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Understanding Rolle's Theorem

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This paper reports on an experiment studying twelfth grade students' understanding of Rolle's Theorem. In particular, we study the influence of different concept images that students employ when solving reasoning tasks related to Rolle's Theorem. We argue that students' "container schema" and "motion schema" allow for rich concept images.

Introduction

Advanced mathematical concepts are characterized by complex interactions between intuitive and rigorous reasoning processes (Weber & Alcock, 2004). Learning calculus, which involves processes pertaining to advanced mathematical thinking, has been a subject of extensive research. One of the significant conclusions arising out of this research is that students typically develop routine techniques and manipulative skills rather than an understanding of theoretical concepts (Berry & Nymann, 2003; Davis & Vinner, 1986; Ervynck, 1981; Parameswaran, 2007; Robert, 1982; Sierpinska, 1987).

The subject of calculus is rich in abstraction and calls for a high level of conceptual understanding, where many students have difficulties. Ferrini-Mundy and Graham (1991) argue that students' understanding of central concepts of calculus is 'exceptionally primitive':

> Students demonstrate virtually no intuition about the concepts and processes of calculus. They diligently mimic examples and crank out homework problems that are predictably identical to the examples in the text. Misconceptions exist as a result of student attempts to adapt prior knowledge to a new situation. Research suggests that students have a strong commitment to these misconceptions and that they are resistant to change and direct instruction (p. 631-632).

While teaching calculus to a wide range of students, it is more practical to appeal to students' intuition when conveying mathematical concepts and ideas, building on what they have already learned without making heavy demands on their aptitude for abstract and rigorous mathematical understanding. Some researchers argue that an introductory calculus course should be informal, intuitive, and conceptual, based mainly on graphs and functions (Koirala, 1997); formulas and rules should be carefully and intuitively developed on the basis of students' previous work in mathematics and other sciences (Heid, 1988; Orton 1983). One of the guiding principles of teaching calculus could be the 'Rule of Three,' (Hughes-Hallet, et al., 1994) which says that, whenever possible, topics should be taught graphically, numerically, and analytically. The aim is to balance all three of these components to enable the students to view ideas from different standpoints and develop a holistic perspective of each concept.

There has been extensive research into the difficulties that students encounter in understanding limits, functions, differentiability, continuity, and so on. However, there is not much literature on students' understanding of other concepts in calculus. Apart from the cognitive obstacles that arise in the learning of calculus concepts due to the complexity of the subject matter, students sometimes encounter difficulties inherent in mathematical reasoning. For example, deductive reasoning is a fundamental tool for mathematical thinking; however, students reveal serious difficulties developing such reasoning skills. Orsega and Sorizio (2000) propose the mental model theory of Johnson-Laird and Byrne as a cognitive framework to analyze students' difficulties in deductive reasoning. Orsega and Sorizio argue that a didactical model should be designed to enable undergraduates to overcome the fallacies of their deductive inferences. They consider a teaching method that enables first-year undergraduates to make explicit the tautologically implicit properties in the hypothesis of Rolle's Theorem and to reflect on them.

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The purpose of this paper is to report on an experiment carried out to study twelfth grade students' understanding of Rolle's Theorem and its relationship to the closely related Mean Value Theorem. In particular, I set up tasks designed to study (1) the learner's ability to state the theorem and apply it to reasoning tasks, (2) the influence of concept images in his or her reasoning about the theorem, and (3) the learner's ability to perceive the relationship between Rolle's Theorem and other related mathematical concepts.

For the reader's convenience, we recall below the statement of Rolle's Theorem.

> Let f be a function that satisfies the following three hypotheses: (1) f is continuous on the closed interval [a,b], (2) f is differentiable on the open interval (a,b), and (3) f(a) = f(b). Then there is a number c in (a,b) such that f'(c) = 0 (Stewart, 1987).

Why Rolle's Theorem?

