Exercising Mathematical Authority: Three Cases of Preservice Teachers' Algebraic Justifications

Priya V. Prasad and Victoria Barron

Students' ability to reason for themselves is a crucial step in developing conceptual understandings of mathematics, especially if those students are preservice teachers. Even if classroom environments are structured to promote students' reasoning and sense-making, students may rely on prior procedural knowledge to justify their mathematical arguments. In this study, we employed a multiple-case-study research design to investigate how groups of elementary preservice teachers exercised their mathematical authority on a growing visual patterns task. The results of this study emphasize that even when mathematics teacher educators create classroom environments that delegate mathematical authority to learners, they still need to attend to the strength of preservice teachers' reliance on their prior knowledge.

Self-efficacy in mathematics depends on having a sense of mathematical authority (Keazer & Menon, 2016; Lloyd & Wilson, 2000; Webel, 2010); that is, learners of mathematics need to rely on their own sense-making in mathematics in order to develop conceptual understandings of mathematics. It is of particular importance that mathematics teacher educators provide preservice teachers (PSTs) opportunities for developing mathematical authority for two reasons: (a) As future teachers, PSTs need to be able to trust their own mathematical self-efficacy (Ball, 1990); and (b) experiencing a mathematics class in which mathematical authority is shared can encourage PSTs to share authority with their students in their own future classrooms. However, elementary PSTs may have limited experience with sense-making in their own schooling, causing

Priya V. Prasad *is an assistant professor at the University of Texas at San Antonio. Her primary research interest is mathematical knowledge for teaching, with a current focus on understanding how issues of status and agency in the classroom mediate how teachers think and act.*

Victoria Barron *is a high school mathematics teacher at Boerne Champion High School. She is interested in researching the conceptualization of mathematical topics through student-centered instruction.*

them to rely on formulas presented by instructors or in a textbook instead of developing conceptual meanings (Blanton & Kaput, 2005; Keazer & Menon, 2016). By structuring tasks and classroom discourse in ways that allow PSTs to develop their own mathematical authority in the process of solving problems and justifying their reasoning, mathematics teacher educators can provide opportunities for PSTs to develop their own internal sources of mathematical authority. In this study, we discuss three cases of small groups of PSTs working on an algebraic task about growing visual patterns. Each case is an instantiation of PSTs' use and placement of mathematical authority, whether in their own previously gained procedural knowledge or from their collaborative sense-making and justification of their work. This study informs the instructional recommendations for mathematics teacher educators who would like to delegate mathematical authority to PSTs in mathematics content courses by highlighting the importance of PSTs' previous conceptions of mathematical concepts in their exercises of mathematical authority.

Background

When students are doing mathematics, they rely on a mathematical authority to drive their decisions about their course of action. Depending on where that authority is located in a classroom or how students confer it or defer to it, the exercise of mathematical authority by students can vary. Although previous scholars have not precisely defined the term *mathematical authority*, most agree that it can be conceptualized as being situated internally or externally. Gresalfi and Cobb (2006) described mathematical authority not as "'who's in charge' in terms of classroom management but 'who's in charge' in terms of making mathematical contributions" (p. 51).

When mathematical authority is situated within the learner (i.e. internally), Schoenfeld and Sloane (2016) described the result to be a "personal ownership of the mathematics they can certify" (p. 62). Povey and Burton (2003) labeled this way of knowing mathematics as *author/ity,* in which learners and instructors co-create knowledge in the classroom. In this way,

learning mathematics is personal and reflective of the needs of students in the classroom, giving students opportunities to practice the development of their own knowledge. Reinholz (2012) proposed three aspects to the exercise of an internally-situated mathematical authority: Students who rely on an internal mathematical authority (a) *explain* their reasoning, (b) *justify* their conjectures, and (c) *assess* their work once they find a solution, just as "mathematicians use these skills to derive authority from the logic and structure of mathematics, rather than relying on some other authoritative source" (p. 242). Reinholz conceptualizes these three aspects as "mutually supportive skills" that form the foundation of students' development of an internal mathematical authority. Reinholz further differentiates between self-assessment (when a student assesses his or her own thinking) and peer-assessment (when a student assesses the thinking of a another student); for this paper, we will allow assessment to mean both types of assessment. Moreover, Reinholz's appeal to the disciplinary norms of mathematics inform how we interpret *explaining*, *justifying*, and *assessing*, as we elaborate later.

Another way of knowing mathematics that Povey and Burton (2003) proposed is through *external authority*. In this way of knowing, learners see mathematics as something they cannot control. The learner views mathematics as a subject that is fixed and unchangeable. Learners with this view leave mathematics to the experts, and they see mathematics as a collection of already-known facts that are decided upon by mathematicians in the field. In addition, the "authority for the learner rests in the content," rather than internal judgement (Povey & Burton, 2003, p. 244). Povey and Burton conclude that most college students do not rely on an internal mathematical authority due to the dependence on an external (likely human) authority figure—the instructor. Similarly, Schoenfeld and Sloane (2016) found that students would propose their own conjectures and then wait for them to be corrected or justified by the expert. In this way, the mathematics in which the students are engaging is external to them. According to Schoenfeld and Sloane, when students rely on an external authority, "students propose; experts judge and certify" (p. 62). If students do not

have opportunities to engage in sense-making in the classroom, they are forced to rely on an external source of mathematical authority such as a textbook, a teacher, or a formula (Reinholz, 2012).

