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Expert Mathematicians' Approach to Understanding Definitions

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In this article I report on a study of the cognitive tools that research mathematicians employ when developing deep understandings of abstract mathematical definitions. I arrived at several conclusions about this process: Examples play a predominant role in understanding definitions. Equivalent reformulations of definitions enrich understanding. Evoked conflicts and their resolutions result in improved understanding. The primary role of definitions in mathematics is in proving theorems. And there are several stages in developing understandings of mathematical definitions (Manin, 2007; Tall & Vinner, 1981; Thurston, 1994). I also include some suggestions for pedagogy that are found in the data.

Definitions play a pivotal role in mathematics. Research on students' understandings of mathematical definitions reveals that learners encounter different types of obstacles. According to Vinner (1991), serious difficulties in comprehending definitions can be attributed to the dichotomy that exists between the structure of mathematics as conceived by professional mathematicians and the cognitive processes involved in concept acquisition by learners. Hence, it is instructive, based on the resulting pedagogical implications, for both mathematics teachers and educational researchers, to understand how professional mathematicians view mathematical definitions and what cognitive processes they employ when they attempt to understand definitions created by their peers.

With this goal in mind, I conducted a qualitative research study with 12 professional mathematicians. The mathematicians in this study have made lasting contributions to mathematics, and many have won several international awards and honors. For the purposes of this article, I define a professional mathematician as a mathematician who is actively involved in mathematical research. An expert mathematician has additionally been internationally recognized by peers based on his or her profound findings. In this research study, I only solicited participants who are expert mathematicians in their respective fields. The participants' responses provided insight into their world of mathematics, particularly with respect to definitions. At times, their responses went beyond their understandings of mathematical definitions, allowing important themes to emerge, such as the role of examples, conflicts, models, abstraction, intuition, and generalization. In order to adequately represent the emergent themes from participants' responses, I have also relied upon writings on the nature of mathematics alluded to by some of our respondents. In the following sections I present the views expressed by the mathematicians on each of the aforementioned themes.

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Theoretical Perspective and Literature

The theoretical framework of Pirie and Kieren (1994) is relevant to this study, for it describes the development of understanding within the learner's mind when a mathematical concept is learned. This theory describes the dynamic growth of understanding over a period of time. The essence of their theory is that understanding is not always a linear, continuous process; learners often revert back to their previous ways of thinking, only to emerge forward with more sophisticated and deeper understanding.

Pirie and Kieren propose an *onion-layer* model to depict eight different levels of understanding within the learner (see Figure 1). The innermost level of the model is referred to as *primitive knowing,* for this level describes the process of initial attempts to understand a new concept (such as functions) through actions involving the concept (adding or composing functions, evaluating a function at a point, etc.) or representations of the concept (such as the graph of a function). In the next level, the learner develops images out of these effective actions. This level is called *image making*. Continuing outward, the next level is called *image having*. At this level the learner is able to refine and manipulate the image (such as the image of a conic associated to a quadratic function) without having to work out particular examples, and, hence, this level represents the learner's first level of abstraction.