As observed by Berlinski (1995), "Rolle's Theorem is about functions, and so a theorem about processes represented by functions, an affirmation among other things about the coordination of time and space. ... The constraints deal with the two fundamental mathematical properties of continuity and differentiability..." (191–192). Berlinski further observes that: "Rolle's Theorem establishes a connection between continuity and differentiability. Continuity guarantees a maximum; differentiability delivers a number. Fermat's Theorem [which says that if f has a local extremum at c and if f'(c) exists, then f'(c) = 0] supplies the connection between concepts" (196). The statement of the theorem involves multiple hypotheses, the universal quantifier (for all) and the existential quantifier (there exists). Also Rolle's Theorem offers the opportunity for pictorial, intuitive, and logical interpretations.

The knowledge components required for the understanding of this theorem involve limits, continuity, and differentiability. The proof of the theorem is given using the Fermat's Theorem and the Extreme Value Theorem, which says that any real valued continuous function on a closed interval attains its maximum and minimum values. The proof of Fermat's Theorem is given in the course while that of Extreme Value Theorem is taken as shared (Stewart, 1987). Hence we appeal to the learners' intuition rather than be rigorous in our approach.

The beauty of this theorem also reveals itself in its connection with real life. A ball, when thrown up, comes down and during the course of its movement, it changes its direction at some point to come down. Rolle's Theorem thus can be used to explain that the velocity of the ball which is thrown upwards must become zero at some point (Berlinski, 1995).

Theoretical Background

We shall use the concepts introduced by Weber and Alcock (2004) in their research on syntactic and semantic proof production. A proof is referred to as syntactic if it involves only manipulation of facts and formal definitions in a logical manner without appealing to intuitive and non-formal representations of the mathematical concepts involved. The prover need only to have the ability to make formal deductions based on the relevant definitions and concepts. Such knowledge and understanding is called syntactic knowledge or formal understanding. On the other hand, if the prover uses instantiations to guide the formal inferences, he or she is said to possess semantic or effective intuitive understanding. Instantiation is described as "a systematically repeatable way that an individual thinks about a mathematical object, which is internally meaningful to that individual" (p. 210). "Semantic or effective intuitive understanding is described as the ability on the part of the prover to explicitly describe how she could translate intuitive observations based on instantiations into formal mathematical arguments" (p. 229). While formal understanding is at the superficial level, effective intuitive understanding lies at a deeper level and is characterized by the following features, which we illustrate in the context of Rolle's Theorem:

- Instantiation "One should be able to instantiate relevant mathematical objects" (p. 229). For example, in the case of understanding Rolle's Theorem, the learner instantiates the statement of the theorem if he recognizes its applicability (or non-applicability) in the case of a "typical" function -- not merely that of a quadratic-- possibly in terms of a graph.
- Richness "These instantiations should be rich enough that they suggest inferences that one can draw" (p. 229).
- Accuracy "These instantiations should be accurate reflections of the objects and concepts that they represent" (p. 229). In our context, the examples should not be too special as to suggest properties not implied by the theorem. For example, one may be misled to believe that there is a unique point where the derivative vanishes if one always instantiates the graph to be a parabola.
- Relation to formal definition "One should be able to connect the formal definition of the concept to the instantiation with which they reason" (p. 229).

In their foundational work, Vinner and Tall (1981) have provided a framework for analyzing how one understands and uses a mathematical definition. According to Vinner and Tall, a concept definition and a concept image are associated with every mathematical concept. Concept image is the total cognitive structure associated with the mathematical concept in the individual's mind. Depending on the context, different parts of the concept image may get activated; the part that is activated is referred to as the evoked concept image. The words used to describe the concept image are called the concept definition. This could be a formal definition and given to the individual as a part of a formal theory or it may be a personal definition invented by an individual describing his concept image. A potential conflict factor is any part of the concept image that conflicts with another part of the concept image or any implication of the concept definition. Factors in different formal theories can give rise to such a conflict. A cognitive conflict is created when two mutually conflicting factors are evoked simultaneously in the mind of an individual. The potential conflict may not become a cognitive conflict if the implications of the concept definition do not become a part of the individual's concept image. The lack of coordination between the concept image developed by an individual and the implication of the concept definition can lead to obstacles in learning because resolution of the resulting cognitive conflict is crucial for learning to take place.