Rather than having to confer authority solely on the instructor, students should be able to construct their own personal meanings of the mathematics, allowing them to rely on an internal authority (Povey & Burton, 2003; Schoenfeld & Sloan, 2016). When students view instructors as the sole arbiter of mathematical correctness, they are surrendering their authority to the instructor and distancing themselves from the material they are learning. In order to decrease students' reliance on external mathematical authorities and foster their dependence on internal ones, instructors must shift their roles in the classroom (Dunleavy, 2015; Gresalfi & Cobb, 2006; Stein et al, 2008; Webel, 2010). Rather than instructors presenting themselves as "'dispenser[s] of knowledge' and arbiter[s] of mathematical 'correctness,'" they should take on the role of the facilitator or "engineer of learning" to allow students to construct their own understanding of the mathematics (Stein et al., 2008, p. 4). The instructor is no longer expected to present procedural problems and evaluate answers as correct or incorrect; instead, the instructor's role is to monitor students' solution processes and to investigate students' sense-making. By shifting the sources of mathematical authority in the classroom from external to internal, instructors can help students gain a deeper conceptual understanding of the mathematics and support students to see themselves not as outsiders or doers of mathematics, but as constructors of mathematics (Webel, 2010). Dunleavy (2015) determined the site of authority in the classroom by identifying the sources of mathematical contributions: "Delegating" mathematical authority requires "students to convince their peers that their solutions make mathematical sense" (p. 63). When teachers delegate their authority to their students rather than maintaining sole authority in the classroom, students become responsible for their own learning. In addition, students are encouraged to actively participate in the construction of mathematical knowledge in the classroom. When mathematical authority is distributed to

students, they are given opportunities to "argue, evaluate, and confirm the validity of their mathematical ideas" (Dunleavy, 2015, pp. 63–64). In order to develop PSTs' own mathematical self-efficacy and encourage them to cede mathematical authority to their students in their own future classrooms, mathematics teacher educators should model the type of classroom described above in mathematics content courses.

For this paper, we chose to focus purely on the PSTs' development of *intellectual authority* as a group. Langer-Osuna's (2016, 2017) recent work has highlighted the importance of inter-group social dynamics (i.e. *social authority*) in mediating students' development of intellectual authority. Although the instructors whose classes are included in this study attend and attempt to mitigate the role of status hierarchies within the group's work, in this study, we focused on the opportunities built into each lesson for students to collaboratively develop a sense of mathematical (intellectual) authority.

Methodology

Context and Research Question

PSTs, especially at the elementary level, like the other college students that Povey and Burton (2003) studied, lack experiences that encourage them to rely on their internal mathematical authorities (Schoenfeld & Sloane, 2016). Thus, when PSTs enter mathematics content courses, many are likely to search for an external mathematical authority to which to defer, be it the instructor, a textbook, or a peer with high mathematics status (Cohen & Lotan, 2014). The data analyzed in this study came from a larger ongoing project that followed the continuous improvement model proposed by Berk and Hiebert (2009) for iteratively improving elementary mathematics content courses for PSTs. The research team, comprised of four mathematics educators (three of whom regularly teach elementary mathematics content courses for preservice teachers), investigated PSTs' engagement with mathematics tasks that addressed concepts that have been

historically challenging to PSTs at the large southwestern university where the study was conducted. The researchers created, either facilitated or observed, and revised tasks in accordance with the continuous improvement model. For each focal mathematical concept, the research team implemented the following cycle: (a) design a task that targets a particular mathematical conception or deepens understanding of a particular mathematical idea (as well as prompts pedagogical reflections by embedding the tasks in teaching scenarios); (b) develop hypotheses about anticipated student responses; (c) collect data such as student work, formative assessments, and recordings of classroom discourse and analyze these data sources for evidence of the desired student learning outcomes; and (d) record the information collected and use it to revise the task for use in subsequent semesters. For this paper, the primary sources of data were audio recordings and synchronized written records of PSTs' work from small-group interactions collected during a single semester.

When the research team wrote each lesson, the following assumptions about effective student learning in mathematics undergirded their thinking:

1. Students need to have opportunities to collaborate and communicate about mathematics in order to learn conceptually and deeply (Stein & Smith, 2008).
2. Good mathematical tasks for group-work must be group-worthy (Lotan, 2003) and student-centered.
3. Students bring their own funds of knowledge to the mathematics classroom, and this previous knowledge should be honored and leveraged (Aguirre et al., 2013).

Therefore, each mathematical task that was developed in this project was implemented in a similar way: The instructor gave (at most) a brief introduction to the task, arranged the PSTs into groups of three to four to work on the task, and then allowed PSTs to work in groups while circulating around the classroom. The instructors closed most lessons by facilitating a whole-group discussion.
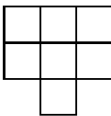
The growing-visual-patterns task described in this paper was implemented in this way in two sections with two different instructors. This implementation structure allowed for the delegation of authority from the instructor to the PSTs in the ways advocated by research in the field (Dunleavy, 2015; Gresalfi & Cobb, 2006; Stein & Smith, 2008; Webel, 2010). Additionally, due to the third underlying assumption, instructors did not direct students to disregard any previous knowledge that they might bring to their engagement with the task. The research group believes strongly in modeling for PSTs the practice of eliciting and leveraging students' existing funds of knowledge. However, when the research group observed the implementation of the first iteration of the growing-visual-patterns lesson, they noticed that many groups of PSTs were not fully exercising their mathematical authority by explaining, justifying, and assessing, even though the structure of the task created opportunities for all three. As many mathematics teacher educators do, the instructors in this study structured their classrooms and wrote mathematical tasks in ways that would support PSTs' reliance on their internal mathematical authorities. However, since PSTs still deferred to their own procedural prior knowledge, this led us to ask the following question: What preempts the full exercise of mathematical authority by PSTs *as a group* on tasks that are structured to encourage it?
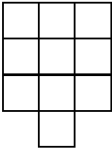
Growing Visual Patterns Task

