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Mathematics at Hand

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The emerging field of mathematics educational neuroscience provides researchers with new approaches to understanding mathematical development, as well mathematics itself. This paper focuses on the role of the hand in constructing mathematics through activity. We rely on Piaget's distinction of three kinds of activity: sensorimotor activity, internalized actions, and interiorized operations—to review results from neuroscience studies. These distinctions and related neuroscience findings contribute to a new sense of mathematical embodiment. They also provide implications for mathematics instruction.

In a scene atop the Sistine Chapel, Adam languidly gestures a finger toward an eager God, who rides a cloaked wave of cherubim to meet his touch (see Figure 1). Aside from theological implications of the work, scholars have speculated Michelangelo's intentions to convey insights into the human anatomy. Specifically, physician Frank Meshberger (1990) conjectured that "The Creation of Adam" depicts the human brain within God's cloak and that the small gap between the fingers of Adam and his maker represent a synapse within the brain—a conjecture supported by striking similarities between the human brain and the outline of the cloak, as well as the fact that Michelangelo rigorously studied the neural anatomy of cadavers. The manner in which God delivers the spark seems especially appropriate in light of modern day neuroscience and

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findings related the emerging field of mathematics educational neuroscience (Campbell, 2006).

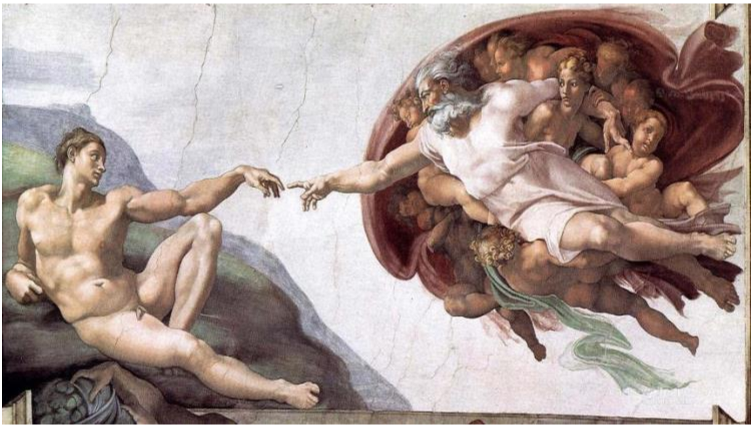

Figure 1. Michelangelo's "The Creation of Adam" (from Wikipedia, retrieved November 29, 2015, from https://en.wikipedia.org/wiki/The_Creation_of_Adam).

Number theorist Leopold Kronecker (1634) famously quipped, "God made the integers, man did the rest" (as cited in Hamming, 1980, p. 84). However, research on children's mathematical development has revealed that the construction of integers requires years of human labor that relies on the coordination of acts of pointing with number words (Baroody, 2004; Piaget, 1942; Steffe, 1992; Ulrich, 2015). As such, we find a connection between the construction of number and activity with the hand, and it is not merely coincidence that binds fingers and numbers within the same word, "digits." We can attribute the proliferation of base-10 number systems, across millennia and continents, to the fact that Homo sapiens have ten fingers on which to count. Even the base-20 of the Mayans had a sub-base of 10 (not to mention their toes), and although the number system of the Oksapmin people in Papua New Guinea is not base-10, it nevertheless relies on counting fingers, in addition to other locations on the body (Saxe, 2012). This connection between fingers and numbers persists in the neural anatomy of humans long after adults stop relying on their fingers to count. As evidence for this lasting connection, Rusconi, Walsh, and Butterworth (2005) found that both finger recognition (gnosis) and number magnitude processing were impaired among adults when a disruption was introduced to the neural functioning of their left angular gyrus (an areas of the brain within the parietal lobe, discussed later in this paper).

The purpose of this paper is to elaborate on the role of the hand in mathematical development by drawing upon neuroscience findings (cf. Norton & Bell, 2017). We introduce a neo-Piagetian interpretation of embodiment to assimilate these findings, accounting for the role of activity in constructing mathematical objects. Following Piaget (1972), we distinguish three kinds of activity: sensorimotor activity, internalized actions, and interiorized operations (see Table 1). We use these distinctions to frame current perspectives on mathematical development and its neural correlates (Ansari, 2008; Gallistel & Gelman, 1992).