The Study

Participants

The students in our study were in twelfth grade in a school affiliated with the Central Board of Secondary Education of India. The twelfth grade is the terminal grade in senior high schools in India and its successful completion qualifies one for university education. After completion of tenth grade, mathematics is an optional subject for eleventh and twelfth grades, but is required for pursuing a degree in science or engineering in universities. The mathematics curriculum is of a high standard and covers a wide range of topics. In eleventh and twelfth grades, the students learn algebra, trigonometry, elementary two and three dimensional analytical geometry, complex numbers, differential and integral calculus, differential equations, matrices and determinants, Boolean algebra, set theory, theory of equations, statics, dynamics and probability theory.

The experiment was conducted after the students had been taught differential and integral calculus over a period of six months, forming part of a year-long twelfth grade curriculum. Rolle's Theorem was part of the curriculum. The rigorous proof is omitted in the course, while graphical interpretations and explanations are offered as to why the statement is valid.

Research Method

Our experiment was comprised of two written tasks followed by interviews. The second task was conducted one week after the first and the interviews were held two days later. Thirty students participated in our study, out of which two students were selected for interviews based on their clarity of expression. They had responded to all the tasks given. The errors that they had committed in the tasks were also committed by many other students of the sample. In view of the depth and detail of analysis we planned, resource restriction limited us to only two participants.

The first task was descriptive in nature and it aimed to probe students' understanding of the statement of Rolle's Theorem. In the second task, we gave the students four graphs along with questions related to Rolle's Theorem. The purpose of this task was to find out if the students were able to connect Rolle's Theorem with problem situations that were presented graphically. Considering the teaching methods adopted, it seemed appropriate to test their understanding using the tasks. These questions had been designed by the researcher to test the intuitive understanding because the material had originally been taught formally.

At this point, two students were selected for interviews. As the interview progressed, our questions built on their responses, so as to gain a better understanding of the student's conception of the theorem. The questions also aimed to provoke a considerable amount of reflection on the part of the students. As a result of the interview, we hoped to ascertain the obstacles, if any, that students face in understanding Rolle's Theorem.

Data Analysis

First Task

The first task consisted of two questions about Rolle's Theorem. The first question was "Explain in your own words what you understand of Rolle's Theorem." Representative responses were identified and are given below.

Abhi (all names used are pseudonyms) exemplifies the response of a student who has a sufficiently rich concept image.

Response 1. (Abhi): [She had drawn two graphs of functions and labeled them (i) and (ii).] “When we draw a graph like graph (i), we can draw tangents at every point on the curve and these tangents will have different slopes. But at one point, say M, if a tangent is drawn, its slope will be equal to the slope of the x-axis, [i.e.,] this tangent is parallel to the x-axis...”

Abhi's response shows a preference for viewing analytic properties geometrically. Her pictorial representation was that of a downward parabola intersecting the x-axis. The hypotheses and conclusion of Rolle's Theorem are stated geometrically and perhaps understood thus. Her notation for the point M on the graph is suggestive of the maximum point.

The following responses are illustrative of inaccuracies in the concept images:

Response 2. (Sweta) [The drawing is given in Figure 1.] “Two points, A, B, are such that they lie on the x-axis: y-coordinates equal to zero. f(A) = f(B) = 0. A curve passes through them such that AD = BD. The graph is continuous since it can be drawn without a gap. It is differentiable, i.e., it can be differentiated at all the points. When all these conditions are satisfied then there lies a point c such that f'(c) = 0. On satisfying these Rolle's Theorem is said to be verified.”

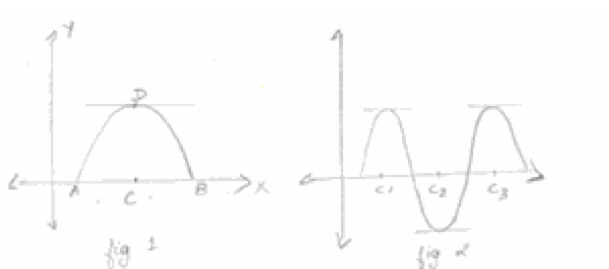

Figure 1. Sweta’s response to the first question.

