The Mathematics Educator
2008, Vol. 18, No. 2, 3–10

Guest Editorial…

Motivating Growth of Mathematics Knowledge for Teaching: A Case for Secondary Mathematics Teacher Education

Azita Manouchehri

There is consensus within the teacher education community that effective teaching hinges upon several factors. Those factors include the teacher's knowledge of the subject matter, ways the subject matter could be manipulated to be made meaningful and accessible to learners, a deep understanding of learners and their developmental trajectories, and a perspective on short and long term trajectory of curriculum. Teachers need to learn how to select appropriate strategies by reflecting on what factors influence the adaptation of particular approaches when teaching specific concepts. They also need to develop a disposition of inquiry and a professional attitude that allows them to continue to learn from practice (Hiebert et al., 2003). A major challenge in mathematics teacher education is fostering prospective teachers' knowledge base in all these domains. As a means to meet this challenge, scholars have proposed that case-based tasks can serve as a powerful vehicle for advancing teacher learning and nurturing the desired dispositions (Richardson, 1996). It is suggested that as teachers examine dilemma driven tasks and analyze teaching actions they not only learn about teaching but also develop conditional knowledge that is crucial to effective practice (Kishner & Whitson, 1997). In light of these perceived benefits, the use of written, video, or animated case studies in methods courses designed for teachers has gained considerable momentum in the past decade (Merseth, 2003). Certainly, sound analysis of teaching actions calls for deep reflection on the subject matter, the structure of the discipline, and its associated ontological and epistemological obstacles and issues. This specialized body of teaching knowledge can be better nurtured when the contexts for learning are presented to students of teaching at the appropriate time and juncture. As such, content courses for teachers present an ideal environment for raising teachers' awareness of the complexities of teaching the subject matter to children. When used in a mathematics content course however, the tasks need to be crafted carefully so as to ensure that mathematics is treated soundly while allowing for the development of insight in both areas. Our research was an attempt at first developing and then examining the potential of the type of case-based tasks that could be used in a content course designed for teachers. One research question guided our research efforts: What impact do case-based tasks have on prospective teachers' mathematics learning when used as instructional tools in a content course required for prospective secondary mathematics teachers?

Azita Manouchehri is a professor of mathematics education at Ohio State University. Her research has focused on the study of classroom interactions and evolution of mathematical discourse in presence of problem based instruction.

In this article, we will first describe the task we designed and used as the research instrument in our study. Drawing from data collected from two teaching experiments, we will outline ways in which the task seemingly enhanced mathematical and pedagogical development of the participants.

Task Design Issues: Goals and Considerations

Our task design was guided by two prominent scholarly voices as they pertain to the scope, goals, and the audience of this project: recommendations of literature for development of contexts for learning in mathematics teacher education and a situated cognition perspective on task design. In listening to the voice of mathematics teacher education scholars, both mathematical and pedagogical goals were considered. Mathematically, we wanted the task to motivate reflection on connections among various (seemingly disjointed) mathematical ideas, engaging them in mathematical problem solving and critical mathematical analysis. Pedagogically, we wanted to increase teachers' awareness of ways in which children's work could impact instruction and curriculum decision making. The content of the task was chosen, bearing in mind these recommendations. The structure of the task was chosen so as to align with the constitutive elements of situated learning. We wanted it to be authentic, dilemma driven, in order to be conducive for the development of discourse, collaboration, reflection, and critical thinking.

In search of the center: Analyzing construction algorithms

The following problem was assigned to a group of geometry students who had used GSP in exploring geometry concepts in class. The students had spent five instructional periods learning about and working on explorations concerning centers of triangle. Additionally, the group had learned in the previous session that in a circle, if a chord is the perpendicular bisector of another chord, then it is a diameter of the circle. The teacher of this group posed the following problem in order to see whether students could use their knowledge of diameter in a novel context.

Problem:

The center of a circle was accidentally erased. Define a procedure for locating the center.

A list of responses offered by several students during a whole group sharing time of strategies is presented below.