Table 1
Three kinds of activity

Activity	**Description**	**Example**
Sensorimotor	Kinesthetic activity, which involves muscular movement, including movement of the eyes	Fair sharing a whole, with the goal of producing equal parts while exhausting the whole
Internalized	Actions that are carried out in imagination, without the need for muscular movement	Imagining a line segment partitioned into a specified number of equal parts
Interiorized	Operations that coordinate internalized actions all at once, with no need to run through the activity, even in imagination	Conceptualizing fractions as numbers that can be acted upon and composed with other numbers, as in fraction multiplication

We begin with an overview of Piagetian theory as an activity-based theory of cognition, which leads to our neo-Piagetian interpretation of embodiment. Then we review neuroscience literature and interpret it through that framework. Finally, we consider instructional implications of mathematics educational neuroscience when framed in this way.

Piagetian Framework

In a Piagetian theories of mathematical development build upon the idea that mathematical objects arise from coordinated activity, which becomes organized within structures for composing and reversing that activity (Piaget, 1970). Over the past four decades, Steffe and colleagues have carried out Piagetian research programs to elucidate children's constructions of number (Steffe, 1992; Steffe & Olive, 2010). Even children's initial conceptions of whole numbers, like 7, result from the coordination of at least three activities: the internalized action of unitizing (delineating individual items in the sensory field that will be counted as units of 1), the sensorimotor activity of pointing to (or otherwise noticing) each item in turn, and the sensorimotor activity of reciting (or otherwise running through) a verbal sequence ("one, two, three, four, five, six, seven"). As such, number is primarily a property of the counter's activity and not the items themselves (Piaget, 1942). Note that the internalized action of unitizing also arises through coordinated sensorimotor activity (resulting in the construction of units of 1) during early childhood, as described by von Glasersfeld (1981).

Steffe and colleagues (e.g., Steffe & Olive, 2010) have demonstrated ways children build upon their whole number knowledge to construct fractions. This process involves reorganizing the internalized actions (which result from coordinated sensorimotor activity) used to construct whole numbers, such as unitizing and iterating, along with new actions, such as partitioning. Children construct fractions as numbers by coordinating actions of unitizing, partitioning, and iterating, where partitioning and iterating act as inverses of one another (Norton & Wilkins, 2012; Steffe & Olive, 2010).

The unit fraction, 1/5, results from partitioning a continuous whole unit into five equal parts and unitizing one of those parts. Thus, the whole unit can be recreated from that unit fraction by iterating (making identical, connected copies of) it five times, thus establishing a 1-to-5 relationship between the two units. Once children have constructed unit fractions, they can begin to construct non-unit fractions as "numbers in their own right" by iterating a unit fraction some number of times, while maintaining its partitive/iterative relationship with the whole (Hackenberg, 2007). For instance, 7/5 is the number created by iterating a 1/5 unit seven times, where 1/5 has a 1-to-5 relationship with the whole unit.

The internalized actions of partitioning and iterating are derived from sensorimotor activity. Partitioning gradually develops as children learn to coordinate sensorimotor activity to satisfy two competing goals: creating equal parts and exhausting the whole (Piaget, Inhelder, & Szeminska, 1960). Before children learn to coordinate that activity, they will sometimes break a whole into unequal parts, or produce a specified number of equal parts with an additional leftover part. After children learn to coordinate their activity, they can begin to perform partitioning in imagination without carrying out the associated sensorimotor activity.

Piaget (1972) described further coordinations students could make, which he called *interiorized* operations, starting from internalized actions. Returning to the prior example, children learn to organize partitioning and iterating within a system for composing and reversing those actions (Wilkins & Norton, 2011). In that system, partitioning and iterating act as inverses of each other. For instance, partitioning a whole into five equal parts results in a part that, when iterated five times, reproduces the whole. This coordination provides a basis for constructing fractions as numbers, and as numbers, children can operate upon them further (Norton & Wilkins, 2012). For instance, children can learn to take fractions of fractions, as in fraction multiplication (Hackenberg & Tillema, 2009)—an instantiation of what Piaget (1972) referred to as "operations on operations."