Sweta assumes f(a) = f(b) = 0. She incorrectly believes the point D = (c, f(c)) divides the graph into two equal parts AD and DB. It is probably because the specific instantiation she has of (the graph of) a function (such as a parabola symmetric about the y-axis) has this property. See Figure 5. If this is so, her instantiation is not sufficiently rich. Since her example is a specialized one, her figural representation of Rolle’s Theorem forces her to assume properties that are not implied by the theorem.

Response 3. (Sheela) “The given part is a function which by nature is continuous and this function is differentiable and our claim is that there exists a point c on that function for which f'(c) = 0 and in a way this point c divides the function into two halves.”

Sheela had omitted the condition that f(a) = f(b). The phrase “in a way this point divides the function into two halves” indicates that she probably imagines the function to have only one extremum, suggesting lack of accuracy in her instantiations.

The following response indicates confusion in distinguishing between hypothesis and conclusion. Like Sweta, she explains through an example.

Response 4. (Anita) “Rolle’s Theorem is satisfied for a function in [a, b] only if the following conditions are satisfied: (a) The function should be continuous, i.e., it could be drawn without lifting your hands, (b) f(a) = f(b), (c) c ∈ (a,b) which means that c must be between a and b, where f'(c) = 0. For example, f(x) = cos(x) ∈ [-π/2, π /2]. (i) It is continuous since it is a trigonometric function. (ii) f(π /2)=0, f(-π /2)=0. Therefore f'(c) = 0. f'(x) = sin(x) implies f'(c) = sin(c) = 0, c = 0 ∈ (-π /2, π /2).”

We observe that Anita includes as part of the hypothesis the existence of an element c ∈ (a, b) such that f'(c) = 0. Nine students committed similar errors. Perhaps this is due to recent exposure to “if, then” mathematical propositions, particularly those involving quantifiers. Yet they do understand what to do when asked to verify Rolle’s Theorem in a specific context as exemplified by her verification in the case of the sine function on [-π /2, π /2], although the syntactical error still persists.

Observe that Anita writes ‘only if f(a) = f(b)’ although examples abound where f'(c) = 0 for some c ∈ (a,b) (e.g. f(x) = x^3 on [-1, 1]). We conclude that the instantiations she carries with her are not sufficiently rich.

Response 5. (Siva) [The drawing is given in Figure 2.] “We draw a tangent P on the curve AB such that tgt is parallel to the curve at the x-axis. This is satisfied only if (a) f(a) = f(b), (b) the graph is [continuous], and (c) there is a derivative for the graph.

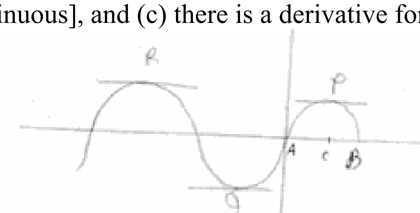

Figure 2. Siva’s response to the first question.

Siva prefers to express the conclusion in terms of tangents at extrema. Again, we note that he states the converse to Rolle’s Theorem as evidenced by the phrase “only if.”

The second question of the first task asked students “How are Rolle’s Theorem and the Mean Value

Theorem related?" Typical responses were identified and are presented here.

Response 1. (Abhi) "Mean Value Theorem is different from Rolle's Theorem because in Mean Value Theorem it is sufficient that the function [be] continuous and differentiable, but in Rolle's Theorem besides the function should be continuous and differentiable, it should also satisfy f(a) = f(b)."

Response 2. (Sweta) "In Rolle's Theorem f(a) = f(b) but in mean value theorem f(a) is not equal to f(b). In Rolle's theorem f'(c) = 0 while in mean value theorem f'(c) = (f(b) – f(a))/(b – a).

Response 3. "Mean value theorem is different from Rolle's theorem in only one way. The condition f(a) = f(b) is not necessary to be proved in mean value theorem, but it is a condition in Rolle's theorem."

Response 4. "Mean value theorem is not completely different from Rolle's theorem. The similarity is that in both the theorems we have got to check whether the graph is continuous. The dissimilarity is that in Rolle's theorem f(a) = f(b) whereas in mean value theorem f(a) need not be equal to f(b)."