One of the first tasks created for this project required PSTs to analyze growing visual patterns and to develop expressions that reflected their growth, which is a particularly rich context in which students can explore the concept of mathematical generalization (Liljedahl & Zazkis, 2002). Our expectations for our PSTs on the task were informed by the results of work that had been done by Warren and Cooper (2008) with third-grade students. Warren and Cooper found that although third-grade students were able to think functionally, they expressed visual patterns algebraically as a relationship between subsequent steps rather than between the term of the visual pattern (i.e., the output) and the step number (i.e., the input). They concluded that

students' understanding of generalization relies on having adequate opportunities to generalize patterns; that is, by exposing students to visual growth patterns, they will be prepared to construct algebraic expressions based on their internal mathematical authority. The initial task (see Figure 1) created by the research team was aimed at expanding PSTs' experience with generalizing visual patterns. Before creating the task, the research team identified the following learning goal for this task: PSTs will be able to use a variable to represent an index in a sequence of visual patterns and build an expression in terms of the variable to represent an arbitrary step in the sequence.
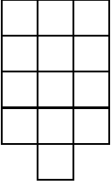

Pattern I

Step 1   Step 2   Step 3

In groups:
  a. Draw the next three steps of the pattern.
  b. Draw the 10th step of the pattern.
  c. Express in words a general rule describing how to create any step in the pattern, given the step number.
  d. Using a variable, write a general expression that tells how many tiles are in a step of the pattern, given the step number.

Figure 1. First iteration of the growing visual patterns task given to PSTs.

PSTs were given two more patterns (shown in Figure 2) to investigate and prompted to answer the same questions; all three patterns were adapted from Boaler (2015). Different groups started with different patterns, but eventually all groups explored all three patterns. The small-group discussions were recorded with the use of smart pens that captured audio recordings of the PSTs' discourse and synchronized the audio with a digital record of their written work.
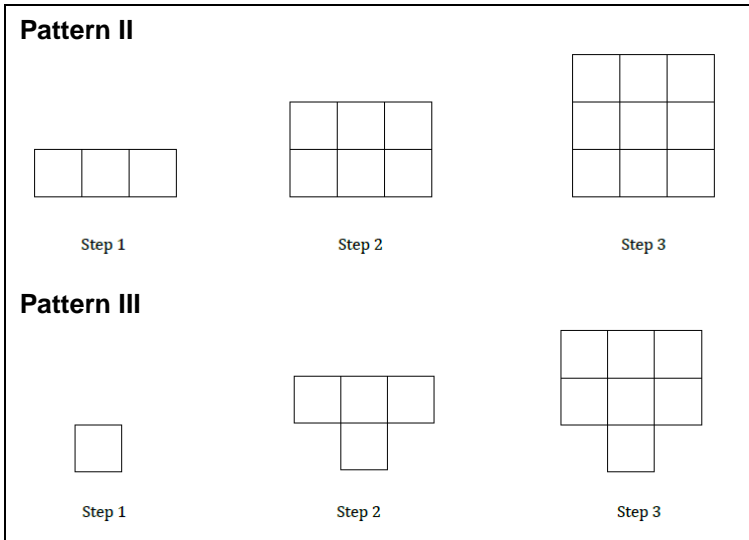
Figure 2. Second and third iterations of the growing visual patterns task given to PSTs.

Data Analysis

The data were analyzed first from a perspective of content analysis (Krippendorff, 2012). Three cases (i.e., three groups of preservice teachers) were selected as instances that typified different modes of reasoning about the task, and the associated recordings were transcribed. Then, the authors identified three major categories of discourse based on Reinholtz's (2012) three aspects of authority:

1. *Explaining reasoning*. In the context of this task, PSTs explained their reasoning most often as they were attempting to communicate to their group members their conjectures about how the pattern would develop.
2. *Justifying conjectures*. PSTs would justify their conjectures by attempting to make meaning for the variables or terms in their expression, often by referring back to the growth of the visual pattern. Justification was identified generally by using the word because or after a prompting question of why.

3. *Assessing solutions*. PSTs would assess the validity of their expressions by numerically testing them against their previously established numerical or visual patterns.

Data were coded sentence by sentence (with phrases being considered sentences if the speaker stopped without finishing the sentence or started a new thought after a pause). Table 1 provides examples of utterances that fell into each category, with the excerpts shown as exchanges in order to provide context. From coding the discourse in the transcripts in this way, the authors could characterize each group's overall exercise of mathematical authority based on the general trends of the group's interactions. For example, even though a group may have an isolated utterance coded as justification, the group was not necessarily considered to be justifying in a larger sense (see Exchange 1 in Table 1). Moreover, the distinctions between explaining, assessing and justifying were made from a mathematical perspective; for example, PSTs may have felt they were justifying a conjecture by checking it against the numerical growth of the pattern, but from a mathematical perspective, this would be considered assessing the conjecture. Additionally, because the focus was on how PSTs exercised their mathematical authority, the instructor's talk was left uncoded.

Table 1
Examples of the Three Categories of Mathematical Authority

Exchange	Example	Code
1	PST 2.2: Because the difference is 4.	Justifying
	PST 2.2: We already know that.	Explaining
	PST 2.2: And then the first step is 7.	Explaining
2	PST 2.2: So, I was using this [*gesturing to the formula*], and then this is your first number so on here, the first number is 7 and it is $n-1$ so that'll whatever step you choose.	Explaining
	PST 2.1: Yeah.	Assessing
	PST 2.2: And it is, the difference and we already knew it was going up by 3 every time.	Explaining

Exchange	Example	Code
3	Instructor: What would happen if you have a *n* instead of a *n* minus 1 there? What would that mean in the picture and pattern?	(*uncoded*)
	PST 1.1: It would be one term past what it needs to be.	Justifying
4	PST 3.1: So, it's like we are deleting step 1, but like for step 2, would be your step 1, in the equation.	Justifying
	PST 3.2: It's almost there is no 1 in a sense.	Justifying
	PST 3.2: So, step 3, we would take 3 times 2 plus 1.	Explaining
	PST 3.2: Step 4 we would take 3 times 3 [*pause*]	Explaining
	PSTs 3.1 and 3.3: Plus 1.	Assessing
	PST 3.2: Plus 1.	Assessing
	PST 3.3: Yeah.	Assessing
	PST 3.2: So, you would just have to get it one number behind.	Justifying

Note. PST 1.1 = preservice teacher, group 1, student 1.

