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Undergraduate Students' Meanings for Central Angle and Inscribed Angle

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Contributing to research on students' multifaceted meanings for angles (e.g., angles as ray pairs, as regions, and as turns), we report on three undergraduate students' meanings for central and inscribed angles in circles. Specifically, we characterize how these meanings govern their mathematical activities when engaging in a circle geometry task, including their experienced perturbations and reconciliation of those perturbations. Our conceptual analysis reveals that some meanings are productive for students to conceive of a reflex angle in a circle and the correspondence between a central and an inscribed angle, while other meanings are limited.

Angle and angle measure are critical topics in mathematics curricula. Writers of the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) specify angle-related content in Grade 2 through high school, starting from identification of angles in planar shapes to radian angle measure in trigonometry. Correspondingly, mathematics curricula in the United States convey a variety of angle definitions, such as angles as geometric shapes formed by two rays that share a common endpoint, angle measures as turns, and angle measures as fractional amounts of a circle's circumference. Despite fruitful research findings on students' and teachers' understandings of angles and angle measures (e.g., Clements & Burns, 2000; Devichi & Munier, 2013; Hardison, 2018; Keiser, 2004; Keiser et al., 2003; Mitchelmore & White,

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1998, 2000; Moore, 2013), not many researchers have given attention to students' understandings of angles in the context of circle geometry at the secondary and post-secondary levels. Since concepts such as central angles and inscribed angles are essential for students' subsequent learning of trigonometry and geometric theorems and proofs, students' understandings for these concepts deserve a closer investigation.

In this study, we characterize three undergraduate students' meanings for angles and how such meanings affected their ability to identify a central angle corresponding to a given inscribed angle in a circle. We conclude by discussing what angle meanings are propitious for and what meanings are deleterious to these students' construction of central and inscribed angles.

Background and Research Questions

We place extant studies on individuals' angle conceptions into two broad categories: (a) studies on students, teachers, or mathematicians' understandings of the angle object itself and (b) studies on how they imagine the object being measured. In this section, we first review the literature of these two categories. Then, we specify the research questions that guide our inquiry in the present study.

Angle as Ray Pair, Region, and Turn

Researchers have investigated individuals' understandings of angles and angle measure with various population groups, including mathematicians in history (Keiser, 2004; Matos, 1990, 1991), elementary students (Clements & Battista, 1989; Clements & Burns, 2000; Devichi & Munier, 2013; Keiser, 2004; Mitchelmore, 1998; Mitchelmore & White, 1998, 2000), middle schoolers (Fyhn, 2008; Mitchelmore & White, 2000), high schoolers (Hardison, 2018), undergraduate students (Moore, 2013), and mathematics teachers (Kontorovich & Zazkis, 2016; Tunc & Durmus, 2012). Due to their relevance to our study, we only focus on findings concerning individuals' planar angle meanings as opposed to three-dimensional angles.

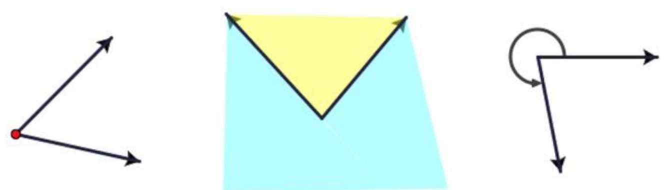
We summarize three viewpoints of angles that repeatedly occur in this group of literature: (a) an angle as a ray pair, (b) an angle as a region, and (c) an angle as a turn. A number of researchers have explored how children recognized physical situations (e.g., wall corner, roof, ramp, scissors, road junction, Spanish fan) as angle situations corresponding to these three viewpoints (Mitchelmore, 1998; Mitchelmore & White, 1998, 2000). Some other researchers have focused on classroom activities or technology designed for supporting students' understandings of angles as ray pairs, regions, or turns (Devichi & Munier, 2013; Fyhn, 2008; Simmons & Cope, 1993, 1997). Common to these researchers' inquiry was their intention to support or gain insights into particular angle understandings with students. In the present study, we are more interested in how students understand angles heuristically in terms of these three aspects. Therefore, we attach more attention to the literature with this focus.