Almost all of the thirty students had written that the Mean Value Theorem is different from Rolle's Theorem. They had all given the reason that while Rolle's Theorem has f(a) = f(b) as one of its constraints, it was not present in the Mean Value Theorem. Twenty students had also added that the similarity between the two theorems lies in the fact that the functions have to be continuous and differentiable for both theorems.

We note that none of the students had said that the Mean Value Theorem was a generalization of Rolle's Theorem. This prompts several questions: How do students view relationships among abstract statements? What are the relationships among abstract mathematical propositions, besides identity, that students in twelfth grade are aware of? At what stage in one's mathematical development do they perceive containment relationships among abstract mathematical statements? One should distinguish here between mathematical objects and propositions. For example, these students surely know and are aware that all right triangles are triangles and all squares are rectangles. Further research is needed to explore these questions.

We note also that for some students, the focus seems to be what one should do: For Rolle's Theorem one should check f(a) = f(b), whereas it is not necessary to do so in the case of Mean Value Theorem. Perhaps it is due to excessive emphasis on an algorithmic approach to doing problems in lower grades.

Second Task

The second task is aimed at the following: (1) To explore whether the learner is able to apply Rolle's Theorem when the function is not explicitly specified by a formula. What does it mean, geometrically, to say f'(c) = 0? (2) To investigate whether students are able to identify a non-example of Rolle's Theorem when it is presented to them in the form of a graph that is rich enough to suggest inferences that one can draw. (3) To see whether students are able to relate the given instantiation to the formal definition.

The second task consisted of the following questions:

1. Verify Rolle's Theorem for the function whose graph is given in Figure 3.

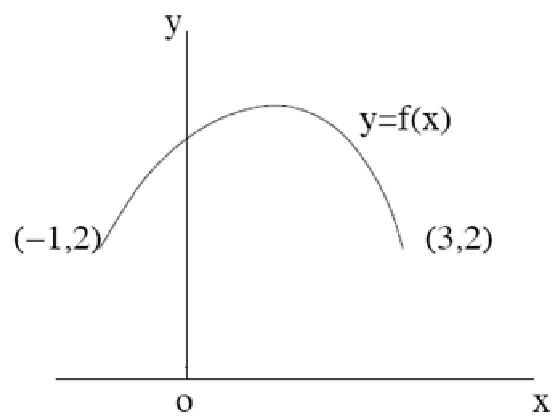

Figure 3. Graph for first question.

2. Is Rolle's Theorem applicable to the function whose graph is given in Figure 4? Give reason.

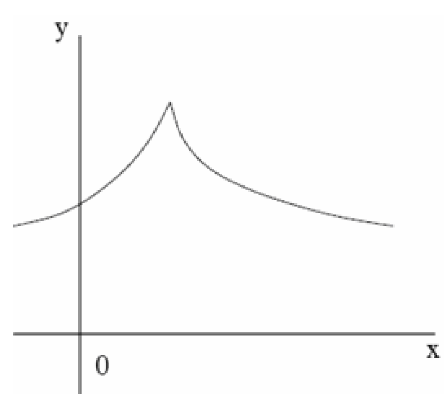

Figure 4. Graph for second question.

3. For the function whose graph is given in Figure 5, show that f(c) = 0 for some c in (0, 6).

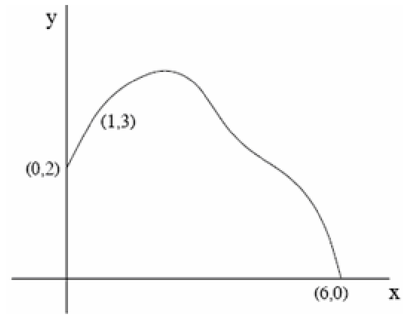

Figure 5. Graph for third question.

4. Consider the function whose graph is given in Figure 6. The graph has horizontal tangents when x = 0 and x = 3. Show that there exists a point $c \in (0, 3)$ such that $f''(c) = 0$.