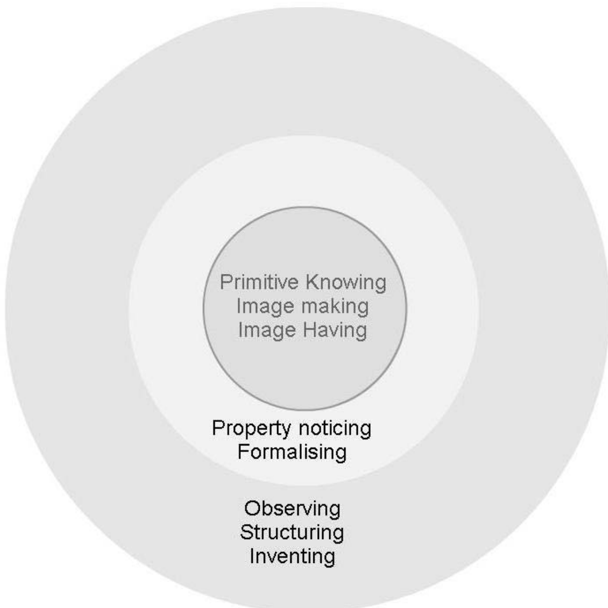

Figure 1. Pirie-Kieren's model

The next level is called *property noticing*, in which the learner is able to examine these images for properties, distinctions and so on. At this level the learner may notice, for example, that certain conic sections such as circles and ellipses are bounded, whereas hyperbola and parabola are unbounded. The model's next level is *formalizing*: The learner thinks consciously about the noticed properties and is able to generalize by abstracting the important features of the mathematical concept. At this level, the mathematical concepts become defined for the learner and begin to exist as an independent entity.

At the level titled *observing*, the learner tries to achieve consistency in his or her thought processes by trying to accommodate existing knowledge structures to fit with the newly acquired knowledge. For example, the learner may base his or her properties of functions on a modification of the properties of numbers, noticing that functions, similar to numbers, can be added and multiplied, and yet, unlike numbers, functions can be composed. The level called *structuring* takes place when the learner is able to place his or her thought processes into an axiomatic structure. At the outermost level, appropriately named *inventing*, the learner is able to freely create new mathematical structures with the previous knowledge structures acting as the initiating ground. At this final level, in which the highest level of recursion occurs, the learner begins to function independently. It is important to note that the levels do not correspond to levels in mathematics, but, rather, levels in understanding. Thus this theory reflects understanding as a personal knowledge construction process.

Vinner and Tall (1981) provide a framework for understanding how one understands and uses a mathematical definition. This framework is foundational to this study in that it explains the dynamic interaction between the concept image and concept definition in addition to being influential in subsequent research on the role of concept-images in understanding formal mathematical definitions. According to Tall (1980), each mathematical concept is associated with both a concept definition and a concept image.

> Concept image is regarded as the cognitive structure consisting of the mental picture and the properties and processes associated with the concept. Depending on the context, different parts of the concept image may be activated. At any given time the portion of the concept image that is activated is called evoked concept image. Quite distinct from the complex structure of the concept image is the concept definition which is the form of words used to describe the concept. (pp. 171–2)

The mathematical definition could be formal and given to the individual as a part of a formal theory or it may be a personal definition invented by an individual describing his or her concept image. A *potential conflict factor* (Tall, 1980) describes any aspect of the concept image that may conflict with any part or resulting implication of the particular concept definition. Factors in different formal theories can give rise to such a conflict. A cognitive conflict is created when two mutually conflicting factors are evoked simultaneously in the mind of an individual. The potential conflict may not become a cognitive conflict if the implications of the concept definition do not become a part of the individual's concept image. The lack of coordination between the concept image developed by an individual and the implication of the concept definition can lead to obstacles in learning. This has been corroborated by the work of several researchers including Cornu (1991) and Edwards and Ward (2004).

The influence of concept images on the understanding of mathematical concepts has been extensively studied. Nardi (1996) observes that novices are often obstructed by their previous unstable knowledge. Alcock and Simpson's (2002) research observes that students generally do not consult definitions to resolve conflicts because they do not understand either the relevance or importance of definitions. When learning certain concepts, learners also face difficulties due to the concept's intrinsic complexity. Bachelard (1938) classifies learning obstacles into several types according to their source: use of particular language, association of inappropriate images, or effect of a previous piece of knowledge that was originally useful but which becomes false in the present context. Thus research reveals that learners, in general, find it difficult to comprehend and use definitions.

Research Methodology

This research began with a pilot study exploring how expert mathematicians approach and understand mathematical definitions. I conducted in-person interviews with two expert mathematicians concerning the cognitive processes involved in their understanding of mathematical concepts and definitions. During the analysis of the pilot study data, the principal themes of the role of examples, conflict resolution, and reformulations and generalizations emerged. The emergent themes provided focus for subsequent research and informed the development and preparation of a questionnaire.

Participants

Twelve expert mathematicians participated in this research study. Although the research interests of the participating mathematicians varied, all participants study and research pure mathematics. Most of them have also taught mathematics for many years.