Individuals who understand an angle as a ray pair conceive of an angle as a pictorial object that is formed by two rays (or two segments of different or identical length) meeting at a common vertex (see Figure 1a). Students who hold this image of an angle may define that an angle is "when two lines meet each other and they come from two different ways" (Clements & Battista, 1989, p. 456). Some other students may emphasize the point where the ray pair meets, defining "an angle is where two vertices meet and make a point" (Keiser et al., 2003, p. 117). Mitchelmore and White (1995) reported that 45% of fourth graders' responses of their angle definitions reflected they understood an angle as consisting of a ray pair, and 22% understood an angle as the intersection point of a ray pair.

Individuals who understand an angle as a region construct an image of an angle as a space bounded by a ray pair (see Figure 1b). In this construction, a ray pair can contain two angles, and the function of a label arc of an angle can be showing which part of a plane is meant as the angle of study (Krainer, 1993). In Mitchelmore and White's (1995) study, 15% of four graders defined an angle as a bounded region.

Individuals who understand an angle as a turn imagine an angle as being formed by a dynamic rotation of one ray from another when the center of rotation is fixed or imagine an angle as describing such a rotation (see Figure 1c). This rotation can be positive (e.g., a counterclockwise turn), negative (e.g., a clockwise turn), equal to or less than a full turn (i.e., a full or a fractional amount of a full turn), or some multiples of a full turn. Many young students do not spontaneously interpret ray pairs as turns or vice versa (Mitchelmore, 1998; Mitchelmore & White, 1998). Clements and Battista (1989) suggested that engaging children in a logo experience supported them in constructing this angle meaning; some third graders in their study defined an angle as “something that turns, different ways to turn” (p. 456).

Figure 1
An Angle as (a) a Ray Pair, (b) a Region, and (c) a Turn



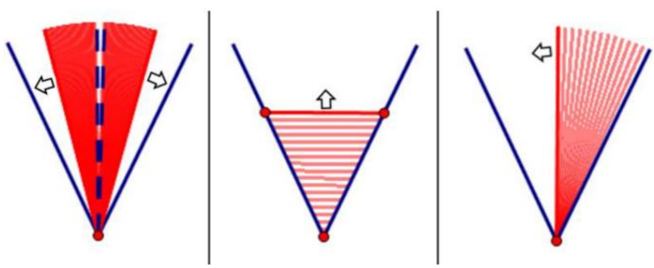
(a) (b) (c)

Angle Measure

As Keiser (2004) stated, “[I]n all of these different perspectives [e.g., ray pair, region, turn], confusion can arise when trying to identify what exactly is being measured when measuring an angle” (p. 289). When students conceive of angles *as* ray pairs or regions, they may conceive of angle measure as measuring the side length of an angle or the area of the sector marking an angle. Some students may also make a judgement of angle size based on the radius of the circle of which the arc used to mark an angle, the length of the arc used to mark an angle, or the quality of sharpness of an angle (Foxman & Ruddock, 1984; Keiser, 2004).

That these students’ angle measure meanings are associated with diverse attributes of the geometric angle objects highlights the need for mathematics educators to support students’ quantification of angularity—a separation between the angle object itself and its measurable attribute and a focus on measuring an angle in terms of its openness (Thompson, 2011). Moore (2013) documented a precalculus student who shifted from conceiving angle measures in terms of geometric objects to constructing an image of angle measure in terms of measuring an arc with respect to a circle's circumference. Moore provided evidence that leveraging the multiplicative relationship between an arc length and the circle's circumference was productive for the student to construct coherent understandings of angle measures across different units (e.g., degree, radian, and other self-created units). Hardison (2018) extended Moore's work by looking into two high school students' quantification of angularity independent of circular contexts. He identified three mental actions involved in these students' construction of angularity (e.g., re-presented opening, segment sweep, and radial sweep; see Figure 2), each of which necessarily involved operations enacted on angular interiors.

Figure 2
Three Images of Quantifying an Angle



Note. From *Investigating high school students' understandings of angle measure* [Doctoral dissertation, University of Georgia], by H. L. Hardison, 2018, Athenaeum, p. 302 (http://purl.galileo.usg.edu/uga_etd/hardison_hamilton_l_201805_phd). Copyright 2018 by Hamilton L. Hardison.