A Neo-Piagetian Interpretation of Embodiment

Theories of embodied cognition posit that cognition derives from our particular embodiment within the world, including our limbs, hands, and opposable thumbs. "Embodiment provides a deep understanding of what human ideas are, and how they are organized in vast (mostly unconscious) conceptual systems grounded in physical, lived reality" (Núñez, Edwards, & Matos, 1999, p. 50). Embodied cognitionists do not usually make distinctions between sensorimotor, internalized, and interiorized activity, and some researchers argue that concepts are strictly sensorimotor: "Conceptions emerge in and through experience, never consisting of anything else but activated prior experiences" (Roth & Thom, 2009, p. 188), and "conceptual knowledge is embodied, that is, it is mapped within our sensory-motor system" (Gallese & Lakoff, 2005, p. 455). In line with other perspectives on embodiment (e.g., Nemirovsky & Ferrara, 2009; Wilson, 2002), we frame internalized and interiorized activity as forms of embodiment essential to mathematical development that are further and further removed from sensorimotor activity—further removed in the sense that they pertain more to cognitive processes (still embodied in the brain) that regulate sensorimotor activity and pertain less to sensorimotor activity itself.

Embodied cognition encompasses varied perspectives on cognition, but at their core, these perspectives all embrace the view that "cognitive processes are deeply rooted in the body's interactions with the world" (Wilson, 2002, p. 625). In other words, sensorimotor experience provides the basis for thinking and learning—a view that fits squarely with Piagetian theories of mathematical development (Piaget, 1972). As an example of the importance of sensorimotor experience in mathematics, Roth and Thom (2009) demonstrated how children's conceptions of geometric shapes, such as cubes, are grounded in bodily experiences, such as physically rotating figures and tracing their edges with their fingers.

Embodied perspectives generally admit imagined activity as well, either as part of the sensorimotor system or closely related to it. Previous sensorimotor experiences can be re-presented in imagination without enacting the associated physical activity. In fact, Nemirovsky and Ferrara (2009) equate mathematics with such re-presentations: "We view mathematics learning as the development of a particular kind of imagination" (p. 159). They insist that imagined activity must be included as part of the sensorimotor system in order to explain mathematical development, even suggesting that sensorimotor activity be reframed as "perceptuo-motor-imaginary activity" (p. 162). To demonstrate the importance of imagined activity, the authors share observational data from students interacting in a high school Algebra class. The authors argued that, in order to make sense of one another's mathematical activity and utterances, students had to engage in imagined activity, which is often indicated through the sensorimotor activity of gesturing. Such imagined activity corresponds to what we call internalized action, not interiorized because the action is still carried out in the brain (especially the premotor cortex—the region of the frontal lobe adjacent to the motor cortex, discussed in the next section) and not sensorimotor because the action is not carried out in the rest of the body.

In examining various perspectives of embodied cognition, Wilson (2002) considered the role of the body during "off-line cognition"; that is, cognitive activity "decoupled from the physical inputs and outputs that were their original purpose" (p. 633). As an example of off-line cognition, Wilson sketched the developmental progression of counting, from sensorimotor activity that depends on finger movements, to imagined activity with no overt physical activity, to *off-line* activity: "Mental structures that originally evolved for perception or action appear to be co-opted and run 'off-line'" (p. 633). When we refer to interiorized operations, we refer to a particular kind of off-line cognition that is not sensorimotor but is embodied nonetheless. In the next section, we relate the progression Wilson described to the distinctions Piaget (1972) made between sensorimotor, internalized, and interiorized activity, and we use those distinctions to interpret neuroscience findings.

Neuroscience Findings Related to Mathematical Development

The notion of mathematics educational neuroscience has arisen as a means to test and refine theories of mathematical development while extracting implications for improving classroom instruction (Campbell, 2006). Beginning with the sensorimotor region, we quickly find connections to other areas of the brain where imagined or internalized activity may occur. We demonstrate that neuroscience provides for a subtle distinction between sensorimotor activity and internalized activity, based on inhibition of activity in the sensorimotor region of the brain. *Interiorized activity* points to a further distinction that is particular to mathematics and logic (Piaget, 1970).