Data Collection and Methods

Because the participating mathematicians live in different parts of the world, only 5 participants took part in personal interviews. All participants responded to the questionnaire. Although no new questions were put forth during the interviews, this approach provided more time to reflect on the questions given.

The intent of the questionnaire (see Appendix) was to understand how expert mathematicians comprehend definitions. Guided by the emergent themes of the pilot study, the questions focused on the roles of imagery, examples, conflicts, as well as the domain to which a given definition belongs.

Analysis

Analysis of the participant responses used grounded theory as defined by Glaser and Strauss (1967). In particular, my analysis followed Glaser and Strauss's method of open coding of data. Using this approach, I constructed meaningful patterns within participant descriptions by looking for structure in the data. Themes emerged as a result of analyzing the responses for commonalities and differences.

Results

Processes Used by Mathematicians in Understanding Definitions

The role of examples. I define *examples* as instantiations of a concept. Every participating mathematician discussed the importance of examples in developing their understanding of mathematical definitions. This research finding supports the contention that having a sufficient number of suitable examples is closely linked with the understanding of the mathematical object. The *image forming* and *image having* levels of Pirie and Kieren's (1994) onion-layer model seem to be related to identifying examples and non-examples associated with a particular definition. By making sense of theorems that rely on a given definition, these images are further strengthened. For the participants, intuition, in any particular area, is also built by having a rich source of examples.

The following responses highlight the different ways expert mathematicians use examples to develop their understanding of mathematical definitions.

> **Response 1:** To comprehend a definition means it usually includes examples, simple counter-examples, and theorems using the definition.

> **Response 2:** I try to see how the definition will exclude/include examples that I already know. For example, if the definition is about groups then I would try to see whether it clearly divides the groups that I know into those that fit and those that do not.

> **Response 3:** In my area of my expertise, it is less of a challenge for me to comprehend a definition, as I may have already built up an intuition in that area, and examples are already swirling in my head upon reading a definition. In an area less familiar, I cannot feel like I understand a definition (or a theorem for that matter) until I have checked myself that it is not an empty definition (or theorem). I immediately try to think of easy examples, try to make a picture or connect it with

some definition I already know and see how it differs in comparison.

Additionally, some of the participants used examples specifically as scaffolds to build their understanding. This use of scaffolding is seen in the responses that follow.

Response 4: Examples are scaffoldings as one tries to build one's understanding of definitions. They are the steps to attain higher and higher levels of understanding. They are also the pillars on which the definitions rest. The more examples one has, the closer one is to understanding the mathematical object. If you want to go from A to B, there may be several ways. Different examples provide the different routes. In fact, examples give approximate shape of the object that is defined. So, complete understanding is impossible. As you make interconnections you get a finer and finer picture of the object.

Response 5: For me, any definition is associated with examples. Definitions cannot stand abstractly without examples. For me, knowledge can be thought of as subdivided into islands. It is examples that connect them. So a definition is a collection of all the examples that will conform to the definition. For me, in some sense, all the examples that satisfy a definition is like a kernel of the definition. The construction of examples and analogies are so important in understanding definitions in all the subjects you mention.

The role of conflicts. Based on the responses of the participants, when one receives external stimuli in the form of a new mathematical definition, one tries to incorporate this new knowledge to one's existing knowledge structure. This can be influenced by distorted images, resulting in a conflict. According to our participating mathematicians, when the same mathematical object is viewed from a different perspective, conflicts may be resolved. Additionally, there are times when the usual meaning associated with a word used in a new definition also can be a cause of conflict, as the first response below indicates.

Response 1: I guess it might also sometimes be required to ignore the usual meaning of the word being defined, but this always came naturally to me.

Response 2: I recall conflicts occurring when I convinced myself that the definition meant a certain something, and then later on I encountered a counterexample! I had to re-evaluate my understanding and re-understand it correctly so that the paradox could be resolved.