1. Test each of the methods presented below, and decide whether the suggested procedure leads to locating the center. Make a list of "mathematical assumptions" that each child has seemingly made and issues that might need to be resolved in each case.
2. How are the students' responses similar? How are they different? Which of these responses draw on the same mathematical concepts? Justify your responses.
3. Rank the student responses along a continuum ranging from Irrelevant to Sophisticated (you may choose your own ranking categories if you find this range inappropriate). Explain the basis for your ranking (as well as your categories, if you have chosen to use a different ranking scheme).
4. Hypothesize about the mathematical issues that the teacher needs to address with the group. That is, what are the central mathematical topics that she needs to bring up and synthesize? What is the basis for your choice?
5. Decide the type of feedback the teacher might provide to each child based on his/her suggested method. How could the teacher expand the thinking of each student and help him/her justify her/his approach? For instance, what questions could the teacher ask? What extensions could the teacher offer? What example could the teacher use to counter the false assumptions?
6. What other methods could be used to find the center of the circle?
7. Examine the three attached chapters from three different textbooks on triangles and circles. How does the content of these chapters differ from the mathematics content that the group is addressing? How do you account for these differences? What assumptions can you make about the teacher of this group and her choice of curriculum? What is your assessment of these assumptions?
8. Study the article: "A journey with circumscribable quadrilaterals" by Charles Worrall in the October 2004 issue of Mathematics Teacher Journal [Volume 98(3)]. How would you compare the investigations that the author described in his article with those of students in this class? If you had to use an activity from that article to use with this group of students, which would it be and why?
9. The NCTM Professional Standards (1991) suggest that the teachers of mathematics must provide opportunities for students to engage in building conjectures, verifying their conjectures and debating the accuracy of those conjectures. In such a setting, the role of the teacher is to facilitate the students' development by orchestrating tasks and assignments that extend the students' understanding of the concepts, while helping them realize the efficiency and elegance of ideas. Imagine that you are the teacher of this group. Write an outline of a whole group classroom discussion that you might lead, along with any tasks that you will use to structure their work. Justify why the exploration would be helpful in addressing the mathematical needs of the group.

<u>Student 1:</u>Inscribe a triangle ABC in the circle. Then construct the perpendicular bisectors of the sides. The point of intersection of these perpendicular bisectors is the center of the circle (the circumcenter of the triangle).

<u>Student 2:</u>First inscribe an equilateral triangle in the circle. Since in an equilateral triangle incenter, orthocenter, circumcenter, and centroid coincide, once we find one of them then that point is the center of the circle. So, construct the medians and mark the point of intersection of the medians. That point is the center of the circle.

<u>Student 3:</u>Locate three points on the circumference of the circle. From those points construct tangent lines to the circle. In this way the circle becomes the incircle of the triangle we drew. Now, if we construct angle bisectors of triangle ABC, the point of intersection is the center of the circle we started with.

Student 4:Consider circle c. Draw chord AB of circle c. From either B or A, construct a line perpendicular to AB to form an inscribed 90-degree angle. Mark the point of intersection of the perpendicular line and the circle, label it as C. Now, we know the side opposite an inscribed right angle is the diameter of the circle that circumscribes the angle. All we need now is another diameter which we can find using the same procedure as before. The point of intersection of these two diameters is the center of the circle.

Student 5:First fold the circle in half. Then fold it again. Now we have two diameters that intersect at the center.

Student 6:Consider circle j and chord AB of j. Find the midpoint of segment AB, label it as M. Construct a circle centered at M with radius MB. What we have now is two circles that intersect at two points. The segment connecting the centers of the two circles is the perpendicular bisector of the line segment that connects the points of intersection of the two circles. So, if we construct a perpendicular to AB from M, we know that line contains the center of j. If we connect A and B, to the point of intersection of perpendicular line and circle j, we get two right triangles (or one isosceles triangle). If we construct the perpendicular bisectors of the sides, we get the center (their point of intersection).

Student 7:Draw a regular polygon in the circle. Actually draw a rectangle. Construct the diagonals, and the point of intersection of diagonals is the center.

Figure 1. Description of case-based task.

In light of these considerations, we designed "Locating the center," a case-based task to be used as the research instrument in our study (see Figure 1). The case contained a range of student responses to a geometry problem. The reader was then asked to analyze the children's work and to hypothesize about how the teacher of the group could proceed with her lesson in the presence of the children's diverse ideas.