The examples of coding from Table 1 illustrate how the coding scheme was implemented. In general, utterances in which PSTs stated or elaborated upon conjectures were coded as explaining. Utterances in which students compared their conjectures to the numerical growth of the pattern or corrected or affirmed each other's thinking (whether solicited or not) were coded as assessing. Utterances such as *right*, *yeah*, or *okay* were ambiguous in terms of whether PSTs were actually confirming their peer's thinking or using the word as a hedge. In these cases, the authors returned to the audio recording to base their decisions on the tone of voice and updated the transcript with a single period if the utterance seemed decisive or an ellipsis if the utterance was drawn out in a more questioning tone. While this process helped code these types of utterances with more confidence, the level of ambiguity that remained led the authors to look for other, more robust instances of assessing in order to include an utterance as part of the characterization of the group's work. Utterances that connected stated conjectures to a generalized sense of how the visual pattern grew were coded as justifying as well as utterances that discussed the meanings of

variables and algebraic terms. This latter example of justification was only coded in this way if the students provided evidence that they had generated these meanings for themselves. In Exchange 4 (coded as justification) in Table 1, the group is moving from discussing just the meaning of a single variable to how that variable operates within a particular term of their conjectured expression. In the first round of analysis, both authors read and discussed the transcripts to develop a portrait of each group's work. In the second round, authors separately applied the coding scheme and discussed any discrepancies in the codes.

Findings

The three cases presented came from three groups of PSTs in a single section of this course taught by a member of the research team who is not an author of this paper. Each of these three groups exercised a different overall pattern of Reinholz's (2012) three aspects mathematical authority in the creation of an expression to represent the growth of a linearly growing visual pattern. Examining the overall pattern of coding of a group's discussion helped the investigators address the research question; the aspects of mathematical authority that PSTs exercised in their group was a function of the level of external authority that they afforded to a previously remembered formula. It should be noted that there were multiple instances of students ceding mathematical authority to each other or developing a shared sense of authority when the entire group agreed with a particular conclusion. Although these instances were fascinating from the perspective of analyzing power relationships and intra-group student dynamics (see Langer-Osuna, 2016), the authors chose to attend to the group's *collaborative* exercise of mathematical authority.

Group One

The first group (made up of two students) started with Pattern II (see Figure 3).
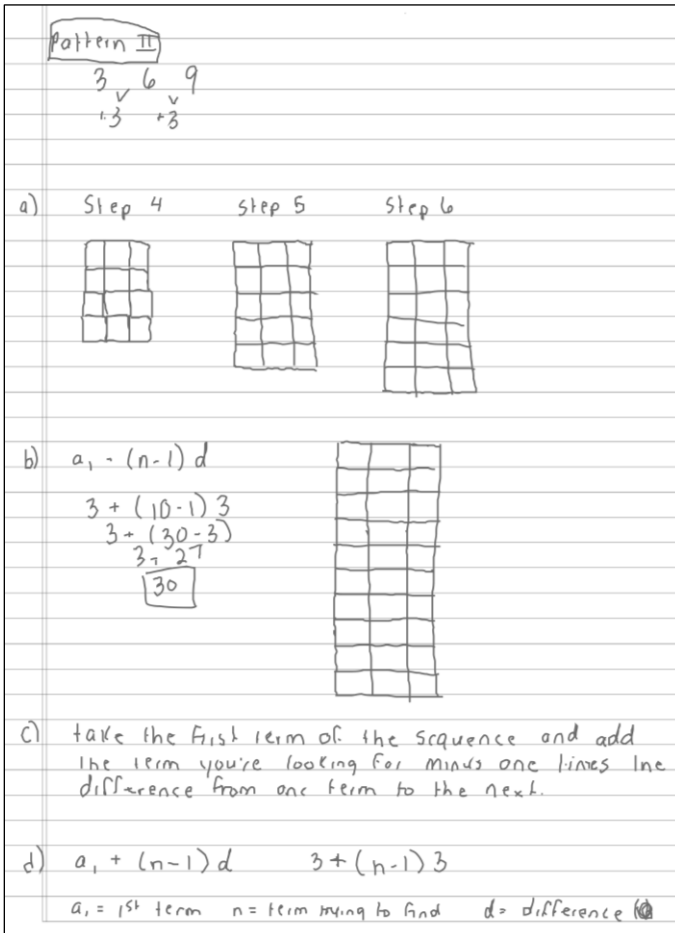
Figure 3. Student work from Group One.

PST 1.2: We need to draw the next 3 steps of the pattern.

PST 1.1: Okay. Then we will just add 3 each time. We are adding one more row on top, right?

PST 1.2: Yeah.

For the first pattern, PST 1.1 identified—and PST 1.2 confirmed—that the pattern was growing by 3 each time, and they connected that growth with the visual representation of the pattern ("We are adding one more row on top"). This

interchange started with an explanation, was followed by a justification that linked their expressions with the visual attributes of the pattern, and ended with an assessment, the last being an instance of peer-assessment when PST 1.1 asked, "Right?" and PST 1.2 confirmed. However, this instance was the only time when this group moved beyond explanation to provide justification or assessment. As they kept working on this pattern, PST 1.1 attempted to recall the formula for an arithmetic sequence to represent a linearly growing visual pattern.