Sensorimotor Activity

Mathematical activity differentially activates the frontal and parietal lobes within the neocortex (Ansari, 2008; Emerson & Cantlon, 2012)—the outer layer of the brain unique to mammals. The frontal and parietal lobes meet at the sensorimotor region of the brain, which includes the (primary) motor cortex at the back of the frontal lobe and the somatosensory cortex at the front of the parietal lobe (see Figure 2). While the motor cortex controls all voluntary bodily movements through neural signals sent to muscles, the somatosensory cortex receives neural signals from the rest of the body through the spinal cord. The premotor cortex sits just in front of the motor cortex, between it and the prefrontal cortex (the foremost region of the frontal lobe). Along with the motor cortex, the premotor cortex is involved in bodily movement, but it is activated before the motor cortex, in preparing for bodily movement (Wise, 1985).

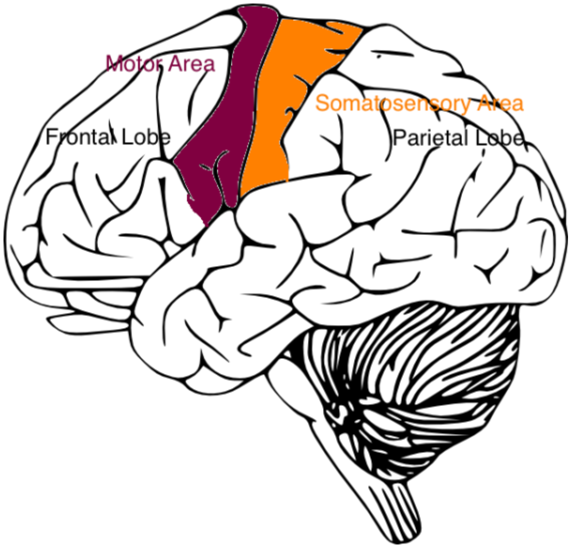

Figure 2. The sensorimotor region.

Within each hemisphere of the brain, the sensorimotor region (including the premotor cortex, motor cortex, and somatosensory cortex) is mapped to the rest of the body as shown in Figure 3. Neuroimaging studies have demonstrated that the premotor cortex is activated even when imagining bodily activity or observing it in others (e.g., Arnstein, Cui, Keysers, Mauits, & Gazzola, 2011). One such study used functional magnetic resonance imaging (fMRI) with 12 young adults to locate specific areas within the participants' frontal and parietal lobes that were activated during various observed activities (Buccino et al., 2001). fMRI provides precise spatial data that can identify very specific areas of activation. In particular, the researchers used the fMRI data to identify which areas of the participants' premotor cortex were activated as they observed another person moving his mouth, hand, or foot. The researchers found that observation activated the specific area of the premotor cortex corresponding to moving the respective body part, as if the subject were planning to move that same part of the body. Moreover, if the observed subject were acting on a physical object, corresponding areas within the participant's parietal lobe were activated as well (Buccino et al., 2001).

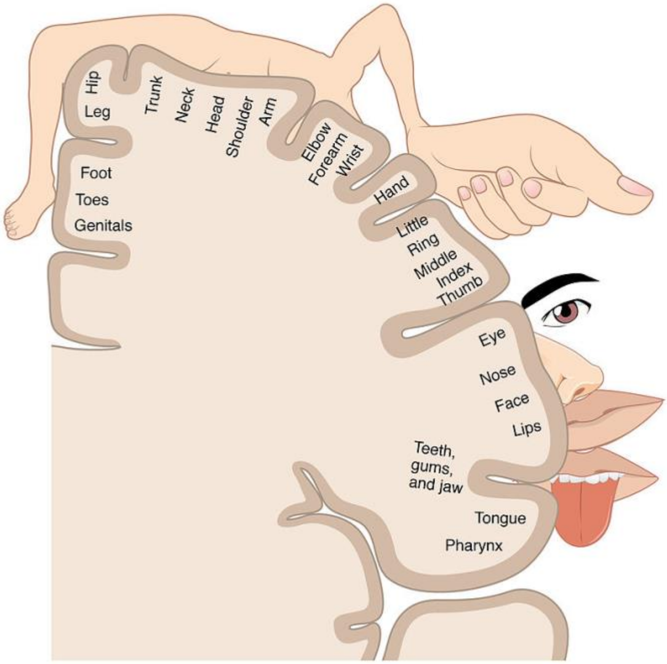

Figure 3. Body map within the sensorimotor region (from Anatomy & Physiology, http://cnx.org/content/col11496/1.6/).