Research Questions

Building on these researchers' findings, we investigate how these different ways of understanding angles and angle measures (e.g., ray pair, region, turn, and arc) are evident in undergraduate students' mathematical activities. We contribute to the extant literature in two ways. First, many of the aforementioned studies have focused on students' definitions of angles; we consider it necessary to draw attention to their mathematical meanings (Thompson, 2013; Thompson et al., 2014) for angles, which consist of a collection of mental actions that govern their articulation of angle definitions and their perception and activity related to angle contexts. Second, despite the critical role of arcs and circles in students' understandings of angles (Moore, 2013), few researchers have investigated the teaching and learning of angles in circle contexts. In the present study, we address these gaps by focusing on students' meanings for central and inscribed angles. We answer the following research questions: What inscribed–central angle meanings do undergraduate students possess, and in what ways do these meanings govern their mathematical perception and activity when solving a circle geometry task?

Radical Constructivism and Scheme Theory

We adopt a radical constructivist perspective to approach the current study. We assume *knowledge* is not a representation of objective ontological truth; instead, it functions and organizes viably within a knower's experience and is idiosyncratic to the knower (von Glasersfeld, 1995). We have no access to anyone else's knowledge, and at best, we can construct hypothetical models of others' knowledge (Steffe et al., 2000). Therefore, what we call students' inscribed–central angle meanings are our hypothetical models (or inferences, or interpretations) that can explain the students' observable behavior evident in the present study. We start with building models of the students' angle meanings, especially those being different from our own, and then we discuss instructional implications sensitive to these models.

Schemes, Assimilation, and Accommodation

To characterize students' knowledge of inscribed and central angles, we use Piaget's notion of *scheme*—"the structure or organization of actions[1] as they are transferred or generalized by repetition in similar or analogous circumstances" (Piaget & Inhelder, 1966/1969, p. 4). An individual's scheme system is a cognitive entity that consists of a collection of mental actions and operations that govern the individual's assimilation, activities, and anticipations of outcomes of those activities (Thompson et al., 2014; von Glasersfeld, 1995). *Assimilation* is the mental process by which an individual's scheme structures their experience and determines what they attend to (Piaget, 1950/1971; von Glasersfeld, 1995). However, there are experiences where the individual's enactment of an existing scheme results in some conflicts due to features not compatible with that scheme. If the individual recognizes the conflict, they experience *perturbations* (or *disequilibrium*). Each individual tends towards an internal equilibrium in the face of perturbations (Ginsburg & Opper, 1988), which necessitates the individual to revise or reorganize their schemes to account for the conceived conflict, (i.e., *accommodation*). Learning occurs as a result of accommodation (von Glasersfeld, 1995).

To illustrate, we consider a student who constructs an angle scheme that exclusively includes an image of an acute angle as a ray pair (see Figure 3a). When asked to label an angle shown in Figure 3b, they may *assimilate* the ray pair as an acute angle (assimilation) and draw a label shown in Figure 3c (an activity). The student also anticipates this labeled angle to have a measure less than 90 degrees (anticipation). However, the student may be perturbed when we present Figure 3d, where a green arc labels an alternative angle associated with the same ray pair. Then, the student may reconcile the perturbation by revising their scheme to consider that a ray pair can have two measures, with one being greater than 180 degrees. We note that students may resolve their perturbations in a variety of ways, depending on the nature of their schemes.

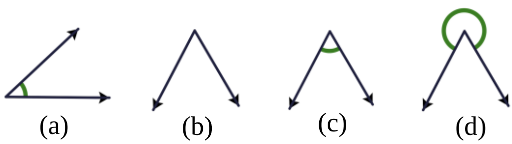
In the current study, we use the term "meanings" in the Piagetian sense of schemes (Thompson, 2013; Thompson et al., 2014). Schemes are researchers' models of individuals' characteristic ways of operating, whose construction requires the

[1] Actions refer to both physical movements and mental actions.

researchers to engage in sustained interactions with the individuals and test and refine those models over time. We do not achieve this goal to provide a comprehensive profile of the participating students' schemes of central and inscribed angles. Therefore, we use the term "meaning" as a subset of mental actions constituting the students' schemes. We assume that the students have constructed all kinds of meanings that are organized in diverse ways from their prior school experience, and our goal is to characterize the available meanings that they use to assimilate the provided mathematical situation in the current study. We also note that our goal is not to attribute the students' observable actions only to their angle meanings or attempt to locate the sources of their difficulties (e.g., why they solved the problem incorrectly). Rather, we use the task as a stimulus to motivate the students to express their meanings in the form of observable behavior, from which we infer (a) the nature, content, and structure of their meanings, (b) how those meanings govern their activity and perception, and (c) what perturbation and temporal[2] accommodation occur in the moment of the study.