> PST 1.1: Okay. Step 4 is 12 blocks, next is 15. How do you want to do this?
>
> PST 1.2: So now we draw the 10th step.
>
> PST 1.1: How do you want to do that? I don't remember the formula for it. [*pause*] I'm writing down the formula. [*writing*] Okay. So a_1 is 3, n is 10 and the difference, d, is 3. It's 30. Do you think that is right? 30. I don't know how many rows I'll need to have 30.

PST 1.1 opened another opportunity for peer-assessment by asking, "Do you think that is right?" but PST 1.2 did not respond to it, meaning that no such assessment occurred. However, this group did ascribe some meaning to the variables in the formula by identifying a_1 as the number of tiles in the first term, n as the step number, and d as the difference in number of tiles from one step to the next, but it is unclear if this is meaning that they have constructed for themselves or just a fact that they have looked up. Additionally, in the dialogue that immediately follows, the students interpreted the directive to "express [a general rule for the pattern] in words" as asking for a verbal description of their formula.

> PST 1.1: I think this might be right. [*Reads the next question.*] Should I just describe the formula now or what? Should I just describe the formula?
>
> PST 1.2: It says to explain it in words.
>
> PST 1.1: Yeah. So, I have to explain like, subtract 1 from the n. Yeah. We will just do that.

After this, the PSTs did not question or justify why the formula requires them to subtract 1 from n. Much of this group's remaining verbal dialogue could be characterized as descriptions; little of it was coded as any of the three aspects of mathematical authority. The only evidence of meaning-making that was found from the group was an explanation of what the variables represented in the arithmetic sequence formula.

Furthermore, we noticed that there were two occasions when the instructor prompted the group to justify their expressions. In the first occasion, the student's response directly appealed to the authority of the formula as her justification.

Instructor: Okay. So why do you take 1 out of the equation [referring to the term $n - 1$)]?

PST 1.1: That's how the formula works.

The instructor then left this group to circulate with the other groups. In the second exchange, when the group was working with Pattern I, the instructor returned and prompted the group more explicitly to justify the use of the $n - 1$ term in their expression.

Instructor: Why is it [n] minus 1 each time and not n?

PST 1.1: Because you have to subtract 1.

Instructor: That's right, but why?

PST 1.1: I don't know.

Instructor: What would happen if you have a n instead of a $n - 1$ there? What would that mean in the picture and pattern?

PST 1.1: It would be one term past what it needs to be.

Instructor: Why? Why can't we have it?

PST 1.1: Because you're not starting with 0.

Instructor: And you're not adding anything to pattern 1, right? That's what you meant by, "You're not starting with 0," right?

PST 1.1: Yeah.

This exchange highlights the PST's appeal to the external authority of the formula. When the instructor asked for justification, the PST implicitly presented the formula as justification, then admitted to not having a justification beyond that formula. PST 1.1 had made some meaning for the term $n - 1$ by realizing that replacing it with n would mean that this would offset the expression by one term. However, the PST did not refer back to the visual attributes of the pattern to provide any justifications for the group's expression; instead she simply validated the instructor's interpretation of the group's thinking. It is not clear that this final "yeah" is an instance of assessment since the PST could simply have been ceding her mathematical authority to the instructor and not assessing the accuracy of the instructor's statement. In other words, the instructor was right by virtue of being the instructor, not because the instructor's statement reflected a conclusion that the PST reached as a result of her own mathematical authority.

Group Two

The second group also started with Pattern II (see Figure 2). For this pattern, the group found that the number of tiles in each step was equal to the step number multiplied by 3. The group designated s to be their variable, and they generalized the statement to find the number of tiles in larger step numbers.

PST 2.3: For [the expression], I think it is just the number of the step times 3. [*pause*] Step 1 times 3 is 3. Step 2 times 3 is 6. Step 3 times 3 is 9. So, just think of the number of the step times 3.

PST 2.1: So, step number times 3?

. . .

PST 2.3: We can use s as our variable, and it can just be $3s$ if you want to find the step number. I don't know if we should define s as our step number?

PST 2.2: Yeah, we should. How should I? Should I just leave it like that or [*pause*]? Okay, $3s$.

PST 2.1: It would equal the number of tiles.

This dialogue was coded mostly as explanation. Later, when the instructor visited the group to check on their progress, she asked them to justify the expression they came up with for Pattern II.

> Instructor: Oh, you already finished the other one? Can I see what your [expression] looked like? [*pause*] Okay. So, explain to me a little bit why that worked. What is 3*s* about? Nothing is wrong, by the way.
>
> PST 2.1: So, like step 1 is 3 tiles, so I was thinking 1 times 3 is 3. So, this is step 2, so that is 2 times 3 equals 6, which is why there is 6 tiles in here. And step 3 is 3 times 3 equals 9.
>
> Instructor: So, you take step 40, what would you do?
>
> PST 2.1: 40 times 3.
>
> Instructor: So, what does *s* represent?
>
> PST 2.1: The step number.