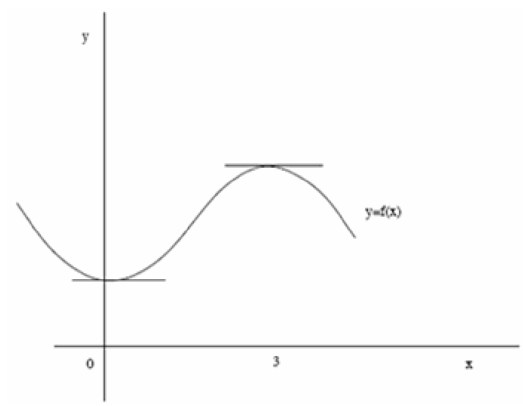

Figure 6. Graph for the fourth question.

Student Responses to the Second Task. The performance on this task can be found in Table 1.

Table 1

Performance on Second Task

Question	Number of Correct Responses	Percentage of Correct Responses
1	19	63
2	18	55
3	4	13
4	5	16

In response to the second question, some students demonstrated a rich concept image of differentiable functions that includes the necessity of a unique tangent. Hence they were able to identify the non-example of a differentiable function based on this criterion. Five students said that the function was discontinuous rather than saying it was not differentiable when giving reasons as to why Rolle's Theorem was not applicable for the given curve.

Compared to the performance on the first and second questions, the students' performance on the third and fourth questions was relatively poor. The third question involved an application of Rolle's Theorem in contexts where its need is not explicitly apparent. Two students unsuccessfully attempted to use the Mean Value Theorem to prove f'(c) = 0 in response to the third question. The fourth question demanded a leap in their thinking by expecting them to apply Rolle's Theorem to f'(x). In other words, the students had to treat f'(x) as an object and perform an action on it (Dubinsky, 1991), which seemed a difficult process for most of the students.

The following are responses to question three by students with a rich concept image.

- (Swamy) "Now f(x) is continuous. It has traveled from y = 2 to y = 3, it then goes back to y = 0 at x = 6. So, between these two points the function becomes 2. Also it is continuous and differentiable and hence applying Rolle's Theorem, we have a root for f'(x) in (0, 6)."
- (Ram) "Since the function is increasing and decreasing the slope of the tangent is zero at some point."

Swamy and Ram imagine the y-coordinate to be moving from point to point. This is a particularly rich mental representation which makes the Intermediate Value Theorem self-evident. Lakoff and Núñez (2000) argue "motion schema" to be a rich source from which many mathematical concepts and truths originate. Swamy recognized that since y is changing from 3 to 0, it must become 2 somewhere. This is a crucial step before one can invoke Rolle's Theorem. Ram observed that the function is increasing and then decreasing. Although he did not make it explicit, it appears that he realized that the slope of the tangent is positive when the function increases and is negative when the function decreases. We surmise that it is evident to him that the slope of the tangent must be zero somewhere. Based on the responses, we conclude that Swamy and Ram possess instantiations of differentiable functions which have richness and accuracy, and capture the content of formal definitions and properties relevant to Rolle's Theorem.

Interviews

Before the interview, we prepared the following questions that would encourage students to reflect on the salient features of Rolle's Theorem. Understanding a theorem or a concept includes being aware of a good supply of examples–instantiations–as well as non-examples. This level of understanding is indicated by the ability to apply the concept to specific problem situations. With this in mind, we began the interview with the following questions:

(1) Give an example of a function (using a graph) to which Rolle's Theorem is not applicable.

(2) Consider the following function: $f(x) = x^3$ if $x \leq 0$, $-x^3$ if $x < 0$. Is f differentiable? Is there an element c $\in$ **R** such that $f'(c) = 0$?

(3) If $f(x) = x^2 + 3x + 7$ is defined on [-2, 2], show that $f'(c) = 0$ for some c in (-2, 2). Are the hypotheses of Rolle's Theorem satisfied? Is there a contradiction? Is the converse of Rolle's Theorem valid?

Abhi's response. In her response to Question 1, Abhi drew the graph of an absolute value function and then said: "In this graph it is possible for us to draw two tangents for one point and hence the slope will not be unique. For Rolle's Theorem to be satisfied, the slope of a tangent at a point should be unique. Hence Rolle's Theorem is not applicable."

For Question 2, Abhi stated: "$f'(x) = 3x^2$ and $f'(x) = -3x^2$. So not equal." We observe that she had given the mathematically correct response to the second question in the second task, which involved identifying a non-differentiable function from its graph. But she seemed to have difficulty in checking the differentiability of the piecewise function $f(x) = |x|^3$ from its formula.