Figure 3

Assimilation, Perturbation, and Accommodation Associated With an Angle Scheme



Methods

Participants

We recruited nine undergraduate students from a large public university in the United States. We chose the students

[2] We note that we were only able to make claims about the participants' temporal rather than sustained, stable accommodation due to our limited interactions with our participants. We acknowledge this methodological limitation at the end of the paper.

from a secondary mathematics methods course on a voluntary basis, and they received financial compensation for their time. At the time of the study, all students had at least completed 30 credit hours' mathematics courses, including a course on Euclidean geometry with emphasis on two-column geometric proofs.

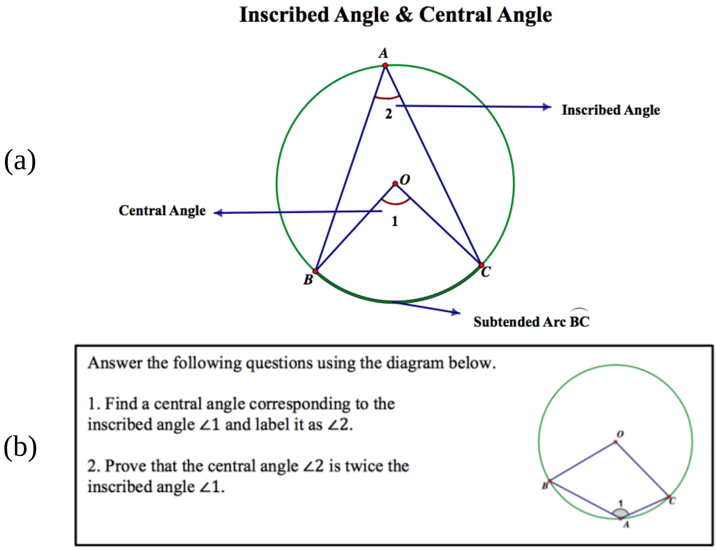
In this paper, we report on the reasoning of Joanna, Hayley, and Jack. Joanna and Hayley were juniors majoring in secondary mathematics education, and Jack was a sophomore majoring in mathematics and secondary mathematics education. We elected to document these three students because their activities established the existence of highly varied meanings for central and inscribed angles among the population of pre-service teachers.

Materials and Data Collection

The study consisted of three related tasks: (a) a pre-test, (b) a reading task, and (c) a post-test followed by a short interview. Each student completed these tasks back-to-back individually during an hour-long session. In the pre-/post-test, we provided each student with a handout that included a graphical definition of central and inscribed angle (see Figure 4a) and asked them to complete a proof (see Figure 4b). We collected each student's written solutions for analysis.

As a normative solution, the central angle that corresponds to the inscribed angle ∠1 is the reflex angle BOC (as opposed to the obtuse angle BOC) with a vertex at the center O because these two angles share a common subtended major arc BC. Our intentions of providing the handout were to remind students about the mathematical terms and, more importantly, offer them initial perceptual material to assimilate. As we illustrate below, the students paid attention to different components of this diagram and conveyed distinct meanings of central and inscribed angles, which did not necessarily include the shared subtended major arc BC.

Figure 4
The Inscribed and Central Angle Definition Handout and the Pre-/Post-Test

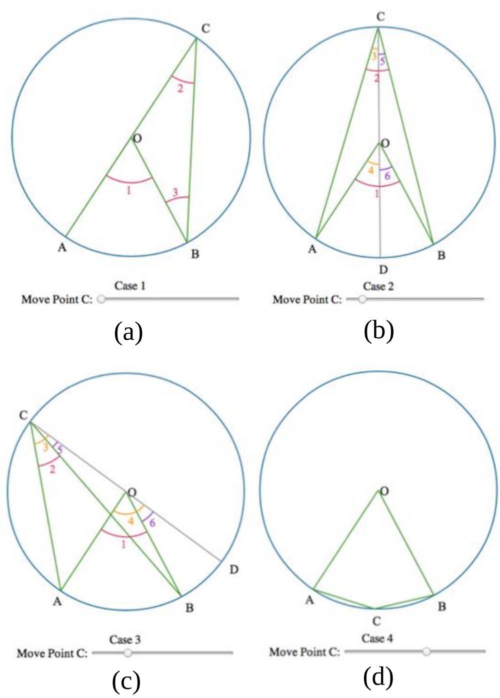


Right after the pre-test, the students worked on the reading task on a computer. We designed two sets of presentations and randomly assigned students to them. We assigned Joanna and Hayley to the static presentation and assigned Jack to the dynamic presentation. The static presentation demonstrated a proof of the inscribed angle theorem corresponding to Cases 1, 2, and 3, along with a static diagram of each case (see Figure 5a–c, without the slider).[3] The dynamic presentation contained a dynamic diagram with a slider (see Figure 5a–d; Manuscript Video, 2019). The slider allowed students to move Point C along the circle so that infinite cases could be seen. Meanwhile, a corresponding proof of the inscribed angle theorem would appear to the right of the figure depending on which static case