Researchers have drawn upon the identification of "mirror neurons" (Rizzolatti & Craighero, 2004) in the premotor cortex to argue for the sensorimotor basis of knowledge in general (Gallese & Lakoff, 2005; Koziol, Budding, & Chidekel, 2012; Nemirovsky & Ferrara, 2009; Roth & Thom, 2009). For example, Gallese (2007) argued that we understand the intentions of others by imitating their activity in imagination. There is no overt sensorimotor activity, but traces of the sensorimotor basis for knowledge are evident in the gestures people employ when attempting to communicate an idea (Alibali & Nathan, 2012; Hostetter & Alibali, 2008; Nemirovsky & Ferrara, 2009; Núñez, 2006). Further evidence for the sensorimotor basis of knowledge comes in the form of motor evoked potentials—electrical signals sent through the spinal cord despite inhibition of motor activity (Fadiga, Fogassi, Pavesi, & Rizzolatti, 1995). When inhibitory control is impaired (e.g., as can be the case with prefrontal lesions), people will compulsively imitate observed behavior, indicating that they may be "unable to generate motor imagery without immediately transferring the imagined action into motor output" (Jeannerod, 2001, p. S107).

Internalized Actions

Here, we focus on mirror neurons related to the hand and their contribution to the development of mathematical knowledge. Mirror neurons help explain an important shift in mathematical development—the shift from strictly sensorimotor activity to internalized actions that are based in sensorimotor experience. Buccino and colleagues (2004) identified a "mirror neuron circuit" associated with internalized activity during imitative learning. Research participants observed a video of someone playing a chord on the guitar and, following a pause, these participants were asked to imitate the chord. This mirror neuron circuit—active during observation, pause, and execution—included the premotor area associated with the hand and the intraparietal sulcus (IPS).

The IPS (see Figure 4) lies between the upper (superior) and lower (inferior) lobules within the parietal lobe and aligns with the area of the somatosensory cortex that is associated with the hand. Buccino and colleagues (2001) found that when participants observed an individual acting on an object with his hand, the IPS was the differentially activated area of the parietal lobe. Subsequent studies have demonstrated that the IPS is also activated when participants observe tools and graspable objects, even when no one is using or grasping them (Mruczek, von Loga, & Kastner, 2013; Valyear, Cavina-Pretesi, Stiglick, & Culham, 2007). More specifically, observing a graspable object (e.g., a doll) evokes stronger activation of the IPS than observing a non-graspable object (a large animal), and observing a tool (e.g., a spoon) evokes even stronger activation within the IPS, especially in the left anterior area (the part of the IPS closest to the sensorimotor region in the left hemisphere of the brain). Because all participants in the studies were right-handed and the left hemisphere of the brain controls the right hand, these findings suggest that simply observing tools calls to mind learned activities in manipulating them with the hand, which may be organized in the left anterior IPS.

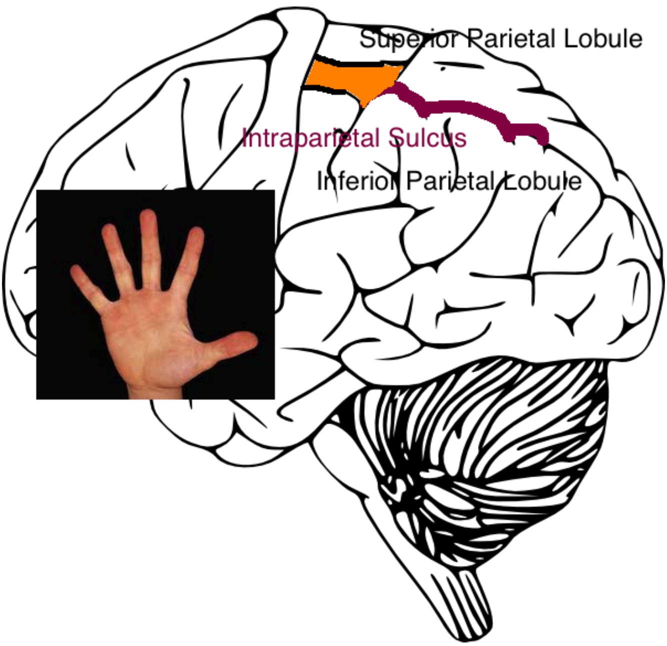

Figure 4. Acting on objects with the hand.

