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A Look Back…

Pólya on Mathematical Abilities[1]

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In April 1978, I interviewed George Pólya about his views on mathematical abilities. I was in California for the annual meeting of the National Council of Teachers of Mathematics in San Diego and arranged to stop by Pólya's house in Palo Alto after the meeting to discuss his views on mathematical abilities as well as the articles on mathematics education to be included in his collected papers (Rota, Reynolds, & Shortt, 1984). The following article is abridged from that interview and focuses on mathematical abilities.

For me, the most unexpected feature of the interview was that although Pólya had obviously reflected throughout his long life on the question of how he and others do mathematics, he had apparently not given much thought previously to the abilities they were drawing on when they did it. Nonetheless, Pólya's wit and charm come through clearly as he patiently struggles with his former student's awkward questions.

JK: What are the qualities that you think make someone capable in mathematics? In other words, what are the mental abilities that distinguish someone who is capable in mathematics from someone who is not so capable?

GP: I couldn't give you a good description, you see. I never made any clear ideas about that. Moreover, there are so many different kinds of mathematicians.

JK: What different kinds?

GP: Well, I wrote a little article about it once where I mentioned Emmy Noether.[i] I made a joke about it. She was for generalization; I was for specialization.

….

JK: Do you think it's important to have good spatial ability to be a mathematician?

GP: To a certain extent, yes, but that's also so different. Hadamard tells about—. Do you know the book of [Jacques] Hadamard?[ii]

JK: Yes, I know the book

GP: If he were here, he would give you much better answers—anyway, more answers. He thinks sometimes you are the "auditive" type, or you are the "visual" type. And he himself is more an auditive type. I don't know. It certainly helps, especially—. There is Jean Pedersen;[iii] she certainly has spatial ability.

JK: What about memory? Do you think mathematicians have a special memory? For mathematical things?

GP: Yes, sure.

JK: Do you have to have a very good memory?

GP: Well, sure, for everything. Horace says in the Ars Poetica, "*Mendacem oportet esse memorem*"[iv]—my Latin still works a little. He says, "A liar must have a good memory." A poet is a liar. He invents everything. He must very well remember what he did before. So a good memory, that is necessary for everything.

JK: A specially organized memory? Do you think mathematicians have a memory that is organized in a different way?

GP: Yes, exactly. What is *organized*? I find, you see, the general terms in which you could describe it, they are either lacking or they are vague.

JK: I can see that. But people have tried to—. Well, one question is whether mathematicians have certain special kinds of abilities, or they just have ordinary abilities, but they apply them to mathematics.

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[1] This interview is abridged from the original transcript, which is available in Portuguese from Guimarães, H. (2010). Jeremy Kilpatrick: entrevista a George Pólya [Jeremy Kilpatrick: interview with George Pólya]. *Quadrante*, *19*(2), 103–119.

GP: The second is probably a little better. No one is completely true, but the second is better. For instance, I can tell you, I have a pretty good memory—. Anyhow, for the mathematics I did, I have a pretty good memory. Well, now it goes downhill like the rest of it, but I could remember pretty much everything what I did. Not what other people did. …But I have also a good memory for poetry and a good memory for jokes. So it is not specialized for numbers. I have a good memory for poetry, but I recall it so: It comes often; I recall it, in between, for any reason or without reason. I just ask you whether you know German. Because I recall something very pretty what Schiller said about it.

JK: And you recall the whole thing?

GP: There are just two lines. He describes very well what he—. I will tell it to you in German. It is very good German. He means it probably for poetry, or possibly, he was also a historian—he wrote history. But it is good for mathematics. I say it in German:

Nur Beharrung führt zum Ziel,

Nur die Fülle führt zur Klarheit,

Und im Abgrund wohnt die Wahrheit.[v]

He said, "Only—." Ah, "*Beharrung*"—how do you say it? "Who always—."

Well, now, I have four languages; it's very difficult to find the right—. "*Beharrung*." So, if you are working all the time in the same direction, you must go ahead all the time. "*Nur die Fülle*"—if you know many things, keep together—"*führt zur Klarheit*"—then you may be clear. If your knowledge is based on many things. "*Und im Abgrund wohnt*"—and the truth is in the deep. You can say the same thing about mathematics, but Schiller certainly meant it for poetry or for history, and not for mathematics. …

….

JK: But different mathematicians have different strengths and weaknesses.

GP: Different people have different strengths and weaknesses.

JK: What are your strengths and weaknesses as a mathematician?

