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Making Sense of Geometry Education Through the Lens of Fundamental Ideas: An Analysis of Children's Drawings

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For many decades, the amount of geometry curriculum worldwide has been cut, mathematics curricula have lacked diversity of geometrical phenomena, and geometry teaching has been reduced to a somewhat eclectic mix of activities. Recently, new trends have begun to counteract these tendencies by framing new curricula around fundamental ideas. The goals of this paper are threefold: (a) to present the structural elements of a coherent geometry curriculum through the lens of fundamental ideas, (b) to develop an analytical tool to determine the fundamental ideas of geometry in children's drawings, and (c) to provide insight into the images primary grade students have of geometry. The results are discussed not only with regard to the latter of these goals, but also with regard to their theoretical and practical implications.

Geometry is one of the earliest established branches of mathematics; it went through a period of significant growth, particularly during the 19th and 20th centuries, becoming well-known for its internal diversity, coherence, and richness (Jones, 2000). Nonetheless, geometry education did not parallelly undergo the changes and growth in its content and structure. On the contrary, in the past several decades, geometry seems to have lost its position in school mathematics developing the reputation of being the "problem child" of mathematics teaching (Backe-Neuwald, 2000). At the same time the overall amount of geometry has been reduced in many national curricula (e.g., Backe-Neuwald, 2000; Glasnović Gracin & Kuzle, 2018; Mammana & Villani, 1998). Furthermore, some researchers

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(e.g., Franke & Reinhold, 2016; Mammana & Villani, 1998; Van de Walle & Lovin, 2006) have made an observation that many geometry curricula have been an eclectic mix of activities without a clear and systematic structure in curriculum, and curriculum focusing on learning terminology. Such trends affect a quality geometry curriculum as it provides the structure for the provision of quality teaching practices and students' learning.

Despite the acknowledged necessity of teaching geometry and its anchoring in the curricula (Franke & Reinhold, 2016; Mammana & Villani, 1998; Van de Walle & Lovin, 2006), there are still great differences in the actual implementation. Those discrepancies can still be found in the classrooms today. For instance, Willson (1977) observed "very wide differences of opinion about what is appropriate subject matter for school geometry and about how to approach it" (p. 19). Hansen (1998) suggested that the geometry curricula should encompass various geometrical phenomena, such as knowledge of plane and space, applications of geometry, presenting milestones in the development of geometry as well as strengthening logical thinking, and deductive reasoning. This diversity of topics in geometry curricula had been especially advocated during ICME-7 (1992) in Québec, which resulted in designing new curricula in many countries worldwide (e.g., Croatia, Germany, and the United States) that reflected the multi-dimensional view of geometry applied to all grade levels (e.g., Franke & Reinhold, 2016; Glasnović Gracin & Kuzle, 2018; Kuzle et al., 2018; Van de Walle & Lovin, 2006).

However, the diversity of topics still does not necessarily guarantee linking the learned objects. Hansen (1998) discussed the problem of lack of coherence in geometry curricula by listing the isolated fragments that are being taught in geometry classes: "small bits of polygon classification, some formulas to measure various shapes, some incidence geometry, a little mentioning of transformations, a few constructions, selected loci, introduction to vectors, and finally dome analytic geometry" (p. 238). Thus, geometry, as a mathematical discipline, offers huge opportunities for diversity and richness in its teaching programs, but these opportunities are still significant challenges to geometry education. The author concluded that in students' eyes the geometry they should learn might look as a "kind of inconsistent 'bazaar'" (Hansen, 1998, p. 238). However, these assumptions should be studied more in depth.

One of the trends to counteract the issues mentioned above focuses on the idea of a coherent geometry curriculum by framing it in terms of "overarching ideas" or *fundamental ideas* (e.g., Van de Walle & Lovin, 2006; Wittmann, 1999). The value of this idea resides upon having a coherent content framework, which is characterized by a high degree of inner richness of relationships, and by gradual and continuous development in every grade (Rezat et al., 2014; Van de Walle & Lovin, 2006). Consequently, having a coherent geometry framework makes it easier to do research that might answer the following questions:

- What geometrical concepts are being taught in geometry lessons nowadays, and to what extent?
- What meanings do students assign to geometry?
- How do these develop over the course of schooling?

The main goal of the inquiry presented in this paper was to provide insight into the images[1] primary grade students have of geometry by using participant-produced drawings. In order to achieve this goal, the study first sought to identify the fundamental ideas of geometry, and to develop an analytical tool to determine the fundamental ideas of geometry in students' drawings before focusing on students' images of geometry.

Theoretical Framework

In this section, we first present the construct of fundamental ideas and introduce different models of fundamental ideas of geometry, with a special focus on the model of Wittmann (1999). Mental images and image-based research using drawings are then discussed. The section ends with the three research questions that guided the study.

[1] Here, we do not refer to an ordinary informal meaning of the word "image." Moreover, we do not use the terms "image" and "drawing" as synonyms. The term image is defined later on in the paper.

Fundamental Ideas

As early as the late 1970s, researchers (e.g., Schweiger, 1992, 2010; Vollrath, 1978) were advocating structuring mathematics curriculum around fundamental ideas, sometimes called overarching ideas. For instance, Freudenthal (1973) claimed that "Our mathematical concepts, structures, and ideas have been invented as tools to organize the phenomena of the physical, social and mental world" (p. 41). This term can be interpreted in many different ways (e.g., Rezat et al., 2014). Winter (1976) defined fundamental ideas as ideas that have strong references to reality and can be used to create different aspects and approaches to mathematics. Schweiger (1992) defined a fundamental idea as a set of actions, strategies, or techniques that (a) can be found in the historical development of mathematics, (b) appears viable to structure curriculum vertically, (c) seems suitable to talk about mathematics, and answers the question what mathematics is, (d) makes mathematical teaching more flexible and transparent, and (e) possesses a corresponding linguistic or action-related archetype in everyday life. In addition, fundamental ideas are characterized by a high degree of inner richness of relationships, and by gradual and continuous development in every grade (e.g., Rezat et al., 2014; Van de Walle & Lovin, 2006). In other words, each fundamental idea represents an independent axis along which competencies build up in a cumulative way.

One of the trends counteracting the decrease in geometry in school mathematics, and the lack of both coherence and diversity of geometry topics in school mathematics focuses on the idea of structuring geometry curricula around fundamental ideas as a means of curriculum development (e.g., Mammana & Villani, 1998; Van de Walle & Lovin, 2006; Wittmann, 1999). For instance, Mammana and Villani (1998) listed several different overarching ideas for the geometry curriculum for the 21st century, such as the idea of measurement, mapping, projection and topology, the idea of geometric figures, simple motions, and transformations, and the idea of connections to arithmetic. *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), on the other hand, provided a content framework for geometry organized around shapes and properties, transformation, location, and visualization (Van de Walle & Lovin, 2006). Similarly, Wittmann (1999) proposed that school geometry be organized around the following seven fundamental ideas: (a) geometric forms and their construction, (b) operations with forms, (c) coordinates, (d) measurement, (e) patterns, (f) forms in the environment, and (g) geometrization (see Table 1). While the fundamental ideas F1–F3 and F6 are specifically assigned to geometry, the fundamental ideas F4, F5, and F7 are intended to illustrate the connection to the content area of measurement, algebra, and number and operations (Backe-Neuwald, 2000).

Table 1
Wittmann's Fundamental Ideas of Geometry

Fundamental idea	Description
F1: Geometric forms and their construction	The structural framework of elementary geometric forms is three-dimensional space, which is populated by forms of different dimensions: 0-dimensional points, 1-dimensional lines, 2-dimensional shapes, and 3-dimensional solids. Geometric forms can be constructed or produced in a variety of ways through which their properties are imprinted.
F2: Operations with forms	Geometric forms can be operated on; they can be shifted (e.g., translation, rotation, and mirroring), reduced or increased, projected onto a plane, shear mapped, distorted, split into parts, combined with other figures and shapes to form more complex figures, and superimposed. In doing so, it is necessary to investigate spatial relationships and properties changed by each manipulation.
F3: Coordinates	Coordinate systems can be introduced on lines, surfaces, and in space to describe the location of geometric forms with the help of coordinates. They also play an important role in the later representation of functions and in analytical geometry.
F4: Measurement	Each geometric form can be qualitatively and quantitatively described. Given units of measure, length, area or volume of geometric forms as well as angles can be measured. In addition, angle calculation, formulae for perimeter, area, and volume, and trigonometric formulae also deal with measurement.

F5: Patterns	In geometry, there are many possibilities to relate points, lines, shapes, solids, and their dimensions in such a way that geometric patterns emerge (e.g., frieze patterns).
F6: Forms in the environment	Real-world objects, operations on and with them as well as relations between them can be described with the help of geometric forms.
F7: Geometrization	Plane and spatial geometric facts, theorems, and problems, but also a plethora of relationships between numbers (e.g., triangular numbers) can be translated into the language of geometry and described geometrically, and then translated again into practical solutions. Here, graph theory and descriptive geometry (e.g., parallel projection) play an important role.