The common association between acting on an object with one's hands and mathematical activity should come as no surprise; after all, children learn to count with their fingers, and providing opportunities for students to manipulate objects with their hands is the pedagogical basis for many learning tools used in mathematics education (Novack, Congdon, Hemani-Lopez, & Goldin-Meadow, 2014; Raje, Krach, & Kaplan, 2013). Research in mathematics education contains an abundance of evidence that children begin to construct mathematical objects through sensorimotor activity, especially involving the hand (Baroody, 2004; Piaget, 1942), which is why manipulatives (e.g., counters) are useful in elementary school classrooms (Carbonneau, Marley, & Selig, 2013). A handful of neuroscience studies explicitly address this connection.

Sato, Cattaneo, Rizzolatti, and Gallese (2007) found motor evoked potentials among fingers in the right hand of right-handed adults as they determined whether a given numeral (1–9) was odd or even. Other studies reveal a neurological connection between finger recognition and mathematical ability (e.g., Fayol, Barrouillet, & Marinthe, 1998; Noël, 2005). Crollen and Noël (2015) investigated this connection by engaging children in counting and addition tasks while they moved either their hands or feet: "In both tasks, the hand movements caused more disruption [in solving the tasks] than the foot movements, suggesting that finger-counting plays a functional role in the development of counting and arithmetic" (p. 37). Penner-Wilger and Anderson (2013) attribute the connection to a repurposing of areas of the brain that had evolved for tool use, now adapted for the purpose of registering numbers.

Interiorized Operations

We have argued that children construct number based largely on sensorimotor activity involving the hands and fingers. In sensorimotor activity, children perform activity on physical material, including tools and fingers. During internalized action, gestures and motor evoked potentials indicate traces of sensorimotor activity, but neural activity in the sensorimotor region is inhibited so that children exhibit little overt behavior (Fadiga, Fogassi, Pavesi, & Rizzolatti, 1995; Hostetter & Alibali, 2008). We draw on additional neuroscience studies to make a further distinction, one between internalized actions and interiorized operations. Namely, internalized actions involve imagined transformations of physical material; interiorized operations act upon products of coordinated activity (Piaget, 1972).

We see traces of sensorimotor activity even among adults as they consider numbers or any ordered series (Andres, Seron, & Olivier, 2007). However, with increased age and expertise, we also see a shift away from sensorimotor activity and associated neural networks. For example, in the Crollen and Noël (2015) study, the disruptive effect of hand movements during counting and addition tasks was less pronounced among fourth-grade children, compared to first-grade children. This finding aligns with a general frontal-to-parietal shift that corresponds with mathematical development (e.g., Ansari & Dhital, 2006; Rivera, Reiss, Eckert, & Menon, 2005).

Rivera, Reiss, Eckert, and Menon (2005) found developmental differences across participants from age 8 to age 19 as they engaged in addition and subtraction tasks. When compared to younger students, older students relied less on neural resources in the prefrontal lobule (associated with attention and sensorimotor activity) and more on neural resources in the parietal lobe, especially the IPS. The researchers concluded, “our findings provide evidence for a process of increased functional specialization of the left inferior parietal cortex in mental arithmetic, a process that is accompanied by decreased dependence on memory and attentional resources with development” (p. 1779). Ansari and Dhital (2006) found similar age-related differences as participants engaged in non-symbolic magnitude tasks, comparing two collections of small squares.

Decreased dependence on attentional resources indicates less reliance on sensory material. This cognitive change corresponds with mathematical development that Steffe (1992) documented in the context of children’s counting schemes: As children develop more and more sophisticated schemes for counting, they shift their attention from physical material, to figurative material, to the coordination of internalized actions themselves, with no further need for sensory material. The shift frees children’s attentional resources to act upon products of their counting activity, as objects (cf. Dubinsky, 1991); for example, in multiplying two whole numbers, they can focus on the internalized action of distributing one whole number across each unit of 1 within the second whole number (Steffe, 1992).