[3] We do not focus on the students' activity related to proof comprehension and proof construction in the current paper, although they count towards a broader goal of the design of the study. We include screenshots of the full presentations in the Appendix in case the reader is interested.

the current state of the diagram belonged to. Since the fourth case (Figure 5d) was in the post-test, we omitted the proof from the static presentation and left blank in the dynamic presentation. The purpose of this task was to understand how the presentation of other cases might (or might not) influence the students' meanings for inscribed and central angles, including their assimilation and accommodation of the fourth case. We screen-recorded each student's activity in this task.

Figure 5
Snapshots of the Four Cases in the Dynamic Presentation



Note. The static presentation contained Cases 1–3 and without the sliders.

Each student took the post-test right after the reading task, and at the end of the session, they participated in a debrief interview. The first author served as the interviewer and asked each student to express their retrospective thoughts regarding the three tasks. Our questioning focused on the following three aspects: (a) the student's reasoning in the pre-test, (b) what the student learned and attended to during the reading task, and (c) the student's reasoning in the post-test. Each interview lasted for about 20 to 30 minutes. We audiotaped each interview, digitized each student's written work, transcribed the interviews, and incorporated figures and annotations.

Design Rationale

We chose the topic of circle geometry due to its potential of offering us a rich context for investigating students' angle meanings. The tasks we designed here included different perceptual components such as segment (or radii) pairs, intersection points, arcs, and quadrilaterals bounded by segment (or radii) pairs, which provided students with rich perceptual materials to assimilate. Thus, this design could afford us insights into what angle meanings (e.g., ray pairs, regions, turns, and arcs) dominated the students' mathematical perception and activity and the interplay between these meanings. Moreover, researchers (e.g., Hardison, 2018; de Matos, 1999) have noted that students tend not to assimilate ray pairs as reflex angles. Our inclusion of a reflex angle in the pre- and post-test (see Figure 4b and Figure 5d) afforded us an understanding of the extent to which the students' meanings were applicable to different angle contexts and the affordances and limitations of those meanings. The diagram we included in the pre- and post-test was not a prototypical illustration for central and inscribed angles in curricula and instruction, and we conjecture it was a novel representation to the students. We were interested in how the presence of this representation might or might not engender perturbation in the students' meanings or necessitate them to leverage and coordinate various angle meanings for assimilation.

Data Analysis

Drawing on the annotated interview transcripts, we conducted a conceptual analysis (Thompson, 2008) to develop hypothetical models of each student's mathematical meanings. Consistent with our theoretical perspective, we positioned the students as rational thinkers and modeled the mental actions underlying their observable actions. By repeatedly reviewing the transcripts along with the students' work, we made inferences about each student's mathematical meanings in ways consistent with their speech and produced diagrams. It was the observed mathematical conclusions the students reached and our hypothetical mental operations that would viably justify those conclusions that comprised our models of the students' meanings. We continually searched the data for instances that either confirmed or contradicted our models. In the latter case, we either modified our models or documented shifts in the students' meanings. In our analysis pertaining to what we discuss in this paper, we drew particular attention to the instances that would offer us insights into the students' meanings for central and inscribed angle and the mathematical consequences implied by those meanings. We kept in mind the prior researchers' findings of students' meanings for angles as ray pairs, regions, turns, and arcs, and we considered how these meanings were evident in the students' activities.

Results

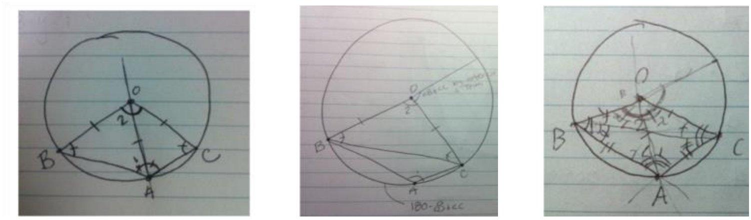
In this section, we first provide an overview of the three students' pre- and post-test solutions and then discuss each student's activities with respect to the three tasks.