Mathematical Goals of the Task

The mathematical focus of the task is on two topics central to the study of Euclidean and Non-Euclidean geometries: triangles and circles. These topics are usually addressed as separate chapters in standards textbooks. Frequently, students leave a geometry course without realizing the connections between them. The goal of the task is to help students develop a sense for how concepts that make up the field are closely related to each other and are sufficiently self-contained. Usually if not always, this kind of conceptual unity is not nurtured in undergraduate mathematics programs. The following list summarizes specific goals addressed by each guiding question.

Question 1	Mathematical problem solving
Question 2	Mathematical connections
Question 4	Content coherence and unity
Question 6	Mathematical analysis
Question 8	Extending mathematical inquiry and content specific pedagogical reasoning

The task contains a deliberate range of learner responses. Each of these responses could lead to a series of important mathematical explorations as listed below.

- Exploring properties of centers of a triangle (Response #1)
- Relationship between the lengths of inradii/circumradii and the area of triangle (Response #1, #2)
- Trisecting an arc (Response #3)
- Construction of tangents to circles and tangent circles (Response #3)
- Relationship between the size of the equilateral triangles and their respective inradius and circumradius (Response #1,2, 3)
- Inversions on circles (Response #4)
- Angular measures of chords of circle (Response #4, 5)
- Properties of tangent circles (Response #6)
- Circumscribed quadrilaterals and their properties (Response #7)

Pedagogical Goals of the Task

Pedagogical goals of the tasks include assisting future teachers to develop an understanding of the connections between student learning and instructional decision making. The task was structured to allow for pedagogical problem solving. The reader is asked to analyze learners' responses, hypothesize teaching actions, design assessment tasks, and develop activities. Furthermore, by asking them to explain, justify, and defend their choice of representations, assessment, and intervention, we envisioned that the task would provide opportunities for the readers to engage in pedagogical reasoning.

Lastly, the task introduces teachers to professional journals and national professional standards (in the US this includes documents published by the National Council of Teachers of Mathematics). The following list summarizes the pedagogical goals addressed by each guiding question.

Question 3	Content specific pedagogical decision making
Question 5	Content specific pedagogical analysis
Question 7	Curricular analysis and connections to student learning
Question 9	Content specific pedagogical reasoning and decision making

Context and Research Design

The primary goal of our exploratory study was to investigate teacher learning in the presence of case-based tasks when used in a mathematics content course. Using a teaching experiment methodology (Steffe & Thompson, 2000), data was collected in two different sections of a course titled Modern Geometry, an advanced mathematics course required of all undergraduate and graduate mathematics majors pursuing a degree in secondary mathematics teaching. Each teaching experiment lasted two and half weeks (five, 75-minute long sessions) and involved 40 prospective secondary teachers. The students enrolled in this course were either of junior or senior academic standing. All students had completed a minimum of 12 hours of coursework in general pedagogy, and a minimum of 21 hours of mathematics coursework prior to taking this class. The teaching experiment commenced during the second month of instruction, after the participants had examined both topics involved in the task.

The teaching sequence in each class consisted of a particular routine. The participants were assigned the task as a homework activity. The follow-up discussion in each teaching experiment included a large group discussion at the beginning of the session. During this time the participants were encouraged to offer their initial reactions to the case, share their responses to the guiding questions, ask any questions that they might have regarding the content and/or expectations of the task, their impressions of the children's work and ways in which the teacher of the group could proceed with her instruction. A small group activity followed the large group discussion. A final whole group discussion allowed for observable evidence of learning including, synthesizing and formalizing processes that could determine a shared level of mathematical and pedagogical analysis by the group. We used the initial sharing time as an opportunity to collect base line data on the participants' initial approach to task analysis and used that data as a means to trace changes in their work as their interactions with the task intensified.

The decision to rely on two different teaching experiments was to verify our interpretations of the participants' work. By collecting data from two different groups we would be in a better position to account for multiple variables that could impact the participants' interactions with the task including: their mathematical background and experiences, interest levels, classroom routines and instructional practices to which they might have been accustomed.