In other words, with counting activity interiorized, students can attend to internalized actions involving the products of that activity (composite units), as in multiplication. In this sense, we might say that mathematics is abstract and builds upon itself.

In his reorganization hypothesis, Steffe (2002) described ways in which children build upon their whole number knowledge to construct fractions knowledge. At earlier stages in their development, children treat fractions as two whole numbers: part and whole (e.g., 1/5 as one part out of five equal parts in the whole). When they can coordinate the 1-to-5 relationship between these whole numbers, as previously described, they can begin to understand fractions as measures, eventually conceptualizing improper fractions, such as 7/5, as numbers in their own right (Hackenberg, 2007). An fMRI investigation provides some indication of the neural correlates for this development and its implications for mathematical embodiment.

Ischebeck, Schoke, and Delazer (2009) used fMRI to study the *numerical distance effect* as 20 adult participants compared two fractions. The numerical distance effect describes a well-established phenomenon in studies involving the comparison of two whole numbers: comparisons are more difficult when the numbers are close together, and discernment usually relies on greater activation in the IPS. For example, Ansari and Dhital (2006) found a distance effect as participants compared two collections of objects—an effect that was associated with greater IPS activation among adults when compared to children. In the fractions study by Ischebeck, Schoke, and Delazer (2009), participants could focus on any of three distances: between the numerators in the two fractions, between the denominators in the two fractions, or between the two fractions as numbers themselves. All three distance effects were observed, but only the distance between the fractions as numbers themselves modulated neural activity in the IPS. This finding supports the idea that the IPS may be repurposed to accommodate fractions as numbers.

Recall that the IPS plays a primary role in tool use—even imagined tool use. It is also centrally involved in comparing non-symbolic magnitudes (Ansari & Dhital, 2006) and in ordering series, even non-numerical series (Andres, Seron, & Olivier, 2007). Piaget (1942) demonstrated that children's conceptions of number develop through coordinated activity that integrates magnitude (cardinality) and order. Now we see evidence that this development corresponds with less dependence on sensorimotor activity and greater specialization within the IPS, first with whole numbers (Ansari & Dhital, 2006; Rivera, Reiss, Eckert, & Menon, 2005) and later with fractions (Ischebeck, Schoke, & Delazer, 2009). Thus, it appears that the IPS plays a recursive role in coordinating activity to construct number: It is involved in constructing whole numbers via coordinated sensorimotor activity involving the hands (e.g., fingers as tools); later, it is involved in constructing fractions via coordinated actions on whole numbers. This recursion may be the neurological sense in which mathematics builds upon itself.

The Embodied Mathematical Mind

Nemirovsky and Ferrera (2009) framed embodied cognition within perceptuo-motor-imaginary activity. That framework acknowledges the importance of activities that people do not physically perform. Wilson's (2002) framing of embodied cognition opens the door for further distinction: "Mental structures that originally evolved for perception or action appear to be co-opted and run 'off-line,' decoupled from the physical inputs and outputs that were their original purpose, to assist in thinking and knowing" (p. 633). Neuroscience introduces opportunities for researchers to elucidate these distinctions.

In the context of mathematical development, we find that sensorimotor activity serves as the basis for mathematical development but that children learn to coordinate internalized actions and then to act on the products of such coordinations in a hierarchical fashion. In this sense, coordinations of actions generate mathematical objects that we might call abstract (Piaget, 1970). Summarizing decades of mathematics education research within a Piagetian program, Steffe and Olive (2010) describe how fractions arise through the coordination of internalized actions on whole numbers, which themselves arise through coordinated sensorimotor activity. We see neurological evidence for such development in the frontal-to-parietal shift that characterizes mathematical development in general (Ansari & Dhital, 2006; Rivera, Reiss, Eckert, & Menon, 2005; Ischebeck, Schoke, & Delazer, 2009).

Specifically with regard to the hand, mathematical development seems to rely heavily on an area of brain evolved for tool use—the IPS (Mruczek, von Loga, & Kastner, 2013). It appears that the role of the IPS within the frontal-parietal network is to coordinate activity, beginning from sensorimotor activity involving tools, extending to sensorimotor activity involved in counting, and later involving the coordination of internalized actions like partitioning and iterating. In this way, the psychological construction of mathematical objects would depend heavily on coordinated activity beginning with the hands. Understanding mathematical objects as coordinated actions holds implications for instruction.