GP: ….. I like to go down to something tangible. And I start from something tangible. From some physics, or even from some everyday things. …. I say the same thing about—have you read it?—about Emmy Noether.[vi]

JK: Yes, I've read the paper.

GP: So there are two kinds of monkeys: up monkeys and down monkeys.

JK: And you're a down monkey.

GP: I'm a down monkey, and she was an up monkey. They are different; so are people.

JK: What were the parts of mathematics that you had the most difficulty understanding?

GP: I don't know. Perhaps, well, oh, I appreciate—. It's not the difficulty of understanding. For instance, I appreciate foundations, but I couldn't work on it.

JK: Why not?

GP: Not my line, you see.

JK: Because it deals with generalization? Because it's too general?

GP: Well—.

JK: Too abstract?

GP: It cannot be expressed in words, you see. It is simply not my line. Oh, I admit it is important, but I just couldn't work on it. It was very, very fortunate, you see. ….[David] Hilbert came to visit Hurwitz in Zurich. He was very old, you see. He felt …he needs a good assistant. And there were proposed two: [Paul] Bernays and myself. It's a great luck that they have chosen Bernays and not me. Because I was not good for foundations, and Bernays was excellent, you see. They wrote the book: Bernays, Hilbert, and [Wilhelm] Ackermann.[vii] It is hundred percent written by Bernays. Of whose thought, I don't know. By Hilbert, you see, maybe it was organized, probably. And it is enormous luck for science and for myself that I was not chosen, you see. It would have been, of course, in a way, it would have been very flattering to be an assistant, but it was much better not to be.

JK: Let's talk about problem solving. Where did the rules and heuristic methods that are in *How to Solve It*,[viii] where did those come from? What's the source?

GP: This I gave in print. ….This is, I think, my first paper about problem solving.[ix] And this is told in detail here in the first lines. I had a kid, a stupid kid to prepare for a high school examination. And

I wished to explain him some—. Almost this problem.[x] And I couldn't do it. And the evening I sat down, and I invented that [representation]. So that was the starting of my explicit interest in problem solving.

JK: So, trying to teach him, you came up with these questions.

GP: No, no, that came afterwards, you see. But just the main thing, the representation by a graph. I didn't know the word graph, and so on, but I invented this representation. Then I made it better. I made a geometric figure. ….And that was the beginning of my explicit interest.

Implicitly, I was probably interested before. I was also interested: How did people discover it? And then Mach, Ernst Mach, he said, "To understand a theory, you must know—. It is really understood if you know how people discovered it." I read his book,[xi] and this influenced me enormously. This brought me from philosophy to physics. …..

JK: The graph came before your questions or your suggestions like, "What is the unknown?" "Can you draw a figure?"

GP: Oh, yes. The graph came first. Then I was also very much interested by Descartes. By the *Regulae*.[xii]

JK: The *Rules*, yes.

...

GP: ….. Oh, have you seen the number of the *Journal of Graph Theory*? …..

JK: No, I haven't seen that.

GP: There are two articles in it.[xiii] The first, by Harary—I don't have a reprint. And the other, by Albert Pfluger. I don't know whether you know who he is.

JK: No.

...

GP: …He was a student. He made his Ph.D. with me. I knew him, his daughter, and so on, and so on.

JK: And he tells the story.

GP: And he pretty much describes the story.

….

JK: When you solve problems, do you use your advice from *How to Solve It*? Consciously?

GP: Yes. Well, even more than that. ….I had the rules, and I tried it out on myself. So, for instance, I edited the works of Hurwitz. ….He had a mathematical diary, and it is beautifully written, you see. It is written very comp1ete1y—not just scribb1ed, but clearly written, well-formulated, you see—where he describes what he thought of: sometimes his conversations; sometimes what he read. And then I thought about editing it, you see. And so, I found among others, this problem which falls me to … this [Pólya] Counting Method, you see. And I chose this counting method just to check my own rules. Whether my own rules would work. …

…

GP: ….. And this problem of Hurwitz, it was just good for that. Obviously an interesting problem because Hurwitz and Cayley had worked on it, and [it is] connected essentially with chemistry. That I like, you see: connected with something important and with the practice. But, on the other hand, very little preliminary knowledge is needed.….

JK: Yes.

…

JK: Some people say that they cannot use the rules. Or that—.

GP: Well, that's okay. People are different. People are different.

JK: Do you think it's possible to develop somebody's ability to solve problems?

GP: I think so.

...

GP: Well, I think it is not so much "develop" as it is "awaken," I would say.