Wittmann's (1999) fundamental ideas of geometry are aligned with ICME-7 study recommendations for new geometry curricula (Mammana & Villani, 1998), which have been adopted by many national curricula. Although mathematics curricula worldwide have been reexamined due to various curricular reforms (e.g., Glasnović Gracin & Kuzle, 2018), it is not clear what influence this may have on images students have of geometry, and whether students recognize the multi-dimensionality of geometry and to what degree.

Drawings, Mental Images, and Image-Based Research Using Drawings

In image-based research, visual methods, such as drawings and photographs, are one of the crucial data collection tools. With visual methods—opposed to surveys and interview contexts which have shown not to be always reliable due to participants' young age (e.g., Einarsdóttir, 2007; Pehkonen et al., 2016)—participants can express things that cannot be easily verbalized (Hannula, 2007; Thomson, 2008), as visual representation requires little or no language mediation. In particular, drawings as a data tool in visual research have been recognized as an alternative form of expression for young students. Drawings can be understood as "visual data that can give insight into how children view things" (Einarsdóttir, 2007, p. 201). For young students, drawing is much more than a simple representation of what they see before them; rather, students use drawings—amongst other—"as a tool for understanding and representing important aspects of their own personal, lived experiences of people, places and things" (Anning & Ring, 2004, p. 26). Thus, drawings are not only effective because of the richness of produced data, but also because of the quality and uniqueness of the data providing a holistic insight into students' everyday lives, lived experiences and their conceptions of mathematics, and mathematics teaching (Einarsdóttir, 2007). Additionally, Kearney and Hyle (2004) found that participant-produced drawings appear to lead to a more succinct presentation of participant experiences, as they inhibit viewing drawings with adult eyes, and enable data triangulation. Still, a drawing as a graphic representation is a construction which cannot be mistaken for the real object, but rather stands for an aspect of reality (Golomb, 1994).

According to Luquet (1927/2001), an image ("internal model") is the starting point of drawing imitation. Here, the presence of a model cues the use of a child's internal model to produce a drawing. Furthermore, Luquet contends that the object of interest must necessarily pass through the mind in "the form of a visual image" before it can be translated onto a paper as a drawing. In other words, a drawing is an expression of the mental image. The extent to which it is similar or different from the external model condition indicates how influential the mental image is. A mental image can be defined in many different manners depending on the theory. In cognitive science, for instance, a mental image is defined as a representation of the physical world (e.g., an object, an event, or a situation) in a person's mind (Eysenck, 2012) whose features are spatially and temporarily organized (Kosslyn, 1988). From the perspective of the theory of imagery, mental images are short-term memory representations generated from long-term memory representations that may be stored in a depictive (pictorial) or propositional (symbolic, language-like) format, regardless of the content (Kosslyn, 1980; Pearson & Kosslyn, 2015). In this paper, the term "image" refers to mental representations of a cognitive structure associated with a particular concept (i.e., geometry), built up over the years through various experiences, which can be stored in both depictive and propositional format, and possess different functional characteristics.

In the last two decades, drawings have been successfully used to access students' beliefs about mathematics (e.g., Rolka & Halverscheid, 2006, 2011), the emotional atmosphere in mathematics lessons (e.g., Laine et al., 2013; Tuohilampi et al., 2016), and students' conceptions of mathematics lessons with respect to social and communicative aspects (e.g., Ahtee et al., 2016; Pehkonen et al., 2016). Only a few studies (e.g., Glasnović Gracin & Kuzle, 2018; Picker & Berry, 2000) focused on students' images of mathematical content, and mathematics teaching and learning. For instance, Glasnović Gracin and Kuzle (2018) conducted an explorative multiple case study with four students (one student per grade level from Grades 2–5) focusing on students' fundamental ideas of geometry using Wittmann's model (1999). The results showed that the four primary grade students mostly depicted the fundamental idea of geometric forms and their construction. Independent of the grade level a square, triangle, and circle disc were presented as the strongest representatives of geometric shapes. Three participants also illustrated several properties of geometric objects. In three cases, the idea of measurement (i.e., length of a line segment, perimeter, area, and volume) was also associated with the participants' image of geometry. The fundamental idea of operations with forms (specifically, line symmetry) as well as the fundamental idea of forms in the environment was depicted by one participant only. The fundamental ideas of patterns and coordinates were not present in the data. During the interview, one participant's drawing was shown to depict the idea of geometrization. Thus, the results of the multiple case study showed that the images the participants have of geometry are strongly related to the fundamental idea of geometric objects and their construction, while the fundamental ideas of operations with forms, coordinates, patterns, and geometrization were minimally represented, if at all. Glasnović Gracin and Kuzle (2018) also reported on the utility of Wittmann's model (1999) when analyzing fundamental ideas in the children's drawings. Though different subcategories of some fundamental ideas emerged, the sample was too small to develop a comprehensive analytical tool. Furthermore, the framework showed weaknesses with respect to clearly categorizing each drawn object to a specific fundamental idea, and to reflecting new developments in geometry curriculum. Thus, the general utility of the model as a research tool appeared to be insufficient with respect to gaining a thorough insight into images students have of geometry.

To summarize, drawings as a data tool in visual research have made an alternative and complementary contribution to conventional research approaches by providing researchers with a less invasive technique when working with young students (e.g., Einarsdóttir, 2007). They opened a nonverbal channel to students' images of mathematics, and mathematics teaching and learning (Ahtee et al., 2016; Glasnović Gracin & Kuzle, 2018) in a multi-dimensional and holistic manner. However, studies focusing on the mathematical content in general as well on a specific mathematical content, such as geometry, using drawings are limited.

Research Questions

In order to gain insight into young students' understanding of geometry, coherent and viable models and techniques are paramount. Wittmann's framework (1999) illuminated students' fundamental ideas of geometry on a global level (Glasnović Gracin & Kuzle, 2018). However, this kind of classification does not provide a comprehensive and thorough picture of students' images of geometry. What concepts are students relating to each fundamental idea of geometry, and to what extent? Thus, we first needed to create an approach to analyze young students' drawings in a comprehensive and holistic way. With this achieved, it is possible to address the question of students' images of geometry through the lens of fundamental ideas taking Wittmann's framework (1999) as a foundation, but at the same time expanding on it on the basis of both the students' data and literature. With these goals in mind, the following research questions guided the study:

1. How can an analytical tool be developed that would provide insight into students' images of geometry from the perspective of fundamental ideas of geometry?
2. What fundamental ideas of geometry can be seen in the primary Grade 3–6 students' drawings?
3. What similarities and differences in students' drawings exist among elementary Grades 3–6 from the perspective of fundamental ideas of geometry?

Method

Research Design and Subjects

For this study, an explorative qualitative research design using participant-produced drawings was chosen. The study participants were Grade 3 to 6 students. This age group was optimal for the purposes of the study as this is an important period for the development of geometric thinking (e.g., Mamanna & Villani, 1998; van Hiele, 1959/1984). In total 114 primary grade students[2] from multiple urban schools in the federal states of Berlin and Brandenburg (Germany) participated in the project (see Table 2). Typical case sampling as a type of purposive sampling was utilized as a way of collecting rich and in-depth data (Patton, 2002).

Table 2
Participant Sample

Grade	Participants
3	25
4	33
5	28
6	28

[2] In the federal states of Berlin and Brandenburg, primary education covers Grades 1 to 6.

Data Collection Instruments

The research data consisted of (a) audio data, (b) document review, and (c) a semi-structured interview. The audio data were comprised of the students' unprompted verbal reports during the drawing process, and prompted verbal reports after the drawing process. For the document review, an adaptation of the instrument from the work of Rolka and Halverscheid (2006, 2011) was used. It involved drawing an individual image of geometry. The students were given a blank piece of paper and instructed to draw their image of geometry. In addition, the students answered three questions, which were on the reverse side of the sheet:

- In what way is geometry present in your drawing?
- Why did you choose these elements in your drawing? Why did you choose this kind of representation?
- Is there anything you did not draw, but still want to say about geometry?

Depending on the age of the student, these questions were answered either orally or in written form. When answers were given orally, the student answers were audio-taped, otherwise the students wrote down their answers. After the student had finished drawing, the drawing was used as a catalyst for a semi-structured interview in accordance with participant-produced drawing methodology (Kearney & Hyle, 2004). Multiple data sources (i.e., data triangulation) were used to assess the consistency of the results and to increase the validity of the results (Patton, 2002).