Finally, when responding to Question 3, Abhi said: "$f(x) = x^2 + 3x + 7$ is defined on [-2, 2]. f(-2) = -1 and f(2) = 17. So f(a) is not equal to f(b). Hence the hypothesis of Rolle's Theorem is not satisfied. The contradiction here is f(a) is not equal to f(b)...: The converse of the theorem is valid. We can see it in any example."

To gain a better understanding of how Abhi was conceptualizing Rolle's Theorem, we probed deeper into her understanding of Rolle's Theorem:

I: Can you state the converse of Rolle's Theorem?

Abhi: If $f'(c) = 0$ in a graph like this [She gestures towards the paper and draws the following], then the three conditions f(a) = f(b), the graph is continuous and the function is differentiable will be satisfied.

I: Can you draw the graph of $y = x^3$?

[Abhi sketches the graph by plotting some values.]

I: Are the conditions of Rolle's Theorem satisfied?

Abhi: (Thinks for some time.) Yes....... but, no two values are same.

I: So,..

Abhi: (Thinks for some time.) Yeah, ...the first condition is not met.

I: What about the derivative at x = 0?

Abhi (Pauses.) Yeah, got it. The derivative exists. The converse is not true!

Abhi believed that even when the hypothesis was not satisfied, the statement of the theorem was false and hence she perceived a contradiction. At the end of the interview, she was convinced about the fact that the converse of Rolle's Theorem is not true.

Sweta's response. In her response to Question 1, Sweta drew the graph of the sign function and stated "This is not continuous."

For Question 2, Sweta said: "f(x) is not differentiable. $y = |x^3|$ and that is because it does not have a unique slope. No, there is no point such that $f'(x) = 0$. Let me check...[She sketches the graph of $y = x^3$ and $y=-x^3$] The graph is like that of modulus function. I know that modulus function [the absolute value function] is not differentiable. So this is also not differentiable. $f'(x) = 3x^2$ $f'(x) = -3x^2$. So not equal." She was not able to proceed further because the moment she sees the function $f(x) = |x^3|$ she connects it to her pre-existing knowledge of the modulus function not being differentiable.

Finally, in her response to Question 3, Sweta said "Rolle's Theorem is not satisfied. f(a) is not equal to f(b). Hence a contradiction." When asked if the converse of Rolle's Theorem was valid, Sweta thought for a minute, and then responded "I think it is true."

Sweta also had difficulty with what constitutes a contradiction to Rolle's Theorem. Both Abhi and Sweta seemed to instantiate a non-differentiable function to be that of an absolute function. Both Abhi and Sweta demonstrated difficulty in writing the converse of Rolle's Theorem. This is probably due to the fact that the theorem has multiple hypotheses. Also they had trouble with what would, or would not, contradict Rolle's Theorem. However, the writing tasks and the follow-up interviews seem to have enriched their understanding of Rolle's Theorem.

The teachers who used this message design logic also expressed a genuine desire to encourage student learning. They talked about saying what needed to be said in order to accomplish specific learning goals. The teachers, not the students, defined the direction of classroom discussion and activity. These teachers assumed that they knew what the students needed to hear to move students closer to the desired outcome.

Conclusion

We aimed to study students' understanding of Rolle's Theorem by setting up specific tasks which involved stating the theorem, relating it to the Mean Value Theorem, and using it to solve problems involving graphs. Two students, Abhi and Sweta, participated in interviews during which further problems and questions were put forth to help us gain deeper insight into their understanding of Rolle's Theorem.

In response to the first task, which asked them to state Rolle's Theorem, nine students stated the converse. Although there were syntactic errors in stating the converse, it did not seriously affect their ability to solve simple problems as they know what to do to arrive at the desired solutions. Recall from Table 1, that 63% of the students answered the first question of the second task correctly.

The instantiations in the context of Rolle's Theorem seem to involve, in most cases, familiar functions or graphs such as the parabola. For some students (e.g. Sweta), this leads to a misunderstanding or misinterpretation of the hypothesis or conclusion of the theorem. Their instantiations are too specialized: they lack richness and accuracy.