PST 2.1's response to the instructor alternates between explanation and assessment, with the latter referring solely to the numerical growth of the pattern. In fact, the group's representation of the tenth step of the pattern shows that they did not attend to the visual attributes at all. They have the correct number of tiles (30), but the tiles are arranged in a 5 by 6 array instead of in 10 rows of 3 (see Figure 4). This group relied on their assessment of the numerical accuracy of their expression and expressed confidence in their expression based on that assessment. However, they did not engage in a full exercise of mathematical authority because they failed to justify their expression in a *mathematical* sense. Thus, while they may have had internally-situated confidence in their conjecture as a result of their assessment of their conjecture, they did not justify their conjecture in a way that would be considered valid in the broader discipline of mathematics.
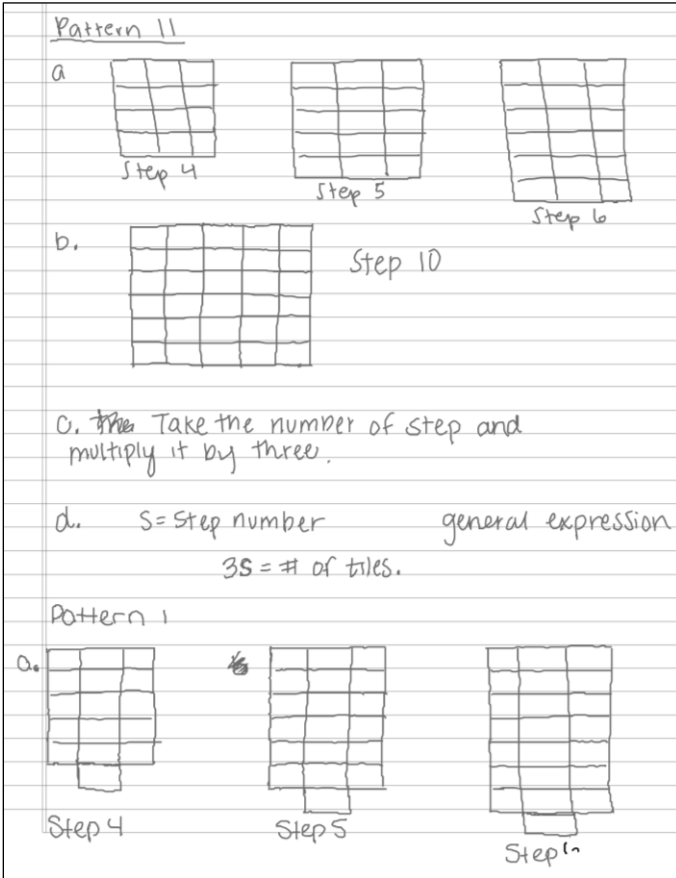
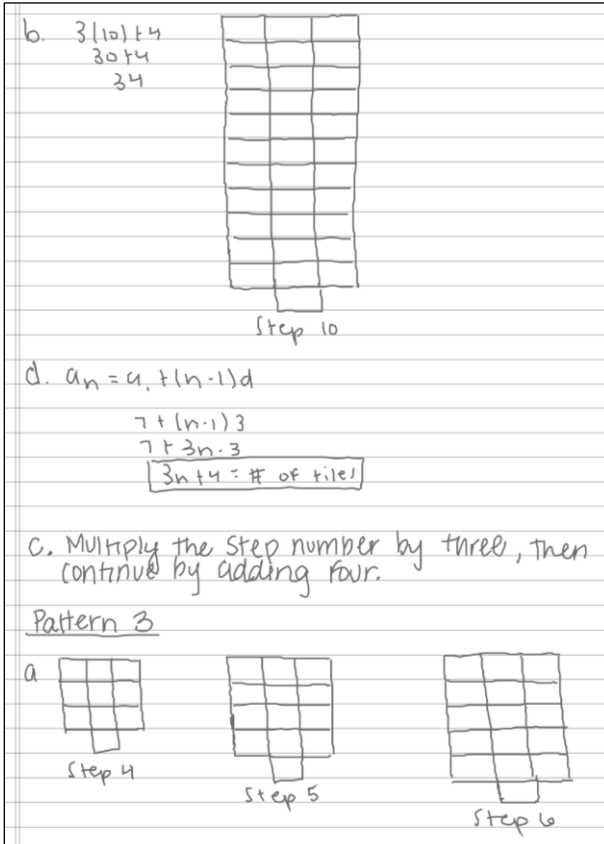*Figure 4*. Student work from Group Two (page 1 of 2).

Figure 4. (continued; page 2 of 2)

When they worked on Pattern I, the group developed a similar template for their discourse of alternating between explanation and justification. The following episode starts as they are discussing the number of tiles in the tenth step:

PST 2.1: Okay. So, then step 10. [*pause*] I feel like having to add 3 like to 10 is not the way to do it.

PST 2.3: I feel like there is a formula.

PST 2.1: I know there is.

. . .

PST 2.1: Yes. So, 4 times 2 is 8 plus 3.

PST 2.2: 11.

PST 2.1: 4 times 3 is 12 plus 3. Is that the right number?
PST 2.2: 13.
PST 2.3: What was the formula you wanted to use?
PST 2.1: Something like this [*writes the arithmetic sequence formula*].
PST 2.2: Okay.
PST 2.1: So, I was just trying to think of that. Because the difference is 4. We already know that. And then the first step is 7.

As shown in Figure 4, the group used the arithmetic sequence formula to build their expression for Pattern I and then went back to finalize their picture for the tenth step of the pattern. Their use of the expression to find the number of tiles in the tenth step can be seen in the top right corner of the page. As they continued discussing their expression for Pattern I, PST 2.2 made an appeal for justification of the group's expression.

PST 2.1: Thirteen. 16. 19. Oh because [*pause*]. Okay. So, $3n + 4$. Three times 3 is 9 plus 4 is 13. Three times 2 plus 4 is 10. That's it. We got it.
PST 2.2: Okay. So, yeah. Explain to me how you figured that out.
PST 2.1: Oh, I just did the homework and that's how I knew that.
PST 2.2: Oh. Okay.
PST 2.1: So, I was using this [formula], and this is the first number is 7 and it is $n - 1$ so it is whatever step you choose, and it is the difference and we already knew it was going up by 3 every time. So, then you just like 7 plus $3n$.
PST 2.2: Okay. That makes sense.

PST 2.1's response explains her use of the formula by describing what each variable represented. Because there was no connection between these representations and further meaning-making within the formula, we did not consider this an example of justification.

This group's investigation of Pattern III followed the scheme developed for their first two investigations (which was prompted by the structure of the task). Once the group developed a conjectured expression, they assessed its validity by referring strictly to the numerical growth of the pattern. As for the previous patterns, after the expression was explained and assessed, the group moved on.