Procedure and Data Analysis

The research data were collected in a one-to-one setting between a student and the first author of the paper. It was briefly explained to each student that we were interested in geometry. Each student was given a blank piece of A4 paper with the following assignment: "Imagine you are an artist. A good friend asks you what geometry is. Draw a picture in which you explain to him or her what geometry is for you. Be creative in your ideas." The students took as much time as needed, usually about 10 to 15 minutes. Afterwards, the student was asked to answer the questions on the reverse side of the sheet. If a student had difficulties reading the questions or writing his or her answers down, this was done by the researcher or the student answered them orally and the answers were audio-recorded. Lastly, the drawings were used as an entry to a semi-structured interview. Each student was asked to describe what he or she had drawn. This procedure gave each student the opportunity to frame their own experiences, and interpret their drawing. This last part lasted about 15 minutes in total.

The drawings were analyzed after all the data had been collected. The analysis of the drawings was understood as interpreting the meanings that the students had given to the situations and objects they had presented (Blumer, 1969). As suggested by Patton (2002), multiple stages of the analysis using an analytic approach were performed. In the first step, the first author of the paper and another expert in geometry focused on developing an inventory to determine the fundamental ideas of geometry in the students' drawings. This process contained the following steps: (a) transcribing audio data, (b) analysis of drawings with respect to Wittmann's (1999) model of fundamental ideas of geometry, (c) confirmation of the interpretation and coding of other conceptions included in the students' oral or written data, and interviews, and (d) developing subcategories for each fundamental idea by clustering similar concepts. The first researcher transcribed the audio data. We both analyzed the drawings separately using Wittmann's (1999) model (see Table 1). Wittmann's (1999) model provided descriptions of each fundamental idea as well as different aspects pertaining to each fundamental idea. Moreover, it offered specific examples that are typical for geometry lessons. This allowed us to assign a particular fundamental idea to items that were present in the students' data. However, taken the generality of the model—as reported by Glasnović Gracin and Kuzle (2018), we revised his framework by structuring and expanding it with the goal of developing a multi-faceted inventory. Concretely, each category as well as description of each fundamental idea of geometry was reexamined, refined, or expanded, if necessary, and subcategories of each fundamental idea were developed, refined and/or defined on the basis of students' data taking into account different expression forms, which allowed us to get a rich insight into images primary grade students have of geometry.

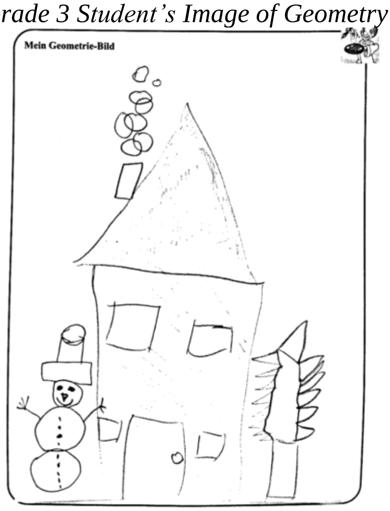
Specifically, we first assigned one of Wittmann's (1999) categories to each item taking into account any form of expression chosen by the child (i.e., drawing, written and/or oral data, or interviews). If a descriptor was not given, the researchers discussed the nature of the descriptor before assigning a particular fundamental idea to the item. The interrater reliability was high (97% agreement). Nevertheless, we discussed the differences in coding taking into consideration both the students' products as well as the mathematics curriculum (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015). In that manner, the fundamental idea descriptors were reexamined and refined. Adjustments were subsequently made to our coding, after which the interrater reliability was 100%. Afterwards, the same researchers focused on separately developing an inventory with subcategories for each fundamental idea by going through all the drawings starting with Grade 3 and ending with Grade 6. The inventories were discussed (89% agreement) to obtain full agreement. Concretely, the nature of each subcategory was discussed, which allowed to refine each subcategory descriptor, and new subcategories were developed on the basis of data to allow for a more fine-grained analysis of the data. Consequently, this allowed developing a very detailed and refined inventory to analyze students' fundamental ideas. All procedures and decisions were recorded in an audit trail, which also ensured trustworthiness and rigor (Patton, 2002). This procedure was used to answer the first research question.

To answer the second and third research questions, we used the developed inventory and coded the drawings once again. We assigned codes to each drawing separately using the inventory, followed by a discussion of the results. For the within analysis, each grade level was treated as a comprehensive case, whereas cross-analysis was used to compare the particular cases against each other. The interrater reliability was high (100% agreement). Thus, analyst triangulation contributed to the verification and validation of qualitative analysis (Creswell & Miller, 2000; Patton, 2002). Afterwards the descriptive statistics were calculated.

Figure 1 illustrates the coding. The drawing does not represent a prototypical drawing, but rather has been selected on the basis of data richness and versatility. In the description of the drawings we used the coding presented in the Appendix. For instance, F6 refers to the fundamental idea of geometric forms in the environment. Here, each real-world object was coded as a whole (F6). Given that three real-world objects (i.e., a snowman, a house, and a tree), F6 was coded three times. Additionally, the real-world objects are composed of 1- (F1b; e.g., curved and straight-line segments) and 2-dimensional figures (F1c; e.g., circles, squares, rectangles, and triangles), which reflect the fundamental idea of geometric forms and their construction (F1). If the same geometric object (e.g., squares in Figure 1) was drawn several times, it was coded once. Different 2-dimensional figures were coded once for each object. The number in brackets gives the absolute frequency of the category and the subcategory.

Figure 1

Grade 3 Student's Image of Geometry With Codes



The child drew three real-world objects, namely a house, a snowman, and a tree, consisting of different geometric forms.

Coding:
F1b: curved line segment; straight line segment
F1c: circle; square; rectangle; triangle
F6: snowman; house; tree

Summary of the coding:
F1(6): F1b(2), F1c(4)
F6(3)

Results

This section is divided into two parts. The first part focuses on the development of the analytical tool that would provide insight into students' images of geometry from the perspective of fundamental ideas of geometry. The second part focuses on the evaluation of the distribution of fundamental ideas in the learning groups by using drawings.

Fundamental Ideas of Geometry: An Analytical Tool

The inventory that emerged from the students' drawings, oral or written responses, and interviews is explained here (for more details see Appendix). The first category is related to Wittmann's fundamental idea of geometric forms and their construction (F1), which refers to both basic and composite figures of different dimensions, their properties, and their constructions. From the data nine subcategories emerged: 0-, 1-, 2-, 3-dimensional objects, geometric properties, drawing and drawing/construction tools, non-geometrical tools for creating geometrical objects, angles, and composite figures. All subcategories except for non-geometrical tools for creating geometrical objects and angles were listed in Wittmann's framework (1999).

When differing between 2- and 3-dimensional objects, other data (i.e., students' oral and written responses or data from the interviews) was needed. As shown in Figure 2, the student named each solid as well as surface shapes, whereas in Figure 3 depth of a rectangular prism was shown by using dashed lines. With respect to Figure 2 the student wrote: "Living and funny bodies are geometry for me. Spheres, cones, cubes, cylinders, and surfaces are represented." With respect to Figure 3 the student said:

> I drew a compass, a circle, a protractor, a ruler, a rectangular prism, and a cube, because I think that these things belong to a geometry lesson. When I think of geometry, I think of exactly these things and that is why I drew them.

Figure 2

Grade 4 Student's Drawing With 2- and 3-Dimensional Figures (3D Solids: Sphere, Cone, Cube, Cylinder; 2D Shapes: Square, Rectangle, Circle, Triangle)

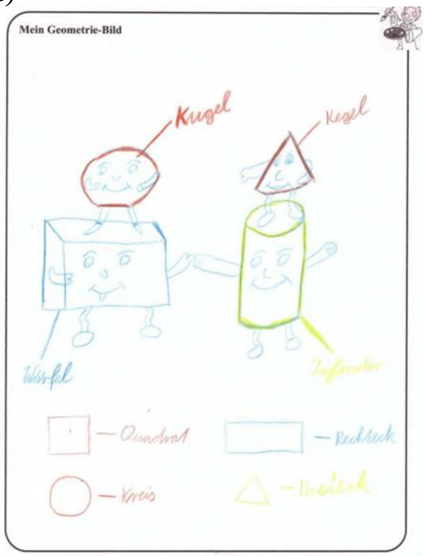
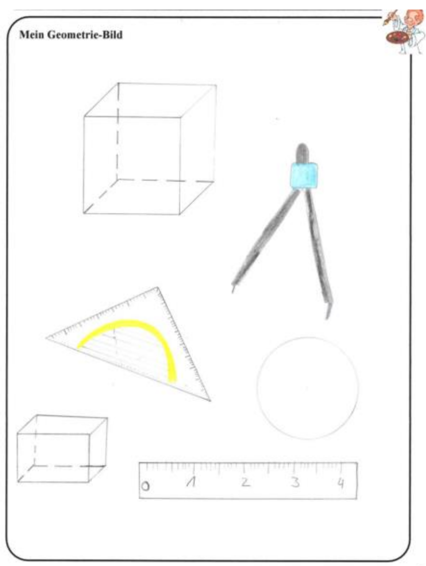


Figure 3

Grade 6 Student's Drawing With 2- and 3-Dimensional Figures and Drawing Tools



With respect to properties of geometric figures, the students either described the figure by writing down “A square has 4 right angles,” or four right angles were illustrated in the drawing of a square. When a student drew a geometric tool, it was important that the function of the tool has been explicitly mentioned or implicit from the data. With respect to the former, a student wrote, “In geometry we use a ruler, compass or protractor to draw figures,” or a student in the interview referred to a ruler as a drawing tool as opposed to a measurement tool, whilst with respect to the latter the tool was present in the figure with which the forms were drawn, and no aspects related to measurement were present (see Figure 3). Other than in Wittmann’s (1999) framework, the students also used non-geometrical tools (e.g., wooden shapes or modelling clay) to create composite figures by using different techniques (e.g., building or printing). Additionally, the first category was expanded by the “Angles” subcategory, as the students’ drawings, written data, or interviews revealed a figurative angle aspect (e.g., angle as a turn or angle as a wedge).