We note that none of the students mentioned, in response to Question 2 of Task-I, that Mean Value Theorem is a generalization of Rolle's Theorem. Also, these students believe that f(a) should not equal f(b) in order for the Mean Value Theorem to be applicable. Again, this could be attributed to viewing theorems as tools for solving problems rather than as mathematical entities which can subsume one another.

Many students had difficulty with the last two questions of Task-II (only 13% and 16% gave correct responses). This was because Rolle's Theorem could not be directly applied to the function given as in Question 3. This particular example first required invoking the Intermediate Value Theorem. In Question 4, the theorem had to be applied to the derivative of the function whose graph was given. Both of these questions required linking Rolle's Theorem to other calculus concepts.

Because many students confused the statement of Rolle's Theorem with its converse in response to Question 1 in Task-I, we asked questions related to the converse of Rolle's Theorem during the interviews. Abhi and Sweta's response to these questions led us to some interesting observations. We believe that their responses are representative of the other students who participated.

Both Abhi and Sweta had difficulty stating the converse of Rolle's Theorem. The converse of "If P1, P2, P3, then Q" is indeed "If Q, then P1, P2, P3". However, mathematically, the statement of Rolle's Theorem is to be reworded in the form "Assume P1, P2 hold. If P3, then Q." In the converse statement, one assumes that the function f is continuous in [a, b] and is differentiable in (a, b). The assertion, then, is: "If $f'(c) = 0$ for some $c \in (a, b)$, then $f(a) = f(b)$", or more generally, a mathematician would state the converse as follows: "If $f'(c) = 0$, in any interval containing c in which f is continuous and differentiable, there exist a, b such that $f(a) = f(b)$." Of course, as Sweta stated correctly, $f'(c) = 0$ does not imply the function is continuous. She offered suggested the sign function as a counterexample. However, the mathematical subtlety has been missed when one places equal importance on all the hypotheses of a statement. Since the general context for these theorems is differentiable functions, P1 and P2 are taken for granted, and the main point is the equality $f(a) = f(b)$.

Some common misconceptions arose for both Abhi and Sweta. They both perceived a contradiction in a statement if the hypothesis is not valid. In addition, they incorrectly viewed the converse to be valid until we suggested a specific counterexample. They also thought that $f(x) = |x|^3$ cannot be differentiable based on a comparison with the absolute value function, which is not differentiable at the origin. Although there is no significant difference in their performance levels, it appears that there are some qualitative differences in their ways of thinking about concepts related to Rolle's Theorem. While Abhi expresses the statement of Rolle's theorem in terms of slope of tangents, Sweta seems to have misunderstood the hypothesis $f(a) = f(b)$ as saying that the point D (in Figure 1) divides the curve into two equal parts. It appears that Abhi prefers to think more geometrically and relates to examples and non-examples, i.e. instantiations, whereas Sweta seems to view each problem as an independent task.

To summarize, our experiment reveals possible difficulties students have in understanding Rolle's Theorem, which involves making sense of it, relating to other concepts such as the Mean Value Theorem and intermediate value theorem, as well as the ability to use it in situations when the function is not explicitly given.

The notion of 'container schema' elucidated by Lakoff and Núñez (2000) underpins one of the important ways mathematical relationships are understood. Perhaps, the students in our sample have yet to realize that just as one object can be contained in another, a mathematical statement can also be 'contained' in another. For example one student wrote: "Now f(x) is continuous. It has travelled from y = 2 to y = 3, it then goes back to y = 0 at x = 6. So, between these two points the function becomes 2." This is an instance of the student possessing the "Source-Path-Goal schema." In such a schema, we imagine an object to be moving from a source location to a target location, along a specific trajectory. Lakoff and Núñez (2000) argue that these schemas are present everywhere in mathematical thinking. They cite the example of functions in the Cartesian plane, which are conceptualized as that of motion along a path, evidenced by such phrases as "going up" or "reaching a maximum" are used to describe them as an instance of the motion schema. From our study, we observe that students who possess these schemas have rich concept images, which aid them in their reasoning tasks. In other words the ability to transfer everyday thinking to abstract mathematical notions guides them to possess versatile concept images.

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