Group Three

The third group is included in this paper to provide an example of a group that fully exercised their mathematical authority by displaying all three of Reinholz's (2012) mutually supportive skills. The PSTs evaluated the first pattern and tried to find different ways to represent the visual pattern's growth.

PST 3.3: So, any step of the pattern.
PST 3.2: Any step, you are adding 3 more blocks.
PST 3.1: Say that again.
PST 3.3: Yeah. For every previous step we add 3 to each one.
PST 3.1: Okay. So, we are adding 3 blocks to the next [*pause*].
PST 3.2: To the previous step.

Their initial numerical investigation led them to develop an iterative relationship between the steps of the pattern. Once the students realized that they needed to express the pattern using the step number (in comparison to the number of tiles in each step), they proceeded to investigate the pattern in a different way, thus shifting their discussion toward a general perspective of the pattern.

PST 3.1: Is there a way [*pause*] it says to, like, describe a rule [*pause*] so it's saying, like, saying or comparing to the step number not the previous step.
PST 3.3: For every step [*pause*] you are describing how to make any step, so you would say you are like

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constantly adding 3 to every step. It doesn't seem like you are adding 3, it seems like you are going backwards.

PST 3.2: So, if you say step 12, how many blocks are there? I should be able to immediately say that is 3 times 12 which is 36 plus 1 is 37.

PST 3.1: In other words, we are explaining an expression.

After the two instances of explanation from PSTs 3.3 and 3.2, PST 3.2 proposed a way to find the number of tiles for the step, which included a "deleted step."

PST 3.1: So, step 1 is your base. Step 2 starts going into the actual equation. So that would be 3 times [*pause*].

PST 3.2: That's where you would see the pattern form. Between step 1 and 2.

PST 3.1: So technically this is your step 1 [*points to step 2 on the paper*].

PST 3.3: Okay. Okay.

PST 3.1: It's like we are deleting step 1, but your step 2 would be step 1.

PST 3.2: So, step 3 we would take 3 times 2 plus 1. Step 4 we would take 3 times 3 plus one. So, you would just have to be one number behind. We can write that out. Verbally express that. And now we can say step 27 is like 3 times 26 plus one.

PST 3.1 &
PST 3.3: Yeah.

The group found that by finding the product of a number and the previous step number and adding 1 would give them the number of blocks for the current step. Note that the PSTs continually referred back to the visual pattern itself, not just the numerical pattern, even as they simply explained their reasoning. The group then assessed their expression and started to develop their justification for their expression.

PST 3.3: So, you can give an example to make sure they understand what we are saying. So, you can write out step 1, 2, 3, etc.

PST 3.1: Okay to find [*pause*] let's do [*pause*]

PST 3.2: They're going to be able to understand what we write.

PST 3.3: You can write it here. And draw a line.

PST 3.1: Do you want me to use a different step like step 6 or 7?

PST 3.3: It doesn't matter.

PST 3.1: I'm going to use step 7 since we don't have that visually. For step 7 [*pause*]

PST 3.3: We would do 3 times 6 plus one.

Their search for justification was motivated by the fact that they were expected to present their thinking to their peers and the instructor. Their plan for justifying their expression hinged on being able to draw a step of the pattern to illustrate each term of their expression, leading them to realize that instead of using n to represent the step number before the given term, they should incorporate the term $n - 1$ into their expression. They saw that this allowed them to use n to represent the step number of the current step, thus negotiating the meaning of the variable between themselves.

PST 3.1: So, wouldn't the expression be 3 times $n - 1$ plus 1?

PST 3.3: Oh, you could do it that way, but then it would cancel out.

PST 3.1: Let's say step number is 3 minus 1 and that would be 2 plus 1.

PST 3.3: It is the same thing.

PST 3.1: It is the same thing, but you could use the current step number.

PST 3.3: This would not confuse anyone.

PST 3.1: Here is our answer: 3 times $n - 1$ plus 1; n would be the step number. You are multiplying 3 and say we are multiplying step 7, so that would be 3 times 7 minus 1, 6, and then add 1.

Not only did this group explain their reasoning, assess their expression, and justify their conjectures, but they also switched between them quickly and often. In this way their full exercise of mathematical authority did not simply come from displaying all three of the necessary skills, but also from using them to mutually support each other, as Reinholz (2012) argued is necessary.

Discussion

Mathematics teachers and educators often emphasize the importance of students' making sense of formulas by deriving or otherwise justifying them (Blanton & Kaput, 2005; Boaler, 2015; Cuff, 1993). Students can confer mathematical authority onto a previously learned formula if they have not made sense of it for themselves. The cases discussed in this paper fall along a spectrum; each group exercised their own mathematical authority to a different extent, sometimes conferring externally on a previously learned formula and sometimes not. This can be seen from mapping their overall patterns of engagement with the task onto Reinholz's (2012) framework. Throughout most of their work, the first group only went as far as explaining their reasoning, with very little assessment or justification of their expression. The second group developed a predictable pattern of dialogue that alternated between explanation and assessment; that is, it was not enough for them to only form conjectures, but they also felt it was important to test them as well. In contrast, the third group decided to visually justify their expressions by referring back to the visual pattern. Moreover, the third group did not develop a predictable pattern of interaction; instead, they used each skill as necessary when prompted either by the task, by the implicitly defined expectations of the classroom, or by their own need to make meaning. In addition, these cases show that locating mathematical authority is more complex than identifying whether its situation is dichotomously internal or external and that the full exercise of authority can vary within the same classroom episode and with the same students.