The second category is related to Wittmann’s (1999) fundamental idea of operations with forms (F2), which refers to different types of geometric mappings and manipulations with forms, and the properties which are influenced by these. From the data nine subcategories emerged as follows: translation, rotation, dilation, point symmetry, line symmetry, congruence, composing and decomposing, folding and unfolding, and tessellation (see Figures 4 and 5). All subcategories were consistent with Wittmann’s framework except for the last two subcategories, namely folding and unfolding, and tessellation. For instance, the student in Figure 5 said: “In geometry lessons we played with different figures, which have different symmetries. Here you can see a figure with rotational symmetry [points at pink ‘windmill’], and two figures with line symmetry [points at red circle and blue wind kite].” Even though the students rarely illustrated properties of a particular transformation, for the sake of completeness with respect to the mathematics curriculum for primary grades (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015) each subcategory of the inventory was expanded with respect to this aspect. The activities of folding and/or unfolding were most often supported by additional data (e.g., "We folded a paper into a Christmas star" as shown in Figure 4) or information (e.g., arrows).

Figure 4

Grade 5 Student's Drawing Illustrating the Activity of Folding ("Folding and Cutting-Out a Christmas Star")

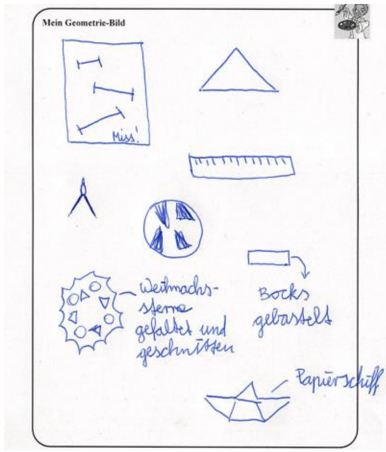
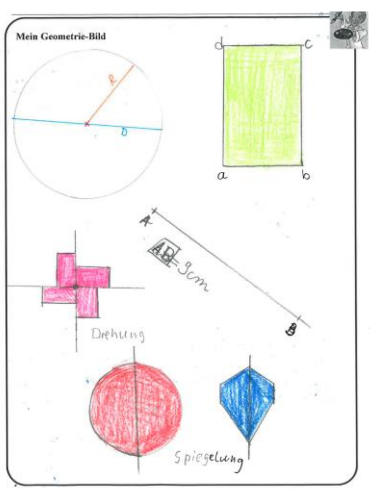


Figure 5

Grade 4 Student's Drawing of a Figure With Rotational Symmetry ("Drehung") and of Two Figures With Line Symmetry ("Spiegelung")



The third category is related to Wittmann's (1999) fundamental idea of coordinates (F3), which was broadened to reflect both curricular trends (e.g., Franke & Reinhold, 2016; Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015; Van de Walle & Lovin, 2006) as well as the students' drawings and interviews. The subcategories that emerged were as follows: coordinate system, positional relationships, orientation and orientation tools, and spatial visualization, relation, and orientation. In that manner, not only location but also position (e.g., above or below) and positional relationship of a geometric object or between geometric objects (e.g., a square lies right from a circle, two lines are parallel to each other as shown in Figure 6) in the plane or space was regarded, which reflected more young students' understanding of the fundamental idea. In addition, the drawings and, especially, interviews included activities that dealt with different aspects of spatial manipulation, such as making a view plan or a building plan of a geometrical composite figure or a cube building, folding a net of a solid mentally. With respect to the latter, one student wrote, "I have a lot of fun making nets of solids or cutting them out and then folding them into solids. But I could not draw that now," which was often reported by the students either in written or oral form. As such, we renamed the fundamental into "coordinates, spatial relationships, and reasoning" to allow for a broader understanding of the fundamental idea than given by Wittmann who limited this fundamental idea to describing location of geometric objects using different type of coordinate systems.

The fourth category is related to Wittmann's (1999) fundamental idea of measurement (F4), which refers to qualitative and quantitative properties used to describe geometric forms as well as calculations of these using different formulae. The subcategories that emerged were as follows: length, perimeter, surface area, volume, angle measurement, measuring tools, estimation, conversion of measuring units, and scaling. Whereas the first five subcategories were also part of Wittmann's framework, the subcategory "measuring tools" often emerged in the students' data (i.e., drawings, oral or written responses, or interviews). The last three subcategories, namely estimation, conversion of measuring units, and scaling were present in the data in a limited manner. Furthermore, these aspects are an important part of the mathematics curriculum (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015), and were likewise revealed in a similar study conducted by the authors. Hence, the inventory was expanded with respect to these three subcategories. In Figures 4, 5, and 6 different subcategories can be seen, namely the activity of measuring the length of line segments ("Miss!", $|\overline{AB}| = 9$ cm), and formula for the area of a right-angled triangle, respectively. Similar to our discussion earlier, when a child drew a geometric tool, it was important that the function of the tool had been explicitly mentioned by the student, either in their written responses or in the interviews (e.g., "I measured the length of a line segment with a ruler") or implicit from the data (e.g., length of a segment is measured which implies that a drawn tool is understood as a measuring tool).

Figure 6

Grade 6 Student's Drawing With a Parallel Projection of a Cube

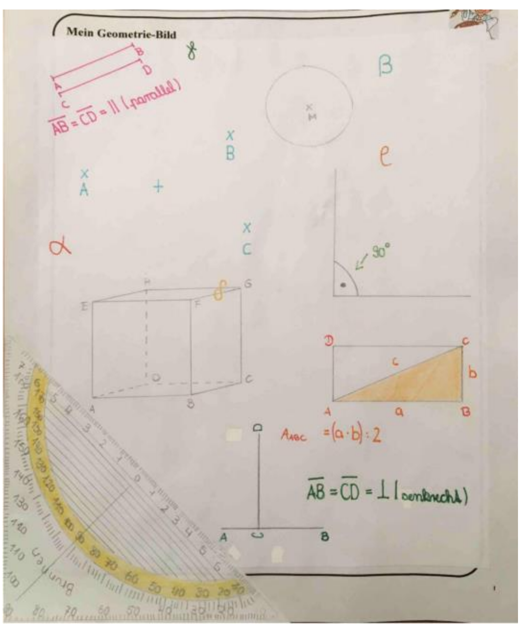
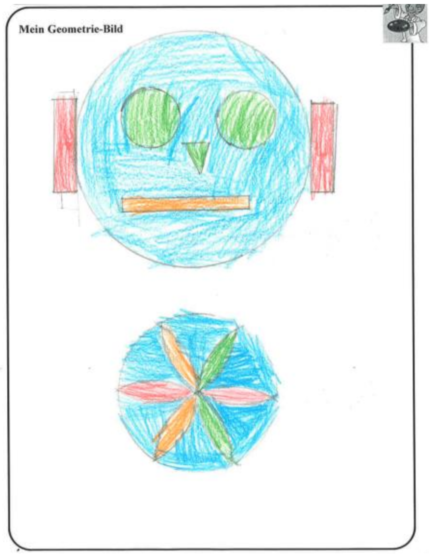


Figure 7
Grade 3 Student's Drawing of a Robot Head and a Six-Petal Rosette Pattern



The fifth category is related to Wittmann's fundamental idea of patterns (F5), which was renamed into "geometric patterns" since the data reflected patterns created by using simple geometric forms. For instance, a Grade 3 student said when asked to describe her drawing: "That in the middle is a pattern. It is made of a square, a triangle, and a circle. And then I repeated them creating a funny pattern." Even though different patterns were illustrated in the drawings, such as frieze pattern and six-petal rosette pattern, patterns formed of geometric shapes (see Figure 7), its small percentage did not allow creating different subcategories.