Methods of understanding. The participants' responses suggest that encountering alternate definitions can increase understanding. Similarly, comparing and contrasting the new definition with an already known definition is a tool used for understanding (see Response 3). Additionally, participants commented on the utility of proving theorems based on a given definition to develop their understanding of that definition. Participants detailed how self-inquiry, or posing questions to oneself, increased understanding.

Response 1: I had once a professor of geometry who told us his main aim was that we understood ….[I don't remember if it was "cube", "tetrahedron", or "projective plane"] as well as we understand what "chair" means. We can recognize a chair, sit on one, or in case of need stand on one to reach a high shelf.

That is what, for me, it is to comprehend a definition. It usually includes examples, simple counterexamples, and theorems using the definition. It happens that "definition" and "theorem" can be interchanged, and sometimes this makes a better understanding.

Response 2: When one meets an elegant result in an area, one usually marvels at the proof. If I would like to understand this at a deeper level, I usually try to formulate a question in which these notions would intervene and this way, I learn to appreciate those concepts as well as the techniques.

If the definition is equivalent, I note it in the back of my mind that this is an equivalent formulation. At times, the alternate way of looking at things proves useful in solving problems.

Response 3: I try to make a picture or connect it with some definition I already know and see how it differs in comparison.

Response 4, below, sums up many of the methods for understanding that were discussed by the participating mathematicians. It is also consistent with the framework developed by Vinner and Tall (1981): The participant hypothesizes that when a learner encounters a new formal mathematical definition, he or she develops a concept image associated with the definition. When this concept image becomes a useful tool, it may replace the definition, or, rather, it may become the concept definition.

Response 4: Understanding (comprehending) a mathematical definition is a process which is in principle open-ended: you can never tell that you understand something completely. It can be conceived as consisting of several stages.

> Stage 1. Understanding the language in which the definition is stated. In mathematics, I will take for granted that this means the language of more or less formalized set theory, which is expressed in metalanguage based on some natural dialect: English, Russian ... All other choices lack universality, conciseness, common acceptability, etc., of Set Theory. However, they might be unavoidable at earlier stages of studying mathematics, e.g., if one is taught Euclidean geometry a la Euclid.
>
> Stage 2. (a) Understanding of the Definition itself as a syntactically correct and meaningful expression of the language of Set Theory. (b) Forming imprecise but intuitively helpful "semantic cloud" of the definition. Let's take as a representative example the definition of a group. Its "semantic cloud" consists in various ideas about symmetry: symmetry of "things", symmetry of patterns, symmetry of physical laws ...
>
> Stage 3. Trying to compile a list of "examples": concrete objects satisfying conditions of the Definition. "Small" objects: groups with one, two, three elements. "Big objects": integers, rationals, matrices ... Can one classify small objects? Describe explicitly groups of 1,2,3,4,5,6 elements up to isomorphism? Here one more definition crops up: that of isomorphism to which the same (up to now, three stage process must be applied).
>
> Stage 4. Studying how the Definition works in various theorems about groups, and in various theories where groups are not the central, but an important part of the picture. Where and how we use associativity, existence of identity, existence of inverse element ? When there is a chance that all these conditions for a composition law will be satisfied, and when not? Does a given theorem remain true if one omits existence of inverse element in the definition? What kind of "group-like objects" [do] we get then?
>
> All stages, but especially Stage 4, is in principle open-ended. It might involve returning to Stage 3, posing and solving classification problems, sometimes marveling at their complexity (classification of simple finite groups). It enriches our grasp of semantics of the basic language and in this way helps to understand further definitions.

The various stages involved in understanding mathematical definitions described in this response also resonate with the levels of understanding given by Pirie and Kieren (1994). Forming a semantic cloud with a good supply of examples related to small and big objects corresponds to primitive knowing, image making and image having levels. Identifying patterns, asking questions and proving theorems related to the definition is similar to the level of property noticing, formalizing and observing. It becomes apparent through the Pirie-Kieren model and the participating mathematician's description that the levels are nested within each other and not necessarily linear.