An Overview of Student Solutions

In the pre-test, the three students identified the same solution angle in response to Question 1 (see "$\angle 2$" in Figure 6a–c). In the interview, Joanna expressed uncertainty about the central angle she identified in the pre-test, pointing to Point O in Figure 6a and saying, "I think I get stuck at this same place where I was trying to find the central angle. Because if I can find the central angle, I could have done it easily. It just like, I did not." Similarly, Jack expressed that he could not complete the proof because he identified a wrong central angle and was ultimately attempting to prove something other than what was intended, pointing to ∠2 in Figure 6c and saying, “I have the wrong angle. I thought this was the central angle, so I couldn’t prove it.”

Figure 6

Diagrams Produced by (a) Joanna, (b) Hayley, and (c) Jack in the Pre-Test

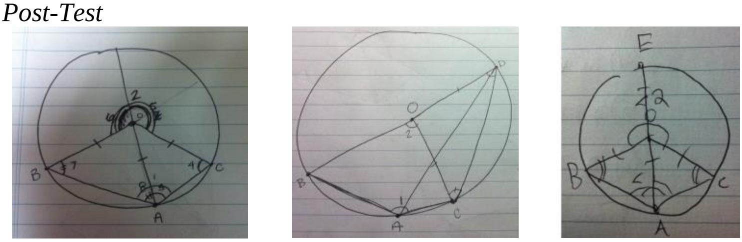


(a) (b) (c)

As the students discussed the post-test, we observed that Joanna and Jack changed their solutions and identified the reflex angle O as the relevant central angle (see Figure 7a and Figure 7c), while Hayley maintained her solution as it was in the pre-test (see Figure 7b). In the following, we characterize each student’s meanings for central and inscribed angles and how their meanings influenced their ability to identify a corresponding central angle.

Figure 7

Diagrams Produced by (a) Joanna, (b) Hayley, and (c) Jack in the Post-Test



(a) (b) (c)

The Case of Joanna

Joanna expressed in the interview that, in the post-test, she realized the central angle she labeled in the pre-test was incorrect, saying, "Well because one has to be outside, one has to be inside. And they are both inside, so then I figured out that can't be right." By "one has to be outside, one has to be inside," Joanna was referring to the figure shown in Figure 4a, in which the central angle was outside the quadrilateral ABOC, and the inscribed angle was inside the quadrilateral. As she applied this idea to assimilate her initial solution (see Figure 6a), where the central angle was inside the quadrilateral enclosing the inscribed angle, she decided to revise her solution to be the reflex central angle (see Figure 7a) so that her meaning was compatible.

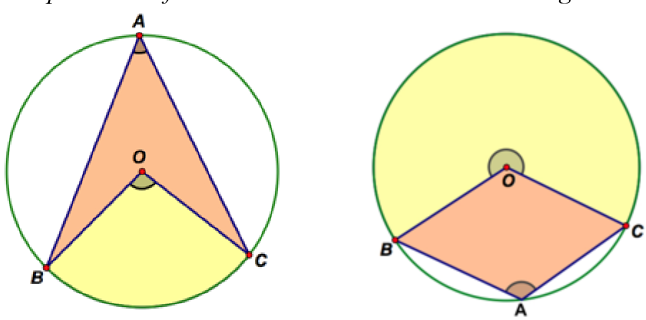
We inferred from Joanna's reference to "outside" and "inside" that she conceived of the relative position of a central angle and its corresponding inscribed angle with respect to the interior of a quadrilateral (colored in orange in Figure 8a and Figure 8b). Namely, a central angle was outside the interior of a quadrilateral and the inscribed angle was within the interior of the same quadrilateral. She also anticipated that this property should hold in both Case 2 (see Figure 5b) and Case 4 (see Figure 5d). At this point, we lacked evidence to conclude what Joanna perceived as the central or inscribed angle itself. As documented in Thompson (2008), she might conceive of the central or inscribed angle as the indicator arc itself (see her labeled arcs in Figure 6a and Figure 7a). However, because Joanna drew the arcs by herself as a way to mark the angles, she was likely attending to a certain attribute (other than the arcs) of those angles in order to label the angles the way she did. We thus provide an alternative interpretation that she conceived of the central angle as the interior bounded by the ray pairs of the angle (see the yellow regions in Figure 8a and Figure 8b). To support this inference, we describe the rest of our conversation in the following.