The sixth category is related to Wittmann's (1999) fundamental idea of forms in the environment (F6), which refers to the description of real-world objects, and operations on and with them by using simple geometric forms. In order to emphasize the core idea of this fundamental idea, we renamed it into "geometric forms in the environment." Figures 1 and 7 illustrate some of the motifs that could be seen in the students' drawings. The student drawing Figure 7 said, “Here is a robot head. It is made of different forms: circle for eyes and head, rectangle for ears and mouth, triangle for his nose.” Despite creative motifs in the students’ drawings (e.g., snowman, tree, house, robot, disco ball, and tent), written responses, oral data, or the interviews pertaining to this fundamental idea, the nature of the fundamental idea did not allow creating different subcategories. Often when students’ drawings included several motifs, the students mentioned other ones in their written responses or interviews.

Lastly, the seventh category is related to Wittmann’s (1999) fundamental idea of *geometrization* (F7), which refers to translation of geometric facts and problems into the language of geometry, their handling with the help of geometric approaches, followed by interpretation of the solution. The subcategories that emerged were as follows: geometric facts, parallel projection, and geometrical problems. In one case only, a geometric fact was revealed during the interview, whereas in all other cases the items were part of students’ drawings (i.e., in Figure 6). With respect to the former, a Grade 6 student said when asked if there is anything she did not draw but still want to say about geometry, “Probably the best-known construction is the construction of the Euler line. There the intersection points of the angle bisectors, the medians, the altitudes, and the side bisectors are located on a straight line.” For the sake of completeness with respect to the mathematics curriculum for primary grades (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015), the inventory includes the subcategory “figurate numbers“ (Wittmann, 1999), but excludes graph theory (Wittmann, 1999) as this is not part of the mathematics curriculum for primary grades (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015). In Figures 3 and 6 parallel projections of a cube and a rectangular prism are drawn.

Fundamental Ideas of Geometry in Primary Education Through Students’ Lenses: Similarities and Differences

Here, the focus was to evaluate the distribution of fundamental ideas by using participant-produced drawings on the basis of the developed inventory. As shown in Table 3, the fundamental idea of geometric forms and their construction (F1) was the most frequently coded fundamental idea of geometry (76.6%). This was independent of the grade level, where all the students' drawings included at least one aspect regarding this fundamental idea. The second most frequently coded fundamental idea was geometric forms in the environment (F6) with 8.7%. This was followed by the fundamental ideas of measurement (F4), and coordinates, spatial relationships, and reasoning (F3), with 4.8% and 4.2%, respectively. The fundamental ideas of operations with forms (F2), geometric patterns (F5), and geometrization (F7) were the three least coded fundamental ideas with 3.2%, 1.3%, and 1.2%, respectively.

With respect to the fundamental idea of geometric forms and their construction (F1), no increase is discernible (see Table 3), even though one might expect a more comprehensive picture from Grade 6 students than appeared in the data. According to the mathematics curriculum (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015), Grade 6 students have covered all of the subcategories listed in the inventory, and should have reached the level of informal deduction (van Hiele, 1959/1984). This was, however, not reflected in the data, since only Grade 4 and 5 students' data exhibited all of the aspects pertaining to F1 (see Table 4). Grade 6 students' data, on the other hand, revealed 7.3 codes on average pertaining to F1, whereas Grade 3 students 4.4 codes, Grade 4 students 6.4 codes, and Grade 5 students 5.8 codes on average. Thus, Grade 6 students' data revealed a deeper and more thorough insight into each subcategory's aspect.

Nonetheless, there were some patterns in the students' answers pertaining to different aspects of this fundamental idea. In all grades, different plane surfaces (F1c) dominated in the data with 36.5%, 38.7%, 42.5%, and 46.9% of codes pertaining to F1 in Grade 6, Grade 5, Grade 4, and Grade 3, respectively. F1c was an aspect mentioned by most students: 97 students (85.1%) gave answers pertaining to 2-dimensional figures (see Table 4). In each grade more than 76% of students mentioned this aspect independently, with a growing tendency from Grade 3 on. The second most often depicted aspect was solids (F1d), ranging

Table 3
Absolute and Relative Frequencies of Students' Fundamental Ideas of Geometry

Grade	*N*	Absolute and relative frequencies of fundamental ideas							Total
		F1	F2	F3	F4	F5	F6	F7	
3	25	111 (78.7%)	6 (4.3%)	9 (6.4%)	2 (1.4%)	2 (1.4%)	11 (7.8%)	0 (0%)	141
4	33	212 (78.8%)	6 (2.2%)	13 (4.8%)	10 (3.7%)	6 (2.3%)	22 (8.2%)	0 (0%)	269
5	28	163 (74.1%)	9 (4.1%)	5 (2.3%)	8 (3.6%)	3 (1.4%)	31 (14.1%)	1 (0.5%)	220
6	28	203 (75.2%)	8 (3%)	11 (4.1%)	23 (8.5%)	1 (0.4%)	14 (5.2%)	10 (3.7%)	270
Total	114	689 (76.6%)	29 (3.2%)	38 (4.2%)	43 (4.8%)	12 (1.3%)	78 (8.7%)	11 (1.2%)	900

Table 4
Subcategories of the Fundamental Idea "Geometric Forms and their Construction" Illustrated by the Most/Fewest Students

		Fundamental idea "geometric forms and their construction"								
Grade	*N*	F1a	F1b	F1c	F1d	F1e	F1f	F1g	F1h	F1i
3	25	1 (4%)	3 (12%)	19 (76%)	12 (48%)	0 (0%)	6 (24%)	3 (12%)	0 (0%)	4 (16%)
4	33	2 (6.1%)	8 (24.2%)	29 (87.9%)	19 (57.6%)	8 (24.2%)	17 (51.5%)	1 (3%)	4 (12.1%)	5 (15.2%)
5	28	2 (7.1%)	6 (21.4%)	24 (85.7%)	18 (64.3%)	6 (21.4%)	14 (50%)	4 (14.3%)	4 (14.3%)	4 (14.3%)
6	28	3 (10.7%)	9 (32.1%)	25 (89.3%)	21 (75%)	8 (28.6%)	20 (71.4%)	0 (0%)	16 (57.1%)	2 (7.1%)
Total	114	8 (7%)	26 (22.8%)	97 (85.1%)	70 (61.4%)	22 (19.3%)	57 (50%)	8 (7%)	24 (21.1%)	15 (13.2%)

from 21.2% (Grade 6) to 27% (Grade 3) of codes pertaining to F1. In total, 70 students (61.4%) illustrated or mentioned this aspect (see Table 4). As in the case of F1c, a growing tendency with respect to F1d was observed, and F1d was illustrated the most by Grade 6 students (75%; see Table 4). Various drawing tools (F1f; e.g., drawing stencil, ruler, protractor, or compass) were the third most frequently coded aspect of F1, ranging from 9% in Grade 3 to 16.6% of codes in Grade 5, and were illustrated or mentioned by every second student (50% of drawings; see Table 4). Likewise, a growing tendency from lower into higher grades was observed, as the topic of 1-dimensional objects becomes more important and diverse (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015).

The fourth most often illustrated aspect was 1-dimensional objects (F1b), which was present in 22.8% of drawings (n = 26; see Table 4). The students most often drew line segments, rays, and lines. In few cases, curved and broken lines were illustrated likewise. The data also reflected a growing tendency from lower into higher grades. Starting with Grade 4, the angle concept (F1h) was present in the drawings in a figurative manner. While Grade 4 and 5 students most often drew a right angle, Grade 6 students mostly drew an arbitrary angle. Furthermore, a growing tendency was observed from lower into higher grades, as the topic of angles becomes more important and diverse (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015).

Most notably, the students differed with respect to the properties of geometric forms (F1e). Though one may expect that properties of geometric forms gain the importance as primary grade students progress into higher grades, this was not reflected in the data. Concretely, this aspect was seen in all students' drawings besides in Grade 3 drawings with 7.6%, 9.2%, and 5.9% of codes pertaining to F1 in Grade 4, Grade 5, and Grade 6, respectively. From another perspective, almost every fourth Grade 4 (24.2%), almost every fifth Grade 5 (21.4%), and almost every third Grade 6 (28.6%) student illustrated or mentioned this aspect (see Table 4). Even though Grade 6 students did not exhibit most of the codes pertaining to

F1e, it was exhibited by the most Grade 6 students compared to other grade levels.

Subcategories F1i (composite figures), and F1a (0-dimensional objects) and F1g (non-geometrical tool for creating geometrical objects) were mentioned by the fewest students, namely by 13.2% (*n* = 15) and 7% (*n* = 8) of students, respectively (see Table 4), which are mostly dealt with in early grades of primary education (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015).