Mathematics educators generally understand the importance of using manipulatives when working with children (e.g., NCTM, 2000) and the increased effectiveness of teaching with manipulatives has been supported by research (e.g., Carbonneau, Marley, & Selig, 2013). However, instruction with manipulatives does not result in increased student learning as consistently or as dramatically as we would like (Carbonneau, Marley, & Selig, 2013). A Piagetian perspective on embodiment, informed by neuroscience, can provide new insights into why the use of manipulatives can be useful and, therefore, how they can best be used.

The current conventional wisdom, drawn from various theoretical perspectives, is that concrete manipulatives facilitate learning in four ways:

> …supporting the development of abstract reasoning, stimulating learners' real-world knowledge, providing the learner with an opportunity to enact the concept for improved encoding, and affording opportunities for learners to discover mathematical concepts through

> learner-driven exploration. (Carbonneau, Marley, & Selig, 2013, p. 381)

Our embodied perspective emphasizes the first kind of facilitation (i.e., the development of abstract reasoning) and provides a more detailed landscape for the process of abstraction. Namely, we see a student's abilities to internalize and coordinate actions as the key to mathematical learning. Our embodied perspective, therefore, would have the following four broad implications for instruction using manipulatives:

1. Manipulatives can be used to carry out and coordinate sensorimotor activity that has not yet been internalized. For example, students can engage in fair sharing activities, such as sharing a fraction strip (thin strip of construction paper) among seven people. Through repeated sensorimotor activity, they might learn to coordinate activity to satisfy two competing goals: creating seven equal parts and exhausting the whole strip. Thus, students might develop an internalized action of partitioning.
2. Manipulatives and visual representations can also play a role in the coordination of internalized actions. For example, Olive and Vomvoridi (2006) demonstrated how teachers can use virtual manipulatives and visual representations—including student drawings—to support the coordination of partitioning and iterating. Through their coordinated activity with this figurative material, students began conceptualizing unit fractions, $1/n$, as having a 1-to-*n* size relation with the whole, rather than simply conceptualizing them as 1 part shaded within an *n*-part whole.
3. Some instructional practices will not be effective, such as when the teacher models a problem solution for the class using actions that students have not yet internalized. Observing a subject acting on an object activates areas of the brain associated with the observer planning to carry out that activity herself (Buccino et al., 2001), so we should not expect visual demonstrations of mathematical

procedures to convey any mathematical meaning for students until they have internalized the associated actions. On the other hand, if students have internalized the associated actions, teacher demonstrations might suggest ways that students can coordinate them, if given the opportunity to do so.

4. Finally, knowing what kinds of actions students need to internalize and coordinate becomes an important consideration. We have suggested partitioning and iterating constitute fundamental internalized actions for constructing fractions. Research in mathematics education from action-based perspectives suggests the importance of numerous other internalized actions in students' mathematical development (e.g., *covariation*, see Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). What kinds of manipulatives afford opportunities for sensorimotor activity, and coordinations thereof, that might induce internalized actions and interiorized operations?

Coordinations of sensorimotor activity and internalized actions introduce unending possibilities for constructing new mathematical objects (Piaget, 1970). A hierarchy of mathematical objects develops by coordinating actions performed on existing objects so that new objects have a reduced dependence on attentional resources (Rivera, Reiss, Eckert, & Menon, 2005). These objects build, not from the integers, as Kronecker suggested, and they do not exist in some Platonic ideal. They begin at the touch of a finger and continually build through human activity (sensorimotor, internalized, and interiorized).

> For the mathematician it is, of course, tempting to believe in Ideas and to think of negative or imaginary numbers as lying in God's lap for all eternity. But God himself has, since Gödel's theorem, ceased to be motionless. He is the living God, more so than heretofore, because he is unceasingly constructing ever "stronger" systems. Passing from "abstract" to "real" or "natural" structures, the

problem of genesis becomes all the more acute. Only if we forget about biology can we be satisfied with Chomsky's theory of innateness of human reason or with Lévi-Strauss' thesis of the permanence of the human intellect. (Piaget, 1970, p. 141)

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