JK: It's there.

GP: It is somewhere there. If there is nothing there, you cannot—. But you can awaken it, you see. A good teacher, and so on, a good opportunity to awaken it, you see. Well, my own case—. I had obviously some probability for it, but it was awakened very lately. I would have been probably a much better mathematician if I had had in the gymnasium a good teacher. It can be awakened—this I think so. This may be too optimistic—. I think even [with] my rules can a teacher, a good teacher emphasizing a little my questions can help awaken it. Alan Schoenfeld has some ideas how to do it. I don't quite agree with what he says, but anyhow, I think so. This I believe. That is no proof, of course. But it would be very difficult to prove or disprove it.

JK: Do you think it is important for the teacher to demonstrate in front of the class how to, to show the class—. Is it important, for the teacher to show in front of the class how to solve the problem? The teacher should be an actor?

GP: The most important for the teacher that he should himself have the experience of solving. In … Belmont [CA], there is a Catholic college, the College of Notre Dame. There we had a meeting. …And there we had Ed Teller, the father of the atomic bomb. He gave a talk, and even a very interesting talk.[xiv] I don't agree with everything what he said, but it was good. He said the most important is the teacher; the teacher should amuse the kids. Mathematics should amuse the kids.

JK: Do you agree?

GP: Yes, sure. To awaken them, the problems should be amusing; the problems should be challenging. They should be amusing—not faraway problems, not "practical" problems: how to pay your income tax.

JK: That's not amusing.

GP: (Laughs.) Definitely not. The Infernal Revenue Service: It's not amusing.

JK: How did you identify the students you had who were best in mathematics? You taught some students who were good in mathematics. How could you tell who were the best ones?

GP: Who was the best one, I can't tell you.

JK: Well, among the best, how could you identify their talent? They were quicker?

GP: Anyhow, they asked good questions. So they found out something by themselves. And so on. There is no simple way—. You see, people are too different. Mathematicians are too different. There is no simple way of describing it. I don't think so.

JK: What about people who are creative in mathematics as opposed to just being able to learn it? What does that take? What does that require? Just great interest?

GP: I don't know.

JK: Not everyone could be creative in mathematics.

GP: I said somewhere, "What is the difference between productive and creative?" If you think about a problem, if you produce a result, then you are productive. If in working you get into a method with which you can solve also other problems, then you are creative. That's the difference. And that is difficult to say. I don't think there are obvious signs to recognize this. I don't think so.

JK: Are these things that kids are born with?

GP: That I am pretty sure: You must have a genetic—. That must be somehow born to it, that is clear.

JK: And it helps if you have a teacher—.

GP: Oh, that helps, to awaken it.

JK: But even if you don't have a teacher to awaken it, you could be—.

GP: Oh, you could.

JK: As your own case.

GP: …. Well, I had Mach as a teacher. A little late, but …Mach said it, and he illustrated it very strongly: "If you wish to understand the theory, you should know how it was discovered." And this I understood.

JK: Do you think that's one of the problems with teaching mathematics in school, that we present it to the kids—? We present mathematics to the kids, but we don't show them how it has been discovered? In other words, teaching should be more genetic?

GP: You should illustrate it, you see. You make a little theatre, and you pretend to discover it. This I printed it even somewhere. You pretend to discover it.

JK: And you think that's important for—.

GP: If you do that well, then they learn much more than just this problem.

JK: You have collaborated with other mathematicians.

...

GP: ….I collaborated with very good mathematicians, better than myself. With Hurwitz, with [Godfrey Harold] Hardy, with [Gábor] Szegö. They are here around me (points to pictures on the wall of his study). Of course, I collaborated most with Szegö.

JK: Does Szegö approach mathematics as you do?

GP: Well, on the contrary—we were to some extent complementary.

JK: How?

GP: For instance, he is an excellent calculator; he is excellent at calculating.

JK: And you're not so good?

GP: Oh, I am not so bad, but he—. Anyhow, we somehow complemented each other. He knew some subjects, for instance, he knew polynomials better than me. About Legendre, and so on. We somehow—. Our interests were sufficiently similar, but also sufficiently different, and I couldn't enumerate all the points, but it was more complementing. We had, of course, some very similar interests, but also different. Also, similar backgrounds. We were both students of [Leopold] Fejér, and so on, but—.

JK: What kind of a teacher was Fejér?

GP: Oh, he was very good, very good. I scarcely had a class by him, but I talked with him a lot. He was excellent. Oh, this is printed somewhere; I have an obituary of Fejér, where I tell about this.[xv] He could tell so good stories.