Fundamental idea “operation with forms” (F2) does not show an increase from Grades 3 to 6, as this fundamental idea was most frequently coded in Grade 3 (4.3%) and least coded in Grade 4 (2.2%; see Table 3). From another perspective, 24% of Grade 3, 15.2% of Grade 4, 32.1% of Grade 5, and 28.6% of Grade 6 students drew an aspect attributed to this fundamental idea (see Table 5). Line symmetry (F2e), and folding and unfolding (F2h) were two aspects mentioned by the most students with 12.3% (*n* = 14) and 7.9% (*n* = 9) of drawings, respectively (see Table 5). The former (F2e) was mainly present in Grade 3 (20% of drawings), whereas the latter (F2h) in Grade 5 (17.9% of drawings). Furthermore, both aspects were exhibited in the data regardless of the grade level. All other transformations were not mentioned very often (once or twice), or not at all. For instance, translation (F2a) and tessellation (F2i) were only present in one Grade 4 and point symmetry (F2d) in one Grade 6 students’ drawings each (see Table 5). Rotation (F2b) was mentioned both in Grade 5 and Grade 6 by one student each (see Table 5). No student drew an aspect pertaining to dilation (F2c), congruence (F2f), and composition and decomposition (F2g; see Table 5).

With respect to fundamental idea of coordinates, spatial relationships, and reasoning (F3), a decrease from the lower (6.4% in Grade 3) to the higher grades is observable (2.3% in Grade 5), but increasing again in Grade 6 (4.1%; see Table 3). Additionally, the drawings qualitatively differed. Lower grade students used prepositions only (e.g., right, left, or below) to describe the position of geometric forms (F3b), while upper grade students used in addition a coordinate system (F3a) for it, which is aligned with the mathematics curriculum

Table 5

Subcategories of the Fundamental Idea "Operations With Forms" Illustrated by the Most/Fewest Students

Grade	*N*	Fundamental idea "operations with forms"								
		F2a	F2b	F2c	F2d	F2e	F2f	F2g	F2h	F2i
3	25	0	0	0	0	5	0	0	1	0
		(0%)	(0%)	(0%)	(0%)	(20%)	(0%)	(0%)	(4%)	(0%)
4	33	1	0	0	0	2	0	0	1	1
		(3%)	(0%)	(0%)	(0%)	(6.1%)	(0%)	(0%)	(3%)	(3%)
5	28	0	1	0	0	3	0	0	5	0
		(0%)	(3.6%)	(0%)	(0%)	(10.7%)	(0%)	(0%)	(17.9%)	(0%)
6	28	0	1	0	1	4	0	0	2	0
		(0%)	(3.6%)	(0%)	(3.6%)	(14.3%)	(0%)	(0%)	(7.1%)	(0%)
Total	114	1	2	0	1	14	0	0	9	1
		(0.9%)	(1.8%)	(0%)	(0.9%)	(12.3%)	(0%)	(0%)	(7.9%)	(0.9%)

(Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015). Furthermore, regardless of the grade level, using prepositions to describe the position of geometric forms (F3b) was the most often coded aspect pertaining to this fundamental idea. Very few students mentioned an aspect pertaining to spatial visualization, relation, and orientation (F3d), which was only seen in Grade 4 students' drawings. A tool for orientation (F3c; i.e., compass rose) was drawn by one Grade 6 student.

With respect to fundamental idea of measurement (F4), the students' drawings show an increase of codes from the lower grades (1.4%) to the higher grades (8.5%; see Table 3). All aspects of this fundamental idea were exhibited (see Table 6). In Grade 3, only length (F4a) and estimation (F4g) were addressed, and each by one student only (4%; see Table 6). In Grade 4, in addition to length (F4a), which was illustrated by four students (12.1%), three other aspects appeared in the data, namely angle measurement (F4e; 3%), measuring tools (F4f; 9.1%), and scaling (F4i; 3; see Table 6). These were, however, illustrated by a few students. Similarly, in Grade 5 in addition to length (F4a; 7.1% of drawings) and measuring tools (F4f; 3.6% of drawings), perimeter (F4b) and surface area (F4c) were illustrated in 7.1% and 10.7% of drawings, respectively (see Table 6). Lastly, Grade 6 students' drawings depicted seven out of nine different measurement aspects. Only estimation (F4g) and scaling (F4i) were not present in the data. Here, measuring tools (F4f) were illustrated by most students (25%; see Table 6). Additionally, F4f was dominant in the students' drawings with 30.4% of all measurement codes. Whilst in earlier grades a protractor was presented as a tool for measuring lengths, in Grade 6 the protractor was assigned another role, namely as a tool to measure angles. Furthermore, perimeter (F4b), and (surface) area (F4c) were only present in Grade 5 and 6 students' drawings, whereas volume (F4d), and conversion of measuring units (F4h) in Grade 6 students' drawings only. Thus, a more comprehensive picture of this fundamental idea appeared in the data as students progressed from lower to higher grades and different aspects win on their relevance (see Table 6). Most notably was the length aspect (F4a), which was seen in all drawings independent of the grade level, and together with measuring tools (F4f) the most

Table 6

Subcategories of the Fundamental Idea "Measurement" Illustrated by the Most/Fewest Students

Grade	*N*	Fundamental idea "measurement"								
		F4a	F4b	F4c	F4d	F4e	F4f	F4g	F4h	F4i
3	25	1 (4%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	1 (4%)	0 (0%)	0 (0%)
4	33	4 (12.1%)	0 (0%)	0 (0%)	0 (0%)	1 (3%)	3 (9.1%)	0 (0%)	0 (0%)	1 (3%)
5	28	2 (7.1%)	2 (7.1%)	3 (10.7%)	0 (0%)	0 (0%)	1 (3.6%)	0 (0%)	0 (0%)	0 (0%)
6	28	4 (14.3%)	2 (7.1%)	4 (14.3%)	2 (7.1%)	3 (10.7%)	7 (25%)	0 (0%)	1 (3.6%)	0 (0%)
Total	114	11 (9.7%)	4 (3.5%)	7 (6.1%)	2 (1.8%)	4 (3.5%)	11 (9.7%)	1 (0.9%)	1 (0.9%)	1 (0.9%)

dominant aspect of measurement idea (9.7% of all drawings). Three measurement aspects, namely estimation (F4g), conversion of measuring units (F4h), and scaling (F4i) were the least coded aspect with 0.9% of all drawings (*n* = 1; see Table 6).

The fundamental idea of geometric patterns (F5) was the second least coded fundamental idea with 1.3% of all codes (see Table 3). Thus, very few students think of this fundamental idea when thinking about geometry. Apart from Grade 4 and Grade 5, where four and three students' drawings or written data, respectively, revealed geometric patterns, only one student in Grade 3 and Grade 6 depicted this aspect. In these instances, different patterns were drawn, such as patterns using basic geometric forms (Grades 3 and 4), frieze patterns (Grade 5), and the six-petal rosette pattern (Grade 6).

As illustrated in Table 3, fundamental idea "geometric forms in the environment" (F6) was the second most often coded fundamental idea. Data revealed an increase from Grade 3 to Grade 5 (from 7.8% to 14.1%), but a decrease in Grade 6 (5.2%). Something pertaining to F6 was illustrated by almost every fourth Grade 3 student (24% of drawings), almost every fifth Grade 4 student (21.2% of drawings), almost every second Grade 5 student (42.9% of drawings), and every fourth Grade 6 student (25% of drawings).

Geometrization (F7) refers to the most abstract fundamental idea, which may explain the small number of codes (1.2%) assigned to it as well as no codes in Grades 3 and 4 (see Table 3). Yet, an increase from the lower to the higher grades is evident, reaching a maximum of 3.7% codes in Grade 6 (see Table 3). Here, all subcategories were elicited apart from figurate numbers (F7d). In Grade 5, one aspect pertaining to F7 was elicited, namely geometrical facts (F7a). Concretely, one student illustrated the sum of the interior angles of a triangle. Drawings and written data of six Grade 6 students (21.4%) showed three different aspects: geometrical facts (F7a), specifically sum of interior angles of a triangle, Euler's line, triangle congruence theorems; parallel projection of a cube and a rectangular prism (F7b); and geometrical problems concerning angle measurements (F7c).

Discussion and Conclusions

In the last section, the key aspects of geometry education through the lens of fundamental ideas we proposed are discussed. Lastly, the limitations of the study are considered, and some possible future research directions are provided.