In each class, two video cameras were set to capture both large group interactions as well as two targeted small groups. The small groups were selected randomly and the same groups were videotaped throughout the teaching experiment. All videotapes were transcribed and used in the data analysis.

Data Analysis

In analyzing the impact of the task on the participants' activities and learning, we considered two intertwined aspects of their work, including: (1) Interactions with the task—Issues that the participants raised about and/or extracted from the activity; (2) Mathematical activities of the participants—Ideas and problems the participants explored. Hence, data analysis was organized around these two key categories.

Participants' Interactions with and Reactions to the Task

In determining the participants' particular approach to task analysis, we focused on their verbal exchanges during the small and large group discussions. We considered whether the participants showed an interest in learning about teaching, the learners, curriculum and mathematics by the type of questions they asked the facilitator or each other. Their comments regarding children's work were also coded in order to trace sensitivity, or lack thereof, to relevant mathematical and pedagogical issues.

Participants Mathematical Analysis and Learning

In seeking evidence of learning, we considered the participants' modes of production (Balacheff, 2000) during the discussions as an indicator of learning. Accordingly, we sought instances of mathematical action, conjecture formulation, and validation processes during each session. We considered situations of action to include instances of problem solving, problem posing, attempts at theoretical constructions, testing a method, or judging merit of ideas by reference to mathematical knowledge. Instances of conjecture formulation included articulation of relationships among mathematical ideas, children's solutions, and suggestions that peers offered. Additionally, in seeking evidence of validation processes, we considered whether the participants referenced mathematical theory when analyzing the problem and its extensions, examining children's work or judging the quality of mathematical arguments offered in groups.

Results

In both sites, the classroom activities followed four phases: Primitive pedagogical theorizing, facilitator modeling, mathematical problem solving and curriculum analysis, and pedagogical inquiry.

Phase I (Approximately 35 Minutes)

During the first phase of the activity, the participants were reluctant to engage in discussions about mathematics, showing a tendency to focus on pedagogical theorizing. Their suggestions, however, were not grounded in evidence or supported by a rationale for choice. They seemed confident in their assessment of children's strategies and rarely elicited explanations and/or guidance from each other or the facilitator. They made brief and trivial references to children's solutions and characterized them as right or wrong without offering a rationale for their choice. None of the participants tried to formalize or justify their assessment beyond stating their personal preferences regarding the classroom environment they felt were conducive to building children's confidence. When asked to comment on how the teacher might decide which of the students' responses to pursue in class, only one participant in both cohorts was willing to commit to a particular method. The facilitators' comments during this phase were aimed at confronting the participants' assessment of children's work and the suggestions they offered for how the teacher might organize subsequent classroom instruction.

Phase II (Approximately 40 Minutes)

In structuring individual and group analysis, the second phase consisted of facilitator modeling. Choosing one of the children's solutions the participants had labeled as incorrect (student 5) as an example, the facilitator spent approximately 40 minutes of the first session in each class describing her interpretation of one child's method, and ways in which it connected to other solutions as well as different mathematical concepts. She listed additional questions the teacher could ask the child either to gain additional insight into his thinking or to advance his work.

Phase III (Approximately 220 Minutes)

Following the modeling episode, the participants were instructed to examine children's solutions again in small collaborative groups. A structure for group deliberation was also set to reach consensus on their assessment of children's work as well as their hypothesis concerning pedagogy. They stated and explored several extension problems that could be shared with children. They also examined different textbook chapters to find places where specific topics could be shared with children in instruction. Three class sessions (210 minutes) were devoted to working on specific problems that different individuals had proposed as extensions to be used with children. The facilitator guided the discussions, offered explanations when asked, and continued to challenge over generalizations that the participants made.

Phase IV (Approximately 75 Minutes)

The last phase of the participants' activities focused on synthesizing and formalizing pedagogical and mathematical analysis of the case. The participants began to ask each other and the facilitator questions about curricular guides that could inform their practice. During the last cycle of the activity, a major component of the participants' discourse included articulation of concerns about their own knowledge of mathematics.

Tables 1 and 2 summarize the different types of comments that the participants made during each phase of their case analysis experience. The major categories of comment types included: Pedagogical, mathematical, eliciting support and feedback, and declarative.