Although from our perspective, Joanna eventually identified a correct central angle, she was uncertain about its accuracy from her perspective, saying, "I figured it out, but I don't know whether that was right, because I don't think it is." She further explained, "Because it was like the way too big to be a central angle." We interpreted that Joanna was experiencing a perturbation due to the central angle she identified being bigger than 180 degrees. We conjecture that her perturbation might result from her imagining central angles as regions. By "too big," Joanna likely referred to the measure of the region enclosed by the central angle being "too big" (see the yellow region in Figure 8b); the reflex nature of the angle might have made it appear to Joanna that the angle was enclosed by the region rather than the reverse. As the conversation continued, Joanna found a way to reconcile her experienced perturbation, commenting as follows:

> When I looked at this [∠1 in Figure 4b], well, that's obtuse angle, so this one [∠2 in Figure 7a] has to be like…really big. So that's what I learned and kept going with this. You know this one [∠1 in Figure 7a], obviously like, pretty big, so then I was like, might be even bigger.

Figure 8

Our Interpretation of Joanna's "Outside-Inside" Meaning



(a) (b)

The idea that the measure of a central angle should be bigger than that of an inscribed angle was implied by the given statement of the inscribed angle theorem, and we inferred Joanna was drawing on this resource to temporarily modify her angle meaning to resolve her perturbation.

Collectively, Joanna held three meanings for central and inscribed angles: (a) a central angle has to be outside the region bounded by an inscribed quadrilateral and its corresponding inscribed angle has to be inside the same region, (b) the measure of an angle (likely as a region) should not be "too big," and (c) the measure of a central angle should be bigger than that of its corresponding inscribed angle. We note that the last two meanings were competing meanings from Joanna's perspective. She continued to use hedge words in the interview, and despite her success of providing a proof of the theorem in the post-test, she was not confident that her central angle was correct. It was nontrivial for Joanna to reconcile her experienced perturbation, and we inferred her difficulty was a consequence of her conceiving angles as regions in general.

The Case of Hayley

Now we turn to describe the story of Hayley, who identified the same central angle in both the pre- and post-tests. In the interview, we asked Hayley about her reasoning:

Int: How do you think about this problem? Where did you get stuck?
Hayley: Umm…the arc part, like finding the first angle…this was the central angle [$\angle 2$ in Figure 6b], right?
Int: What do you think is a central angle?
Hayley: The central angle would be in the middle of the circle [motioning her hands along the two rays towards the center] 'cause this is the center, so that's why I put that, this "O" is the central angle.

Hayley described a central angle as the center of a circle where two radii met, from which we inferred her angle meanings at least included a meaning for angles as ray pairs. Because she only conceived of the radii pair BO and CO as constructing one angle, she related this singular angle to the subtended minor arc BC and considered angle "O" to have exactly one measure as opposed to two (see Figure 9 and her annotation of angle measure underneath the circle).

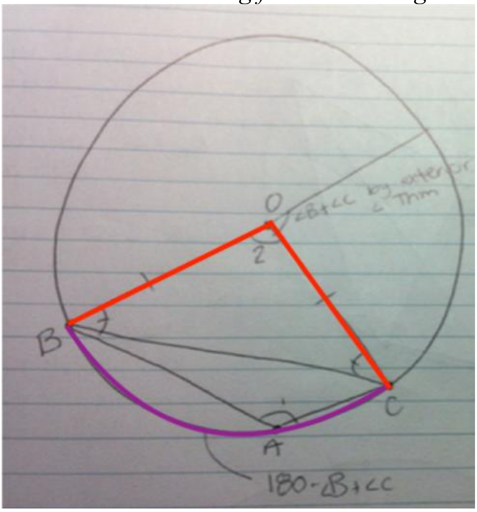
Our inference about Hayley's ray pair angle meaning was also supported by her assimilating activity during the reading task. When asked if the three cases demonstrated in the static presentation helped identify the central angle in the post-test, she responded:

> I don’t know because it is in the same spot. In all those and they are always the same angle in all them [∠1 in Figure 5a–c], because they are the same angle, so, yeah, I don’t think that will be helpful.