Educational Classroom Practices in Primary Grade Geometry

In our study, we used participant-produced drawings as a data source for researching primary grade students' images of geometry. We framed our study around fundamental ideas, which have been advocated by many researchers as a means for curriculum development (e.g., Mammana & Villani, 1998; Rezat et al., 2014; Schweiger, 1992, 2000; Van de Walle & Lovin, 2006; Wittmann, 1999). As it was not obvious whether Wittmann's framework worked for the approach of using participant-produced drawings, in the first step we were concerned with clarifying whether and how this framework can be understood in this context. Since the framework turned out to be suitable, it was used as a basis for developing a multi-faceted inventory which both refined and expanded Wittmann's (1999) theoretical framework of fundamental ideas of geometry. Concretely, on the basis of produced data, we developed subcategories of each fundamental idea illustrating its different aspects in order to get a more detailed and rich insight into current educational practices in primary school geometry. Also, we took different expression forms into consideration. The developed inventory was then used for classifying the students' images of geometry encoded in the participant-produced drawings.

Independent of the grade level, the fundamental idea of geometric forms and their construction (F1) dominated in students' drawings. This focus is not surprising as this fundamental idea predominates throughout the mathematics curriculum (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015). Moreover, there was no noticeable increase from Grade 3 to Grade 6. This is possibly due to the fact that geometric forms are already covered before Grade 3. Naturally, with each grade level, students learn new geometric shapes and solids, and their properties; however, all Grade 3 students were able to make statements in this area. Though properties of 2- and 3-dimensional objects are already covered in the first two grades of primary education, the data of Grade 3 students did not reflect this (see Table 4). This may be due to the limited linguistic abilities of young students. Nevertheless, it is surprising that students mainly associated geometric forms with plane surfaces and solids (see Table 4), even though 0- (F1a) and 1-dimensional objects (F1b) are covered in each grade in the mathematics curriculum (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015). The results showed that these aspects increased from lower to higher grades. This might mean that with time students associate geometry with 2- and 3-dimensional forms, which may be due to the fact that (surface) area and volume calculations are added to the measurement of distances in the higher grades (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015). Additionally, great attention is given to 2- and 3-dimensional forms in the mathematics curriculum (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015), and students develop different ideas of these forms in every grade. Hence, the existing mathematics curriculum may be crucial in developing learners' understanding of geometry and the geometrical concepts.

Interestingly, students associated geometry more with geometric forms in the environment (F6; 8.7% of codes), which is addressed only once per grade level in the curriculum, than with measurement (F4; 4.8% of codes; see Table 3), which dominates throughout the curriculum (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015) as it is a separate mathematics standard. An initial increase from Grade 3 to Grade 4 was expected as this content is explicitly dealt with in Grades 1 to 4. In Grade 5, this content was still highly present, even though this fundamental idea is no longer primarily part of the curriculum (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015). It may be that this content was carried over from Grade 4 or was covered in Grade 5, and thus still present. Similarly, in Grade 6 this content is no longer primarily part of the curriculum (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015), which may explain its low frequency (see Table 3). Rather, in Grade 6 the focus shifts onto a more deductive approach to geometry. This may also explain an increase in drawings addressing the fundamental idea of geometrization (F7), especially with regard to geometrical facts (F7a), parallel projection (F7b), and geometrical problems (F7c). Considering that this fundamental idea is relatively well represented in the curriculum in the upper grades (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015), only 3.6% of Grade 5 (n = 1) and 21.4% of Grade 6 students (n = 6) mentioned or illustrated this aspect. This may be due to teachers who perceive geometry rather as entertainment than important mathematical content (Backe-Neuwald, 2000). In that manner, the development of deductive and logical thinking plays a subsidiary role.

With respect to the fundamental idea of measurement (F4), an increase from lower to higher grades was observable, reaching its peak in Grade 6 (see Table 3). This may be due to the fact that in the higher grades (surface) area (F4c), volume calculations (F4d), and angle measurement (F4e) are added to the measurement of lengths (F4a; Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015). Even though Grade 6 students' drawings depicted almost all geometry measures, on average just one aspect was drawn per student. Only four drawings depicted three or more different measures. Since this fundamental idea illustrates the connection between geometry and number and operations, it may be that not many students perceived this fundamental idea as a part of geometry or were not sure if that was the case. The fundamental idea of coordinates, spatial relationships, and reasoning (F3) was not frequently found in the students' drawings, even though this topic and its different aspects are well-covered in the mathematics curriculum (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015), and are recognized as one of the most important goals of school geometry (Franke & Reinhold, 2016; Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015; Van de Walle & Lovin, 2006). Furthermore, it is very surprising that this content was primarily addressed by

Grade 3 students at a basic level by using prepositions (F3b), even though in Grades 5 and 6 the topic of coordinate systems (F3a), and spatial visualization, relation, and orientation (F3d) are intensively covered. However, there was no increase in Grades 5 and 6. The low results with respect to the fundamental idea of geometric patterns (F5) may suggest that this content is either rarely discussed (Backe-Neuwald, 2000) or does not seem to be directly linked to geometry lessons, but rather to algebra lessons. As a consequence, the students might not have established the connection between geometry and algebra, which is the core idea of this fundamental idea.

Similar to the results of Glasnović Gracin and Kuzle (2018), this study shows that primary grade students' drawings revealed a relatively narrow understanding of geometry with respect to the diversity of fundamental ideas. Concretely, the majority of the students drew aspects pertaining to either one (n = 37, 32.5%) or two fundamental ideas (n = 48, 42.1%). Only rarely did students' drawings present an image containing three or more fundamental ideas of geometry (three ideas, n = 17; four, n = 11; five, n = 1). Although all of the fundamental ideas were depicted in the students' drawings or mentioned in other data sources, the fundamental ideas of geometric objects and their construction (F1), and geometric forms in the environment (F6) were most frequently exhibited. These, however, are just two of the fundamental ideas, and solely focusing on them may result in students developing a narrow understanding of geometry, instead of facilitating the diversity and richness geometry has to offer (Hansen, 1998). Also, placing little or no emphasis on fundamental ideas (i.e., F4, F5, and F7) that connect geometry to other content areas (i.e., measurement, algebra, number, and operations) will resolve in developing a fragmented understanding of geometry.

Limitations of the Study and Future Research Directions

This study was an exploratory qualitative study using purposive sampling. A sample of 114 cases was used, but the results may be limited to the curriculum of two German federal states (i.e., Berlin and Brandenburg), and for that reason may not be widely generalized. These limitations suggest a possible next step in research, namely to conduct a study with a larger data sample in a wider variety of settings (e.g., federal states or countries), so that a researcher could create a more thorough description of the images students have of geometry. In addition, drawings from entire classrooms across different grades and schools may reveal a more complete picture of primary grade students' images of geometry. This would in addition allow for comparisons between different grades and schools. Also, we cannot assume that the drawings offered a complete picture of the development of reasoning ability, so that in future studies connection to van Hiele (1959/1984) levels could be explored. Moreover, a longitudinal study would show whether students' images of geometry change over time and how. Lastly, the study design does not allow us to make direct inferences between students' images of geometry and their classroom practices. This may be a path to explore in our future work by using other data sources, such as observations of geometry lessons. This would not only give researchers a better insight into current educational practices in geometry, but would also provide practitioners a window into their students' thinking and learning (e.g., Anning, 1997; Pehkonen et al., 2016), providing teachers with ideas for modifying their teaching practices with respect to the multi-dimensionality of geometry. Future studies could also evaluate the possibilities for classroom implementation of the inventory, and the practicability of it as a classroom-tool for discussing images of geometry.

Drawings and the processes by which they are made have opened up a new way of gaining insight into students' cognitive processes pertaining to geometry. Nevertheless, there were some drawbacks: some students had difficulties drawing, some did not like to draw, some drew the objects which they found easy to illustrate, and some aspects can be expressed by drawing in a limited way. Concretely, students most often expanded on their image of geometry pertaining to 3-dimensional figures (F1d), geometric properties (F1e), and drawing/construction tools (F1f) in the semi-structured interview, as they found those aspects hard to draw. It is certainly plausible that the students have knowledge of properties of geometric figures which was not elicited in their drawings. Additionally, aspects pertaining to operations with forms (F2), measurement (F4), and geometrization (F7) also proved to be hard to draw. This may also explain very few or no codes are pertaining to different aspects of these fundamental ideas (e.g., F2c, F2f, F2g, F4a, F4b, F4c, F4d, F4g, F7c, and F7d). Here again, additional data sources (e.g., written questions and a semi-structured interview) were necessary. Despite the inventory, the analysis of the drawings has proven to be a challenging task. As Blumer (1969) noted, the analysis of drawings is understood as interpreting the meanings that the students had given to the situations and objects they had presented. Thus, in order to avoid the coder's own interpretation, not only analyst triangulation is needed, but also methodological triangulation such as participant-produced drawings (Kearney & Hyle, 2004), allowing each student to interpret his or her own drawing, which consequently allowed an in-depth understanding of what the student had drawn.