Pedagogical themes included instances of: (1) hypotheses about instructional moves (i.e. the teacher should ask that each child go to the board and explain his method to the group), (2) references to the impact of instruction on children's work (i.e. "Maybe the teacher should have told them to use only one method for solving the problem"), (3) references to the type of evidence they used to support their pedagogical decision making (i.e. The drawing is not accurate so the teacher should be sure to help them draw it right) and, (4) references to the impact of children's work on instruction (i.e. children are not ready to move on to a

Table 1

Typology of Comments: Pedagogical and Mathematical

Comment Type	Group 1	Group 2
Episodes of mathematical question posing (Is there a way to locate the incircle of a quadrilateral?)	61	54
Episodes of conjecturing about new mathematical relationships (I think there is a relationship between the area of inscribed regular polygon and its circumradius)	82	96
Episodes of eliciting explanations concerning connections among mathematical concepts (Can we connect the study of triangles to other polygons?)	75	82
Episodes of eliciting explanations concerning relevant mathematical theorems they could use (Is there a way to find the area of pentagon?)	234	186
Episodes of confronting peer's analysis when discussing children's work or extension problems (But this works for only one case! You need to generalize it)	187	198
Referencing evidence from children's work when hypothesizing about teaching actions (It would be useful to start with the informal approach first, the folding paper part and then connect it to S7's suggestion before moving on to 1st and 2nd methods)	56	61
Offering mathematical explanations on children's solutions (When he says fold it twice he is finding two diameters of the circle, so he is right)	78	72
Statements indicating having gained new mathematical insights—Aha! (I know better why triangles are so important to geometry)	83	72
Number of mathematical problems on which participants worked*	28	25

Table 2

Typology of Comments: Eliciting Support and Declarative Statements

		Phase I & II	Phase III	Phase IV
Eliciting Guidance	Statement of need for additional guidance on pedagogical decision making	M = 2 SD = 4.6	M = 32 SD = 4.2	M = 48 SD = 0.92
	Statement of need for additional information on curriculum	M = 12 SD = 4.2	M = 21 SD = 0.04	M = 52 SD = 4.8
	Statement of need for additional guidance on mathematics	M = 3 SD = 9.08	M = 89 SD = 0.09	M = 22 SD = 0.87
	Statement of need for additional information on learners and how they learn	M = 0	M = 13 SD = 0.19	M = 76 SD = 2.11
Declarative Statements	Statements of concern about the ability to teach	M = 0	M = 12 SD = 0.16	M = 43 SD = 5.03
	Statements of concern about the ability to make sense of children's work	M = 0	M = 0	M = 76 SD = 6.4
	Statements of concern for knowledge about appropriate decision making	M = 0	M = 24 SD = 2.8	M = 32 SD = 4.11
	Statements of concern about finding appropriate resources	M = 0	M = 18 SD = 1.88	M = 29 SD = 2.01
	Statement of concern about the quality of their teacher training	M = 0	M = 19 SD = 0.92	M = 81 SD = 3.3

different topic, the teacher should go back and review her lesson).

Eliciting support included instances of statements of need for additional information on: (1) classroom conditions (i.e. How did the teacher organize classroom activities? What is the teacher's curriculum?), (2) mathematics (i.e. Is this mathematically sound?), (3) learners' thinking (i.e. Is this method common to all children this age?) and (4) pedagogical decision making (i.e. Is this approach developmentally appropriate? Is this question adequate to be posed to this group?).

Mathematical themes included instances of mathematical analysis consisting of: (1) references to mathematical theory in analyzing children's work (i.e. What this student is suggesting is related to the theorem that says the diameter is the perpendicular bisector of a chord), (2) references to the impact of children's work on instruction with a focus on identifying mathematical significance of the ideas presented by a child (i.e. I think the teacher should ask the students to study this solution first cause they can see there are several things that need to be resolved), (3) references to mathematical connections among ideas (i.e. Circumscribing a triangle is the same as finding the perpendicular bisector of three chords, so the sides of the triangle are actually three chords of the circle that intersect at vertices).