The responses in this section can also be linked to Lakoff and Núñez's (2000) work, which demonstrates how mathematicians conceptualize mathematical objects through conceptual mapping. Just as in mathematics, where a mapping (or a function) has a domain and a range or target, the domain of the conceptual mapping consists of examples and the target is the algebraic structure that underlies the set of examples. It is through such metaphorical mappings that mathematicians assign an algebraic essence to an arithmetic structure. They claim that mathematicians tend to think of the algebraic structure as being present in the arithmetic structure. Calling this a metaphorical idea, Lakoff and Núñez propose that this metaphorical idea helps mathematicians to see, for example, the same mathematical structure in addition modulo 3 and rotational symmetries of an equilateral triangle.

Nature and role of mathematical definitions. When mathematicians communicate their results in a formal way (e.g., a journal article), a need arises to introduce a collection of associated definitions in order to facilitate "chunking" and help avoid repetition. According to David Mumford (2001), when one encounters many complex examples, isolating part of their shared structure is the best approach. This is what generating models is all about, and, as he points out, pure mathematics is full of models. This method of generating models is referred to by Mumford as the "bottom-up view". The opposing view is the "top-down view" where different branches of mathematics grow out of one *true* axiomatization of the subject. This "top-down view" is contrary to the ideas of generating mathematical models (pp. 4–5). These two viewpoints also reflect the two different approaches used by mathematicians in solving problems and developing theories. Some mathematicians work from concrete examples, abstract their essence, and generalize theorems. Some mathematicians, by intuition and by virtue of their mathematical experience, conjecture some mathematical statements to be true, verify it by examples and then prove them formally. Mumford points out that, according to Sir Michael Atiyah, the most significant aspects of a new idea are often not contained in the deepest or most general theorem resulting from the idea, but they are often embodied in the simplest examples, the simplest definitions and their immediate consequences (p. 3).

The following participant responses depict additional emergent themes on the nature and role of mathematical definitions in understanding.

> **Response 1:** In some sense, a mathematical definition is an isolated tool obtained from a mass of concepts which is utilizable again and again. From existing mathematical concepts, when a selection is made by rearrangement so that this rearrangement becomes a useful tool, or, when a part of an existing concept is isolated so that it becomes an entity in its own right, a new object is born. It is characterized completely by its definition.

> **Response 2:** Some prefer to start with some examples and then work slowly to an abstract setting. And some much prefer to start with a general abstract definition and find examples as they crop up. I am strongly of the first type and always seek to delineate the abstract concept with an array of examples or else I just can't work with it.

> **Response 3:** Mathematicians in the process of setting up a whole universe of mathematical definitions and abstractions to express their ideas in their generality, fail to communicate the original examples, which led to these ideas in the first place. This leads to difficulties in comprehending these new ideas even for some fellow mathematicians. It is no wonder that students struggling to comprehend new mathematical definitions and theorems exhibit great difficulties.

Discussion and Conclusions

The themes that emerged from the responses of the participating mathematicians led to several conclusions, outlined below. I believe that the emergent themes identified in this study are applicable to most mathematicians and the way they understand mathematical definitions. Nevertheless, the sample size of the mathematicians involved was small and all of them were pure mathematicians. It would be interesting to conduct a similar study with a much larger sample and, perhaps, involve applied mathematicians as well. It is possible that wordings of some questions—specifically questions (1) and (3)—in the questionnaire might have influenced the responses, which in turn might have influenced the conclusions.

Predominant Role of Examples in Comprehending New Definitions

All participating mathematicians described examples as an important cognitive tool they employ during the process of understanding a new definition. One of the processes employed is commonly known as the *inclusion-exclusion principle*. For example, one of the participating mathematicians stated,

> I try to see how the definition will exclude/include examples that I already know. For example, if the definition is about groups then I would try to see whether it clearly divides the groups that I know into those that fit and those that do not.

In other words, a set of mathematical objects can be viewed as specifying a territory. Other mathematical objects can then be differentiated according to whether they belong to a specific territory or not. Examples and counterexamples can be viewed as an approximation of the form or shape of this territory. Therefore, the more numerous and varied the examples, the finer the approximation.