By relating the study results to teaching practice, some implications for geometry teaching can be drawn. In terms of Brunner's spiral curriculum, it seems to make sense to build the children's knowledge successively. It is important to pick up the children from where they stand. The framework curriculum can be an orientation for this (Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, 2015). In addition, the school's internal curriculum may be used to help plan lessons. Furthermore, it may be concluded that the fundamental ideas of geometry that occurred less frequently have also played a subordinate role in classroom instruction. Consequently, since teachers are the most significant influencing factor in students' learning of geometry, their attitude and willingness to teach determine the development of students' content-related and process-related competencies. Further training courses could remedy a lack of didactic knowledge and ensure professional confidence in teaching.

Last but not least, we strongly believe that the new framework of fundamental ideas of geometry presented in this article, and the method, namely drawings, employed in this research, will provide a basis not only for further study of students' images of geometry, but also impact educational classroom practices in school geometry. With the help of the inventory, both researchers and practitioners have the possibility of identifying practices in geometry as seen through their students' lenses. Thus, the inventory may be used as a classroom tool for discussing students' learning in the context of geometry lessons. In that manner, the tool could make students' images of geometry more visible, allowing students as well as teachers to gain insight into geometry thinking in the classroom. If children are allowed to draw, it makes sense to talk to them afterwards (e.g., narrative interviews [Krüger, 2006]) to get an insight into their thinking. On the other hand, it may provide teachers a window into their teaching to see through their students' eyes. Especially the fundamental ideas or the subcategories of these that were not often illustrated by the students, but are part of the mathematics curriculum and were taught by the teacher, may provide the teachers with paramount feedback (e.g., paying more attention to the idea in question, revising the content) and allow the teacher to reflect on his teaching practices (e.g., Why did not the students perceive the idea in question as important?). As such, students' drawings and their interpretations of drawings are productive ways of promoting dialogue about learning between young people and their teachers (Anning, 1997).

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Appendix

Analytical Tool for Analyzing the Fundamental Ideas of Geometry

Code	Title	Comments and/or examples
F1	**Geometric forms and their construction**	**Basic and composite geometric forms of different dimensions, their properties and construction/creation fall into this category.**
F1a	0-dimensional objects	A point as a separate object has been drawn. It can be, but it does not need to be, labeled.
F1b	1-dimensional objects	A segment, a ray and/or a line have been drawn as a separate object or written down. The object can be, but it does not need to be, labeled.
F1c	2-dimensional objects	A geometrical shape (e.g., square, circle, rectangle, triangle) has been drawn as a separate object or written down. The object can be, but it does not need to be, labeled. Other data (e.g., written, oral) is needed to confirm that the child did not draw a 3-dimensional object in 2-D.
F1d	3-dimensional objects	A geometrical solid (e.g., cube, pyramid) has been drawn as a separate object or written down. The object can be, but it does not need to be labeled. The object can also be drawn as a 2-dimensional object. Here, either shading or written/oral data confirms classification.
F1e	geometric properties	A property of a geometrical object is described or illustrated in the drawing. For instance, a child wrote “A square has 4 right angles” or four right angles are illustrated in the drawing of a square.
F1f	drawing and drawing/ construction tools	Drawing/constructing as an activity was mentioned. A drawing/construction tool (e.g., ruler, protractor, compass) has been drawn. The function of the tool has to be explicitly mentioned. E.g., a ruler is explained as a drawing tool rather than as a measuring tool.

F1g	non-geometrical tools for creating geometrical objects	Material, such as inchworms, wooden 3-dimensional shapes, modeling clay, is illustrated or mentioned as a way of creating geometric objects.
F1h	angles	Angle stands for planar objects that can be created by means of concrete representations. The figurative angle aspect is emphasized here. For example, angle as a wedge, angle as a turn.
F1i	composite figures	A composite figure without any reference to real word object(s) (e.g., cube building, net of a cube) made out geometry manipulatives (e.g., wooden cubes, polydron material) or material (e.g., modeling clay) is illustrated or mentioned. Here different techniques are possible (e.g., building, kneading, covering, printing).
F2	**Operations with forms**	**Geometric mappings and other manipulations with forms, and the properties influenced or changed by these, fall into this category.**
F2a	translation	A figure with translational symmetry or a translation of an object with a translational vector is drawn. Properties of translation are documented.
F2b	rotation	A figure with rotational symmetry or a rotation of an object with an angle and a point of rotation is drawn. Properties of rotation are documented.
F2c	dilation	A given geometrical object is either enlarged or compressed. Properties of dilation are documented.
F2d	point symmetry	A figure with point symmetry or a point symmetry of an object is drawn. Properties of point symmetry are documented.
F2e	line symmetry	A figure with line symmetry or a reflected figure of an object with a given line of symmetry is drawn. Properties of line symmetry are documented.
F2f	congruence	Two figures overlap. Properties of congruence are documented.
F2g	composing & decomposing	A figure is decomposed into simpler forms or composed into a larger simpler form.

F2h	folding & unfolding	The activity of folding or unfolding is illustrated. For instance, the activity of folding and/or unfolding a paper is illustrated (e.g., origami). A net of a shape is drawn with an explanation that by folding it one gets a cube or arrows illustrating folding are drawn.
F2i	tessellation	A tessellation of a plane is drawn (e.g., fish tessellation). Properties of tessellation are documented.
F3	**Coordinates, spatial relationships, and reasoning**	**Position and location of geometric forms in the plane or space as well as spatial reasoning about them fall into this category.**
F3a	coordinate system	A coordinate system with x- and y axis or a map grid is drawn. An object is placed in a coordinate system or in a map grid with coordinates given to its constituent parts.
F3b	positional relationships	The subcategory refers to specifying positions and describing relations to other objects. A positional adverb (e.g., above, below, left from) or positional relationship of an object or between object is described (e.g., a square lies right from a circle, two lines are parallel to each other).
F3c	orientation and orientation tools	Mathematical conventions (e.g., labelling vertices in a polygon, labelling an angle) with respect to orientation are illustrated in a drawing (e.g., arrow showing a (counter)clockwise labelling of vertices in a square). A tool for orientation (e.g., a compass rose) is drawn.
F3d	spatial visualization, relation and orientation	The subcategory refers to tasks dealing with different aspects of mental manipulation (e.g., folding a net of a solid mentally, making a view plan of a geometrical composite figure or of a cube building).
F4	**Measurement**	**Qualitative and quantitative properties used to describe geometric forms as well as calculation of these using formulae fall into this category.**
F4a	length	A geometrical object is drawn and the length of at least one constituent part is illustrated. For example, the length of the sides of a parallelogram or the radius of a circle are measured. Units of length are illustrated.

F4b	perimeter	A geometrical object is drawn and its perimeter is illustrated. A formula for a perimeter of an arbitrary figure is written. Perimeter of a figure is calculated.
F4c	surface area	A geometrical object is drawn and its surface area is illustrated. A formula for a surface area of an arbitrary figure is written. Area of a figure is calculated. Units of surface area are illustrated.
F4d	volume	A geometrical object is drawn and its volume is illustrated. A formula for a volume of an arbitrary figure is written. Volume of a solid is calculated. Units of volume are illustrated.
F4e	angle measure	The size of the drawn angle is illustrated (e.g., quarter of a circle with a dot in the middle for a right angle). Size of an angle is measured.
F4f	measuring tools	A measuring tool (e.g., ruler, set square) has been drawn. The function of the tool has to be explicitly mentioned. For example,, a ruler is explained as a measuring tool rather than as a drawing tool.
F4g	estimation	Estimation as an activity is illustrated. For example, a child draws a benchmark for a particular geometrical measure (e.g., 1 cm = 1 small finger width, 10 cm = a hand's width with thumb).
F4h	conversion of measuring units	Conversion of 1-, 2-, or 3-dimensional units is illustrated.
F4i	scaling	A scale drawing of a geometrical object (e.g., a house) has been illustrated with a given scale.
F5	**Geometric patterns**	**Geometric patterns created by using simple geometric forms fall into this category. For example, a frieze pattern, six-petal rosette is drawn.**
F6	**Geometric forms in the environment**	**Description of real-world objects, and operations on and with them by using geometric forms fall into this category.**
F7	**Geometrization**	**Plane and spatial geometric theorems and problems, relationships between numbers (e.g., triangular numbers), and abstract relationships, which can be translated into the language of geometry and then translated again into practical solutions, fall into this category.**

F7a	geometrical facts	A particular geometrical fact (e.g., the sum of interior angles in a triangle, S-S-S theorem) is documented.
F7b	parallel projection	A parallel projection of a particular solid is drawn.
F7c	geometrical problems	A geometrical problem is illustrated (e.g., computing a missing angle measurement in a complex task, computing volume of a composite solid).
F7d	figurate numbers	An example of a figurate number (e.g., triangular, numbers, cubic numbers) is illustrated.