Declarative statements included instances of volunteered remarks regarding self knowledge (i.e. I am not sure I know the subject well enough now; how do we decide which textbook to use?), learning or lack thereof (i.e. I used the last student's method to solve this problem), particular needs (i.e. Why can't we do this type of activity more often?), and projected plans for professional development either in mathematical or pedagogical domains (i.e. Maybe we should start building a resource book; I really should work on learning the software more.)

Following the modeling episode by the facilitator (Phase II), mathematical references that the participants made when they analyzed children's work increased significantly. The participants asked more questions about theories they could use or ways in which children's methods related to other mathematical ideas. Indeed, the participants' declarative statements were indicative of the impact of the task on raising the participants' sensitivity to the quality of their own knowledge and ways in which they could improve their capacity to teach.

The Participants' Mathematical Work

The number of mathematical problems on which the participants worked averaged 21 per group. The sheer number of problem posing, conjecturing, and explaining episodes is remarkable considering that the participants had rarely practiced such processes in their regular classroom. The children's solutions were used as a springboard for extending the study of triangles to circumscribed and inscribed polygons. The following is a partial list of common problems on which the participants worked during the extended problem solving episode.

1. Construction of tangent line to a circle, and tangent circles
2. Constructing three kissing circles
3. Relationship among the radii of kissing circles
4. Properties of circumscribed and inscribed quadrilaterals
5. Finding areas of regular polygons using the measure of inradius and circumradius
6. Star Trek Lemma (formulated by the group)
7. Bow-tie theorem (formulated by the group)
8. Determining the interior angle sum of polygons using the Star Trek lemma
9. Properties of Pedal and Orthic triangles
10. Derivation of the extended law of sines using circumcircles
11. Describing the area of triangle and quadrilateral using the law of cosines

Discussion

Our data indicate that the case-based task successfully engaged teacher candidates in doing mathematical inquiry and pedagogical analysis. Evidence of mathematical learning from the task was manifested not only in the number of problems that the participants explored in the course of their case analysis sessions, but also in the amount of mathematical information shared as they made and verified conjectures. Further evidence was evident in how the participants justified their assessment of children's work. They elicited and articulated connections among solutions and detailed ways in which these connections could be made public in instruction. They elicited information about, and also identified mathematical structures that could be used when discussing specific problems with children.

The participants' comments revealed that their experience with the task helped them to realize connections between a teacher's own mathematical knowledge and his or her pedagogical choices. The significant number of participants' requests for theoretical guidance on both mathematics and pedagogy is a strong indication of their interest in learning. Additionally, the large number of statements of concerns they raised about their own knowledge, the quality of their preparation, and their ability to teach further support our perspective that the goal of the task in raising teachers' sensitivity to complexities associated with pedagogical decision making.

Lastly, a careful analysis of case-based tasks in order to maintain the integrity of both mathematics and pedagogy is a time intensive process. Discussion of the center activity took five class sessions. Indeed, if we had the opportunity to pursue each of the mathematical and pedagogical questions that the participants raised, we could have easily doubled the length of time spent on each case. In making decisions about what to pursue with the teacher candidates, we focused mainly on mathematical objectives of the course. Considering the amount of learning developed as the result of exposure to only one case-based task, we are prepared to conjecture that if used systemically facilitators will be in a better position to strike a balance in instruction when simultaneously addressing mathematical and pedagogical issues in a course.

References

Hiebert, J. , Morris, A. K., & Glass, B. (2003). Learning to learn to teach: An "experiment" model for teaching and teacher preparation in mathematics. *Journal of Mathematics Teacher Education*, *6*(3), 201–222.

Kirshner, D. & Whitson, J. A. (1997). *Situated cognition: Social, semiotic, and psychological perspectives*. Mahwah, NJ: Erlbaum.

Merseth, K.K. (2003). *Windows on teaching math: Cases of middle and secondary classrooms*. New York: Teachers College Press.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Richardson, V. (1996). *The case for education: Contemporary approaches for using case methods*. Boston: Allyn & Bacon.

Steffe, L. P. and Thompson, P.W. (2000). Teaching experiment methodology: Underlying principles and essential elements in Research design in mathematics and science education. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307), Hillsdale, NJ: Erlbaum.