….

JK: When you work on mathematics, when you try to do mathematics or solve a problem, do you find the advice to let the problem go for awhile and—is that good advice?

GP: Not before I did something.

JK: [You need to] try a little. Have you ever had the experience of having a solution come to you in the unconscious?

GP: Oh, yes, sure. There is even—. "Waiting for the good wind"—this is a usual expression.

JK: Have you had the experience?

GP: I don't know by whom I heard it, but I didn't invent it, I am sure. So, if you are a sailor—not if you have a boat with a machine, but if you have a sailing boat—then you have to wait for the good wind. So, "waiting for the good wind"—I didn't invent this expression; that must be somehow traditional in English.

JK: People like Poincare and others tell—.

GP: And that is waiting. Sleep on your problem. That is international. It is said in all languages.

JK: Have you had the experience of waking up with a solution?

GP: Oh, yes, now and then. Even this I describe somewhere in one of my papers.

JK: It came that way to you.

GP: But very seldom. And I heard it from Hurwitz the same. You wake up with a solution, but it is just phantasmagoria.

JK: It's not really a solution?

GP: It doesn't; it is not so. It happened very seldom. That really I wake up with a solution that was so. A simple thing is in the *Inequalities*, one solution for the—. It is mentioned, I think, in one of my late papers.[xvi] (Gets paper.)

...

GP: ….. But once or twice—once I remember it definitely happened; I really dreamt it correctly. I just had to write it out, the details, in the morning. And Hurwitz had the same, I heard. I'm pretty sure it is described there.

JK: Do you draw a lot of figures when you work on problems?

GP: Sometimes, yes. Oh, I draw a lot of figures. Sometimes very carefully.

JK: Even when the problem doesn't require a figure?

GP: Sure. It may be a beginning of the idea. That you come to a figure which is connected with the problem.

…

GP: [The conversation turns back to the talk by Teller] But it was good that somebody told it to the teachers. Especially that the main thing of the teacher should be the interest; he should amuse. He should convince the kids that mathematics is amusing.

JK: How can the kids ever learn mathematical skills, then?

GP: They will learn it. If he plays Nim, he will learn to make additions very quickly. And learn to combine things, and so on. Teller is surely a much greater scientist, and by the way, Teller is not only that. You know there was a mathematical competition in Hungary.[xvii]

JK: Yes.

GP: Teller won this competition as a kid. So he knows it, when he talks about learning mathematics, about the mathematics at high school age, he has real experience, first-rate experience. But Jean Pedersen, who is a very successful teacher, goes to high schools, or they come to the University of Santa Clara. And she shows the kids how to make models. Then they are anxious to make models. And once she photographed each kid with the model he made. So that is also something. That is also a mathematical occupation. They learn geometric figures, and so on. "Learning starts by seeing and doing"—this I also quote somewhere.[xviii]

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[i] Pólya, 1973/1984.

[ii] Hadamard, 1945.

[iii] Professor of mathematics at Santa Clara University.

[iv] The quotation actually comes from Quintilian (*De Institutione Oratoria,* IV. ii).

[v] "Naught but firmness gains the prize, Naught but fullness makes us wise, Buried deep, truth ever lies!" (Schiller, 1796).

[vi] Polya, 1973/1984.

[vii] Hilbert & Ackermann, 1928; Hilbert & Bernays, 1934, 1939.

[viii] Pólya, 1957.

[ix] Pólya, 1919. The improved representation can be found in *Mathematical Discovery* (Pólya, 1981, Vol. 2, p. 9) and inside the front cover of Vol. 2 of the original edition.

[x] The problem is to find the volume of a right pyramid with square base given the altitude and the lengths of the sides of the upper and lower bases (see Polya, 1981, Vol. 2, p. 2).

[xi] Mach, 1883.

[xii] Descartes, 1701.

[xiii] Harary, 1977; Pfluger, 1977.

[xivxiv] At the February 1978 meeting of the Northern California Section of the Mathematical Association of America, held at the College of Notre Dame, Edward Teller's talk was entitled "The New (?) Math."

[xv] Pólya, 1961. See also Pólya, 1969.

[xvi] It was the proof of the inequality between the arithmetic and geometric means given in Hardy, Littlewood, and Polya, 1934, p. 103. See Pólya, 1970.

[xvii] The Eötvös Competition.

[xviii] Pólya, 1981, Vol. 2, p. 103. Pólya's paraphrase of Kant: "Learning begins with action and perception."