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Does 0.999… Really Equal 1?

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This article confronts the issue of why secondary and post-secondary students resist accepting the equality of 0.999... and 1, even after they have seen and understood logical arguments for the equality. In some sense, we might say that the equality holds by definition of 0.999..., but this definition depends upon accepting properties of the real number system, especially the Archimedean property and formal definitions of limits. Students may be justified in rejecting the equality if they decide to work in another system—namely the non-standard analysis of hyperreal numbers—but then they need to understand the consequences of that decision. This review of arguments and consequences holds implications for how we introduce real numbers in secondary school mathematics.

Whenever the equality of 0.999… and 1 arises, teachers can expect a high degree of disbelief from students, and proofs may do little to abate their skepticism (Sierpinska, 1994). This equality challenges students' conceptions of the real line, limits, and decimal representation, but students have a strong historical and intuitive basis for their resistance. The purpose of this paper is to investigate the reasons students reject the equality and to consider the consequences of this rejection. With this purpose in mind, we have organized the paper in the following way: We begin by outlining various arguments supporting the equality and then review some of the pedagogical struggles noted in research that explain students' resistance. Next, we justify students' intuitive resistance by presenting a system of hyperreal numbers in which the equality

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does not necessarily hold. Finally, we consider the implications of adopting such a system, which forces students to choose between conflicting properties; we offer as an example the conflict between the Archimedean property for real numbers and the existence of infinitesimals.

Arguments for the Equality

There are many arguments that support the equality of 0.999… and 1. Here we present four of these arguments.

Relying on the Decimal Expansion for 1/3

A common argument for the equality goes as follows: If 0.333…= 1/3 then digit-wise multiplication by 3 would imply that 0.999…= 1. Of course, this argument relies on students' acceptance of the equality of 0.333… and 1/3. Research has shown that students generally accept this equality, even while rejecting the equality of 0.999… and 1 (Fischbein, 2001). Students might resolve this tension by asserting, "Well, then, maybe 0.333… doesn't equal 1/3."

Subtracting Off the Infinite Sequence

Figure 1 outlines a more formal argument that does not depend on similar equalities. Yet students might still object.

If $x = 0.99\overline{9}$ then

$$\begin{aligned} 10x &= 9.99\overline{9} \\ 9x + x &= 9 + x \\ -x &= -x \\ \hline 9x &= 9 \end{aligned}$$

Therefore $x = 1$

Figure 1. A proof of the equality.

The issue with this argument is whether x can be canceled. Richman (1999) asserted that skeptics might reject the equality by claiming that not all numbers can be subtracted from one another! Moreover, if we consider 0.999… as the limit of the sequence 0.9, 0.99, 0.999, … then we see that the corresponding products, using the standard algorithm for multiplication of 9 by 0.999… produces a limit of 8.999…, which leads back to the same central issue that x might not be 1 after all.

Generating a Contradiction

A third argument for the equality works by contradiction: If 1 and $0.\overline{9}$ are not equal, then we should be able to find a distinct number in between them (their average), but what could that number be other than $0.\overline{9}$ itself? Still, students might argue that some pairs of distinct numbers simply do not have averages; some students have even argued that there *are* numbers between 1 and $0.\overline{9}$—namely, ones represented by a decimal expansion that begins with an infinite string of 9's and then ends in some other number (Ely, 2010). Even when students cannot find fault with the argument, they still might not believe the result. After reproducing the proof illustrated in Figure 1, one frustrated student sought help from Ask Doctor Math (www.mathforum.com): "The problem I have is that I can't logically believe this is true, and I don't see an error with the math, so what am I missing or forgetting to resolve this?"

Defining the Decimal Expansion with Limits

Since Balzano formalized the definition of limits in the early 19th century, Calculus has been grounded in the formal definitions of limits that we teach in Precalculus and many college-level mathematics courses. Figure 2 lays out Balzano's formal $\varepsilon - N$ definition for limits of sequences.

Formally, a sequence S_n converges to a the limit S

$$\lim_{n\to\infty} S_n = S$$

if for any $\varepsilon > 0$ there exists an N such that

$$\left|S_n - S\right| < \varepsilon \text{ for } n > N$$

Figure 2. Definition of the limit of a sequence (Weisstein, 2011)

This definition amounts to a kind of choosing game: Assuming S is the limit of a sequence, $\{S_n\}$, for any positive distance, ε, you choose, I can find a natural number, N, so that whenever the sequence goes beyond the Nth term, the distance between any of those terms and S is less than ε. The definition says that if the tail of a sequence gets arbitrarily close to a number, then that number is the limit of the sequence.

We can think about the decimal representation, 0.999…, as the limit of an infinite series:

$$9/10 + 99/100 + 999/1000 + \ldots$$

Thus, we arrive at the following conclusion:

$$0.\overline{9} = \sum_{k=1}^{\infty} \frac{9}{10^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{9}{10^k} = 1.$$

The equality holds because for any real value of ε that you choose, I can find a natural number N such that 1 is within ε of

$$\sum_{k=1}^{n} \frac{9}{10^k} \text{ whenever } n > N.$$

This means that we have devised a way to answer the question, "How close is close enough?" The answer is that we are close enough to the number 1 if, when given an ε neighborhood extending some distance about the number 1, we can find a number N such that the terms at the tail end of the series are inside that neighborhood. When this happens, we no longer distinguish between the terms of the series and the number 1.