To illustrate, consider the concept of continuous functions. Typically, the first examples of continuous functions students encounter are polynomial functions, followed by trigonometric, exponential, logarithmic functions, and so on. When one encounters the absolute value function and notices it to be continuous but not differentiable, one gains a better understanding of the territory of continuous functions. Therefore, pedagogically speaking, it helps to have a good supply of examples when learning a new concept or definition. Thus, a definition is a collection of all the examples that conform to the definition. According to one of the mathematicians, definitions cannot stand abstractly without examples. He said, "All the examples which satisfy a definition is like a kernel of the definition."

Definitions in one's area of expertise are easily understood because one has at hand a rich supply of objects on which to test the definition. Thus, examples are like pillars on which definitions are built. For some expert mathematicians, understanding a new definition involves the process of continuously molding the definition so that it approximately fits into their area of expertise.

For example, there is a close relation between algebraic geometry and commutative algebra in that many algebraic objects have parallels in geometry. When a mathematician encounters a definition in algebra (concerning, say, projective module), the mathematician may prefer to view the definition as one in geometry (in this case, vector bundles).

Role of Equivalent Reformulations of Definitions in Enriching Understanding

At times, a definition and a theorem can be interchanged. That is, the theorem yields an equivalent reformulation of the definition, leading to deeper understanding. For example, an equilateral triangle may be defined as a triangle in which all sides have equal length. The theorem stating that a triangle is equilateral if and only if it is equiangular provides an equivalent reformulation of this definition. This realization gives new insight into the original definition. Thus, a mathematical object is better understood through characterizing properties.

Role of Evoked Conflicts and Their Resolutions in Improved Understanding

Sometimes when one's understanding of a definition encounters an example that conflicts with one's understanding, it can lead to transformation in one's thinking that helps to resolve the conflict in understanding. For example, one might believe that all continuous functions are differentiable until one encounters an example that is continuous but not differentiable. It appears that understanding of a definition undergoes constant change as conflicts of various kinds are evoked and resolved. One participating mathematician described how some conflicts simply disappear as one views the definition with a new perspective and at other times by performing a simple calculation.

Role of Definitions in Mathematics

One of our participants believed the central role of definitions is to *prove* theorems. For example, the importance of understanding the definition of a continuous function is not found in the ability to apply it in order to determine the continuity of individual functions. The importance of understanding the definition is using it to make broader conclusions. For instance, one can use the definition of continuity to prove that all polynomial functions are continuous.

Stages in Understanding Definitions

The participant responses supported several theoretical models of stages in understanding definitions. In fact, the view of stages represented by Response 4 in "Methods of Understanding" reflects ideas from many of the participants. According to Response 4, understanding a mathematical definition consists of four stages (cf. Manin, 2007). The first stage involves familiarizing oneself with the formalized mathematical language of and the dialect in which it is written. For instance, when one attempts to grapple with the definition of continuous functions, one should be familiar with the notation used to express limit of a function. The second stage involves understanding the definition as a syntactically correct and meaningful expression represented with mathematical language. This second stage also requires development of an intuitive understanding of the definition, a "semantic cloud." It appears that *semantic cloud* is the mathematical aspect of what is termed as *concept image* (Tall & Vinner, 1981). According to Thurston (1994), learning a mathematical topic consists of building useful, non-formal mental models, and this learning cannot be accomplished by studying definitions and rigorous proofs alone. I include these processes as part of the semantic cloud. Using the representative example of continuous functions, its semantic cloud could consist of ideas of graphs without gaps. In the third stage, one acquires a variety of examples, ranging from "small objects" to "big objects". In the case of continuous functions, small objects could include the constant function, identity function, etc., while big objects could include continuous functions that are nowhere differentiable. In the fourth stage, one acquires the knowledge of how the definition is used in theorems and in other related topics. This stage leads to an understanding of why a particular definition is formulated in a specific way. Also, it often leads to an appreciation of the need to characterize mathematical concepts with precise, unambiguous definitions. This understanding may also lead to the construction of new examples and counter-examples. Hence, the third and fourth stages are dynamically nested, representing the complexity of the learning process for mathematical definitions.