In the first two cases, ceding mathematical authority to the external authority of a remembered, but not conceptually understood, formula preempted the groups' full exercise of mathematical authority. The second group's faith in the arithmetic sequence formula prevented them from reaching a more robust justification of their expression. By serving as the mechanism of their justification, the formula obviated the need for the first group to assess their expression numerically. Since the third group did not remember the arithmetic sequence formula nor did they recognize the discrete linear growth of the visual pattern, which might have prompted the use of $y = mx + b$ as a template for their expression, they had to make sense of every part of the expression they built. They did this by engaging with the visual attributes of the pattern more than the first two groups.

This study demonstrates that what can be understood as *mathematical authority* is co-constructed by both the student and the instructor (who functions in some ways as a representative of the formal the discipline of mathematics). For example, the PSTs in the second group had confidence in their conjectured expression and ascribed this confidence to the fact that the conjecture matched the numerical growth of the pattern. However, when the instructor prompted them to justify their work, their answer did not constitute a valid justification in the disciplinary sense. This begs the question of whether they can truly be considered to have developed a full sense of mathematical authority. Based on our interpretation of Reinholz's (2012) framework, we would argue that this is an incomplete sense of mathematical authority. Therefore, both words that make up the phrase "mathematical authority" are important: Not only must students develop an internal sense of authority, but they should also engage in explaining, assessing, and justifying in ways that are seen as mathematically valid.

The researchers' interpretation of the learning goal they had identified at the beginning of the task design sequence requires the task to prompt continued interaction with the visual aspects of the pattern in order to validly justify algebraic expressions that the PSTs developed. However, the unanticipated appearance of the formula for arithmetic sequences interfered with the

success of the task in prompting the type of reasoning that the researchers wanted students to practice. In order to address the fact that PSTs remembered the arithmetic sequence formula, the research team revised the task to have the following features: (a) Only one visual pattern in the task grows linearly (that is, in an arithmetic sequence), and this task is color-coded to encourage students to attend more closely to its visual attributes in their reasoning; (b) the task includes a multiple-choice portion in which multiple patterns grow according to a given expression, but only one matches an accompanying visual description; (c) a second growing visual pattern for which students need to create an expression grows quadratically, which means that PSTs cannot successfully use the arithmetic or geometric sequence formulas; and (d) a task is given where PSTs are asked to develop their own visual patterns based on a given linear expression, reversing the pattern of thinking prompted by the rest of the task. When completing the revised task, it is anticipated that PSTs will be less likely appeal to previously learned formulas (as it is very unlikely that they have learned to apply a quadratic equation in such contexts) and may instead be more likely to construct expressions based on their own reasoning.

Implications

All students, but most certainly PSTs, need to learn the metacognitive skills of understanding their own knowledge. The research group that developed the task (which included the course instructors) felt it was important to model the practice of leveraging and honoring students' funds of knowledge (Aguirre et al., 2013). However, MTEs need to attend to the ways in which PSTs' prior knowledge can interfere with their relearning of mathematical content for greater conceptual understanding (Zazkis, 2011). To PSTs, relying on their previous knowledge can look like using their own reasoning, but it might not be, as seen with Group 2's work. Thus, the onus is on MTEs to develop mathematical tasks that uncover and address PSTs' previous knowledge. Moreover, the instructors' expectation that PSTs engage in justifying their conjectures leads to the question of

whether these PSTs knew what it means to justify in mathematics.

For students to fully exercise mathematical authority, teachers need to provide opportunities for students to use all three mutually supportive skills. Intentional task design can help teachers provide these opportunities. Group-worthy mathematical tasks also allow teachers to step back and remove themselves as a potential source of mathematical authority upon which the students can rely (Lotan, 2003). However, no matter how carefully written a task is, student work on group tasks can be unpredictable. As evidenced by the cases presented, previously held procedural knowledge may interfere in the conceptual development of knowledge for a student. Thus, an iterative task design and revision model, such as the one proposed by Berk and Hiebert (2009) and followed by the researchers in this study, can help teachers attend to such unanticipated student responses and improve the tasks accordingly. Implementing this model follows the recommendations of Cai et al. (2017) in using data collected during classroom implementations to revise and assess tasks.

The dual importance of supporting the development of internal sites of mathematical authority in elementary PSTs—both for their sake as learners and as future teachers—makes it especially critical for mathematics educators' attention. PSTs enter teacher development with a great deal of prior knowledge, most of which has not been conceptually unpacked. This unpacked prior knowledge serves as its own force of mathematical authority, with students taking the results (e.g., formulas and procedures) as absolute truth. In such cases, it is vital that PSTs are given more opportunities to exercise their own mathematical authority, especially with respect to unpacking their prior knowledge. An interesting side note to the cases described in this study is that multiple groups in the class appealed to the arithmetic sequence formula, but only a few PSTs actually remembered it correctly. Most PSTs remembered that such a formula existed, but they needed to solicit help from the internet for its particulars. Thus, having seen and used the formula in a previous course only resulted in PSTs recognizing a situation in which the formula could be used.

For many students, this is perhaps the ideal outcome—they understand the correct contexts for the use of a formula and apply it correctly. However, PSTs need to understand how to make sense of the parts of the formula and be able to discuss why (in the case of this study) the common difference is being multiplied by $n - 1$ instead of n.

Recognizing PSTs' prior knowledge as a possible location of mathematical authority in the classroom is vital to the objectives of mathematics teacher education, as prior knowledge can disrupt the sense-making and justification expected of these future teachers in their mathematics content courses. Revising the task as the researchers did acknowledges the role PSTs' prior knowledge can play in the process of conceptually unpacking elementary mathematics content, allowing subsequent iterations of the task to require PSTs to rely largely on their own reasoning and less on external mathematical authorities. Attending to the location of mathematical authority in the classroom is therefore particularly important for mathematics teacher educators. When MTEs develop prospective elementary teachers' mathematical self-efficacy, those teachers gain confidence in their own mathematical abilities and an open mind toward children's mathematical thinking, thereby better equipping them to support their own students' learning.

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