Why Students Remain Skeptical

There is a historical basis for students' skepticism in accepting any of the arguments above, and researchers have found several underlying reasons for why students reject the equality—some more logical than others (Ely, 2010; Fischbein, 2001; Oehrtman, 2009; Tall & Schwarzenberger, 1978). For example, many students conceive of 0.999… dynamically rather than as a static point; they interpret the decimal expansion as representing a point that is moving closer and closer to 1 without ever reaching 1 (Tall & Schwarzenberger, 1978). Starting from 0, the point gets nine-tenths of the way to 1, then another nine-tenths of the remaining distance, and so on, but there is always some distance remaining (cf. Zeno's paradox). This conception aligns with Aristotle's idea of potential infinity and his rejection of an actual infinity: 0.999… is a process that never ends, producing a decimal expansion that is only *potentially* infinite and not actually an infinite string of 9's (see Dubinsky, Weller, McDonald, & Brown, 2005, for an excellent discussion of historical struggles with infinity and related paradoxes). This issue points to a confusion between numbers and their decimal representations: Would students be inclined to say that one-third is a process that never ends simply because its decimal expansion is 0.333…?

Tall and Schwarzenberger (1978) analyzed student reasons for accepting or rejecting the equality and found that they generally fit into the following categories:

- *Sameness by proximity:* The values are the *same* because a student might think, "The difference between them is infinitely small," or "At infinity it comes so close to 1 it can be considered the same" (p. 44).
- *Infinitesimal Difference:* The values are *different* because a student might think "0.999… is the nearest you can get to 1 without actually saying it is 1," or "The difference between them is infinitely small" (p. 44).

It is interesting that students in the two categories draw different conclusions using the same argument. Each uses a non-standard, non-Archimedean distance from the number one as an argument in their favor. In other words, each believes that there is some unmeasurable space between the two numbers, as in the number "next to" one.

In his research involving 120 university students, Oehrtman (2009) found that mathematical metaphors had significant impact on claims and justifications. With regard to the mathematical equality, 0.999… = 1, Oehrtman found that students were likely to use what he called an "approximation metaphor." Student comments referred to "approximations that could be made as accurate as you wanted" and the "irrelevance" of "negligible differences" or "infinitely small errors that don't matter" (p. 415). Although the students were

asked to explain why 0.999… = 1, most students disagreed with the equality. Many students referred to 0.999… as the number next to 1, or as a number touching 1.

Oehrtman (2009) went on to suggest that there is potential power in the approximation metaphor because this type of thinking closely resembles arguments for the formal definition of a limit. In fact, early definitions of limit by mathematicians such as D'Alembert included the language of approximation: "One magnitude is said to be the limit of another magnitude when the second may approach the first within any magnitude however small, though the first magnitude may never exceed the magnitude it approaches" (Burton, 2007, p. 603). Although the modern definition reflects an attempt to remove temporal aspects (see Figure 2), such ideas still underlie our conceptions of limit. And although students might make incorrect metaphorical statements, these metaphors often provide a gateway for deeper understanding of corresponding concepts.

The Hyperreals

The argument that 0.999… only *approximates* 1 has grounding in formal mathematics. In the 1960's, a mathematician, Abraham Robinson, developed nonstandard analysis (Keisler, 1976). In contrast to standard analysis, which is what we normally teach in K–16 classrooms, nonstandard analysis posits the existence of infinitely small numbers (infinitesimals) and has no need for limits. In fact, until Balzano formalized the concept of limits, computing derivatives relied on the use of infinitesimals and related objects that Newton called "fluxions" (Burton, 2007). These initially shaky foundations for Calculus prompted the following whimsical remark from fellow Englishman, Bishop George Berkeley: "And what are these fluxions? … May we not call them ghosts of departed quantities?" (p. 525). Robinson's work provided a solid foundation for infinitesimals that Newton lacked, by extending the field of real numbers to include an uncountably infinite collection of infinitesimals (Keisler, 1976). This foundation (nonstandard analysis) requires that we treat infinite numbers like real numbers that can be added and multiplied. Nonstandard analysis provides a sound basis for treating infinitesimals like real numbers and for rejecting the equality of 0.999… and 1 (Katz & Katz, 2010). However, we will see that it also contradicts accepted concepts, such as the Archimedean property.

Consequences of Accepting Infinitesimals and Rejecting the Equality

Consider the argument for equality that uses limits outlined in the previous section. What if you were allowed to choose ε to be infinitely small? Then the game is up; one cannot possibly hope to bring the sequence within such an intolerant tolerance! However, you should beware that, in order to win (i.e. choosing a value for ε that makes the limit argument fail, thus proving 0.999… does not equal 1), you have violated the Archimedean property.