Pedagogical Implications

Participants provided the following pedagogical recommendations:

> **Response 1:** Teaching strategies must take into consideration the different challenges posed by each of the stages (given above) in understanding the mathematical definition.
>
> **Response 2:** It is preferable to start with some examples and then work slowly towards an abstract setting. It is important to delineate abstract concepts with an array of examples which tie the idea into their cognitive framework.
>
> **Response 3:** The teaching strategy should aim to convey that a mathematical definition is just as

> tangible as a table or a chair. The student should be able to recognize it, use it for the routine purposes for which it is meant, and perhaps use it in a novel way, just as one can recognize a chair, sit on one, or in case of need stand on one to reach a high shelf.
>
> **Response 4:** Solving well-formulated problems is an important strategy to gain in-depth understanding of the definition.

According to one of our participants, there is a wide gulf between *mathematical thinking* and formal *mathematical writing*. While *mathematical thinking* involves creativity unrestrained by demands of structure, logic, and rigor, guided only by intuition, knowledge, and experience, *mathematical writing*, particularly the development of proofs, does not permit such freedom. According to this mathematician, *abstraction* is a very relative term. He concludes:

> Abstraction is very individualistic. One question students frequently ask is: What can one say or do about a mathematical object when it is so abstract that it can't be even be seen or imagined? I wish to suggest that a mathematical object is its defining property. The more you learn its characterizing properties the better you get to "know" it.

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References

Alcock, L., & Simpson, A. (2002). Two components in learning to reason using definitions. *Proceedings of the Second International Conference on the Teaching of Mathematics*, Crete, Greece. Retrieved from: http://www.math.uoc.gr/~ictm2/Proceedings/pap110.pdf

Bachelard, G. (1938). *La formation de l'esprit scientifique*. Paris: J. Vrin.

Cornu, B. (1991). Limits. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 153–166). Dordrecht, The Netherlands: Kluwer.

Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definition. *American Mathematics Monthly*, *111*, 411–423.

Glaser, B. G. & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. New York: Aldine Transaction.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Manin, Y. I. (2007). *Mathematics as metaphor: Selected essays of Yuri I. Manin*. Providence, RI: American Mathematical Society.

Mumford, D. (2001). *Trends in the profession of mathematics*. Retrieved from http://www.dam.brown.edu/people/mumford/Papers/dmv.ps

Nardi, E. (1996). *The novice mathematicians' encounter with mathematical abstraction*. Unpublished doctoral dissertation, Oxford University, Oxford, England.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How we can characterize it and how we can represent it. *Educational Studies in Mathematics*, *9*, 160–190.

Tall, D. O. (1980). *Mathematical intuition with special reference to limiting processes*. Paper presented at the meeting of the Fourth International Conference for the Psychology of Mathematical Education, Berkeley, CA.

Vinner, S. & Tall, D. O. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Thurston, W. P. (1994). Mathematics education. *Notices of the American Mathematics Society*, *37*, 844–850.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–79). Dordrecht, The Netherlands: Kluwer.

Appendix

Questionnaire

1. How would you comprehend a definition in your area of expertise and a definition in an area less familiar to you? More specifically, is there a specific identifiable intellectual process specific to your individuality which is called up when needed to comprehend a definition. As a part of the process, do you use special examples and then abstract the process, or draw pictures, schematic diagrams, etc.?
2. In the course of your mathematical development, are you aware of any changes in the way you comprehend a new definition? In what ways, if any, are they different from the ones you use now?
3. Do you have a recollection of having understood a definition or a mathematical statement in a particular way which later on resulted in a conflict? If so, how is the awareness of the conflict triggered? Does the awareness occur spontaneously or when working consciously at it?
4. In your experience, if someone recasts your definition in a different way, what method(s) do you use for reconciling or understanding the new definition?
5. Is it possible to evolve general strategies for understanding mathematical definitions based on your research experience or teaching? To what extent are the strategies common or different across the subjects (algebra, topology, geometry, analysis,...).