.

A Mathematical Hike in the Woods

There are at least three different ways to go hiking in the woods. One is to be led down a previously created path, often by an expert who's taken the path before. Another is to follow a path with which you are already familiar, perhaps after being led along the path several times. The third is to be willing to leave a familiar path to try a completely new trail when the need arises (or just for the heck of it).

Being led on the hike is efficient, if your goal is to get to the end of the trail. You can see some sights but only those you are led to. It is probably the most comfortable method for the novice hikers, because it isn't necessary for them to keep track of where they are. However, this also doesn't help them learn to get to places that are not on the trail. Traveling alone on a path with which you are familiar is less efficient, since now you don't have an expert to keep you on the trail. However, it's definitely more interesting, since you can choose when and where to stop and how quickly to walk.

Of course, in order get to a location you haven't already been to, you must be willing to stray off the path and possibly blaze a new trail every now and then. Some of these new locations might be just off the path, while others may be far away from your comfort zone, but these less-traveled sights are often the most interesting (and educational). And each new trail you forge provides you with new locations you know how to get to (and return to later).

But many hikers aren't natural explorers. It isn't likely that someone who has always been led through the woods will stray far from the known path. It takes a rare person to feel confident enough to take over leading the group to the end of a trail (or even back to where they started) or to choose to lead the group entirely off the path just to explore, unless he had been given the chance to explore in the past. Unless it is your responsibility to get everyone back to the trailhead, your mind is usually focused on following the leader.

The second method (taking a path with which you are familiar but without a leader) is more work, since you need to pay closer attention to where you are and which branches of the path you need to choose. You might feel limited to taking the particular path you are on, but the real fun comes when you veer from the known path and explore, knowing that if you choose the left fork and arrive at a dead end, you can always find your way back and choose the other fork. When you're the leader of the hike (or at least an active participant in the decision making), whenever you break off a familiar path and look for new sights to see, you must be aware of where the familiar path is (in relation to your current location) so you don't get lost.

Learning to be comfortable with straying off the path (or becoming a trailblazer) comes from experience, but that experience need not be a solo effort. A good hike leader will point out trail markers and share his or her decisions with fellow hikers, letting them in on the thought processes being used as options are chosen and decisions are made. And novices could be encouraged to take charge under the watchful eye of the hike leader, who allows them to make the decisions. Even if the novices get lost, the leader can keep track of their location and bring them back to familiar territory (or help them solve the problem of finding their way back themselves). As novices become more comfortable (and experienced) with making decisions and realizing that every decision is reversible, they are more willing (and able) to explore on their own.

If novice hikers ever find themselves in unfamiliar territory, they will have a very difficult time finding their way out if their only hiking experiences involved having been led or traveling on familiar paths. Therein lies the key connection with problem solving. If we are to help students learn to solve *new* problems that they haven't seen before, they need experiences—guided and otherwise—that allow them to try and fail, try something else, and eventually arrive at a solution to the problem. They aren't alone, though, because the teacher/hike leader is there as a safety net—not to solve the problem *for* them, but to serve as a mirror, reflecting their strategies and progress, asking probing questions that encourage the novices to think through their options. Experienced hikers, like expert problem solvers, are able to keep track of where they are, where they need to be, and the options available to them at any given moment. By explicitly discussing these options and decisions with novices (hikers and problem solvers), the novices gain an understanding of the process and are more likely to be able to navigate the paths themselves. Pòlya (1954) spoke of the importance of intellectual courage—being ready to revise ones beliefs. I like to expand this notion a little further to include a willingness to persevere even when you don't know whether your strategy will lead to a solution or to a dead end. Isn't that the goal we have for all our students?

Other Analogies of Problem Solving

I leave it to the reader to elaborate on further analogies and to propose new ones. Of course, the list of analogies is endless, for what is life, but a series of "problems" to solve and situations to explore?

Learning to solve/learning to drive. Many of the analogies with hiking can be transferred to finding your way around city streets. I have found many problem solving opportunities while driving in the Boston area. You never know when a road, bridge, exit ramp, or tunnel will be blocked off, often without detour signs to help you find your way back to your usual path. And until you are behind the wheel on your own, you don't realize how little you learn by sitting in the passenger seat and watching someone else drive!

Mathematics as a big Sudoku puzzle. There are many connections with Sudoku and problem solving. As you solve more puzzles, you begin to notice patterns (and analogous configurations), develop useful strategies, and become more comfortable with each succeeding puzzle, recognizing that every puzzle is different and you never know which of the many techniques will be the most useful on the next one. And sometimes you just have to make a guess and keep track of the consequences of your assumption. If it turns out you run into a contradiction, you just backtrack (there's that hiking analogy again!), change your guess, and continue.

Mathematics as a video game. I conclude by stating the first "analogy *of* problem solving" I ever used in my teaching—a statement I've long included on course syllabi:

> Mathematics is like a video game; if you just sit and watch, you're wasting your quarter (and semester).

References

Pòlya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Pòlya, G. (1954). *Mathematics and plausible reasoning*. Princeton, NJ: Princeton University Press.