The Archimedean property states that, for any positive real number, r, we can choose a natural number, N, large enough so that their product is greater than 1. That means any real number is farther from 0 than $1/N$ for some N. To visualize what this means, consider the illustration in Figure 3. No matter how close r is to 0, if we zoom in on 0 enough, the two numbers will be visibly separate. In other words, there is no number "next to 0," or infinitely close to 0. If r were allowed to be an infinitesimal, this would not be the case; r would be less than $1/N$ for all N, or stay perpetually next to 0, which violates the Archimedean property. Thus, the only way to maintain this intuitive property of the real line is to reject infinitesimals, as we have done in the historical development of the real line (standard analysis).

Ely (2010) described a case study of a college student who argued that there is no number next to zero but that there are numbers infinitely close to 0. This argument aligns with nonstandard analysis and presents the greatest challenge to the Archimedean property and other concepts from standard analysis. In particular, the student argued that one could zoom in *infinitely* to separate 0 from an infinitesimal number. Note, however, that the Archimedean property insists that positive real numbers be separable from 0 when zooming by a *finite* value, specified by the natural number N.

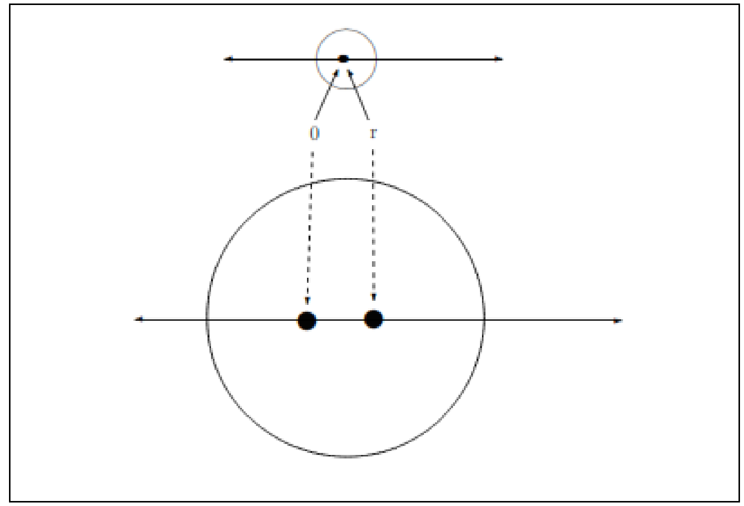

Figure 3. The Archimedean property.

Conclusions and Implications

The Archimedean property captures one of the most intuitive ideas about the real line (Brouwer, 1998). Starting from that property, we can use the definition of limits to show that the equality of 0.999… and 1 must hold. Thus, we can see that the Archimedean property and the formal definition of limits imply the equality. The only way to reject the equality is to reject the property or to reject our definition of limits.

As our investigation affirms, "attempts to inculcate the equality in a teaching environment prior to the introduction of limits appear to be premature" (Katz & Katz, 2010, p. 3). Yet a meaningful introduction of limits at the K–12 level is problematic. Bezuidenhout (2001) discusses difficulties in introducing limits even at the college level. Similar issues arise with the introduction of irrational numbers in the K–12 curriculum. It may be useful for students to recognize that some numbers (such as the length of the diagonal on the unit square) cannot be written as the ratio of two integers, but state standards demandmore. Consider the following example from the *Common Core State Standards* (National Governors' Association and Council of Chief State School Officers, 2010): "In eighth grade, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers." Are middle school teachers prepared to meaningfully address the formation of the real number system, and is this an important requirement for eighth graders?

In the history of mathematics, the development of calculus prompted speculation about the existence of infinitesimals, while motivating the construction of limits (Burton, 2007). Even the Archimedean property arose from a pre-calculus concept—namely Archimedes' method of exhaustion. If history is any guide for motivating and developing ideas in the classroom, then Katz and Katz (2010) draw a natural conclusion in suggesting that we delay the discussion of irrational numbers and infinite decimal expansions until after limits are formally addressed. An equally natural conclusion is that, when we do introduce students to limits, we should take advantage of intriguing problems, such as the (in)equality discussed here, so that students will understand why we might want to reject infinitesimals and, as a consequence, why we need limits.

Whereas *Common Core State Standards* ask students to consider infinite decimal expansions as early as eighth grade, many students are never asked to seriously consider whether 0.999… really does equal 1. Consideration of this equality might generate meaningful discussion about students' intuitive concepts. Imagine a Precalculus classroom full of students who have studied decimal expansions but have never studied irrational numbers except to prove that some numbers (such as the square root of 2) cannot be expressed as a ratio of two integers. Some students might have wondered, but none had formally studied whether this property is related to repeating or terminating decimal expansions. On the first day of a unit on limits, the teacher could ask whether 0.999… equals 1. This paper outlines potential connections students might make through arguments about this equality—connections between decimal expansions, the real number system, and limits. It seems that this kind of discussion does not typically happen because we ask some questions too early and others not at all.

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