Hayley conceived that the three central angles were constructed by the same ray pair, AO and BO, that met at the same Point O (“it is in the same spot”) and that her solution angle shared a compatible feature. In addition, the central angle labeled in each diagram was less than 180 degrees, which fit into Haley’s meaning that each ray pair only corresponded to a minor arc (see Figure 10). We also note that Hayley’s attention to the invariance of the central angles across cases suggested that she conceived of central angles as being absolute rather than coupled with a corresponding inscribed angle. Because she perceived the central angles to be “the same angle” across cases, she chose the same angle as the central angle for the unknown case (see Figure 7b).

Figure 9

Hayley’s Ray Pair and Arc Meaning for Central Angle in the Pre-Test



Note. The ray pair is highlighted in red and the arc in purple.

Figure 10
Hayley's Ray Pair and Arc Meanings for Central Angles With Respect to the Three Static Diagrams

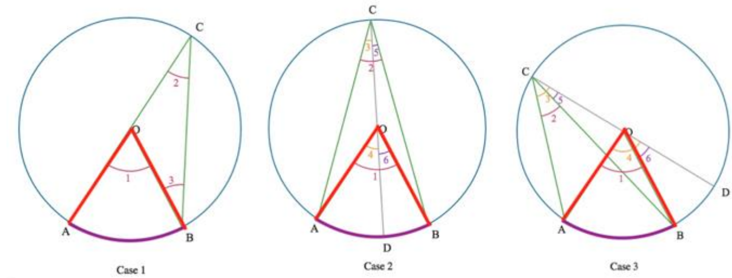
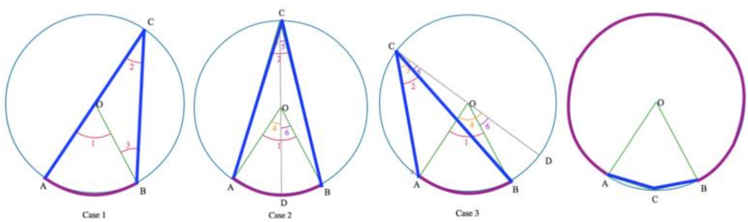


Figure 11
Hayley's Activity of Finding Subtended Arcs of the Given Inscribed Angles Regarding the Four Cases



(a) (b) (c) (d)

Observing that Hayley had constructed an arc image of angle, the interviewer decided to support her in relating a central angle to its corresponding inscribed angle with a shared arc. The interviewer asked her to identify a subtended arc of the inscribed angle in Case 1 and then to find a central angle that subtended the same arc (see Figure 11a). They repeated a similar activity for Case 2 and Case 3 (see Figure 11b–c), and when Hayley looked at the post-test diagram, she used the shared subtended arc to determine the reflex angle ∠AOB as the corresponding central angle for the first time (see Figure 11d). We claimed that Hayley constructed the relationship between a central and an inscribed angle by conceiving a shared subtended arc of the two angles. She at least experienced a temporal accommodation in her central angle meaning, and due to the limitation of the study, we lack information about the extent that this meaning was stable.

The Case of Jack

Jack identified the correct central angle in the post-test and discussed how interacting with the dynamic diagram in the reading task was helpful. Specifically, he slowly tracked the angles as Point C moved along the circle and paid particular attention to the transition between Case 3 and Case 4. He described,

> I realized that it was opening up like this [Figure 5b], and then when I pulled it around, here [dragging the slider to transition to Figure 5c] I was sort of seeing, you know kept on seeing the thing I played [dragging the slider and pointing to the Point C]… it is getting really close, like doing a limit. Like take a limit and you get there [Figure 12], like just…Oh no! [dragging the slider to transition to Figure 5d] Just like, oh I know what was going on, so I know how it went through.

Here, Jack was looking into the details involved in between Case 3 and Case 4 to observe how the angles were "opened up" differently. Later, he described his observation from this activity:

> You like keep track of the angles as they move because you can see here [Figure 12], you know these angles stay the same, the same, but they just flipped over [dragging the slider to transition to Figure 5d], so you can just sort of generalize it.

By "flipped over," Jack might be imagining ∠ACB as if it were existing on a transparency, and it was turned over and flipped upside down when transitioned to Case 4. Similar to the case of Joanna, we did not have evidence to conclude which angle attribute he was attending to. One interpretation was that before C passed A, he perceived the segment pairs CA and CB bounding a region underneath them (see the green and the blue region in Figure 13a), and same for the segment pairs AO and BO (see the green and the yellow region in Figure 13a). After C passed A, since CA and CB bounded a region above them (the green and the blue region in Figure 13b), the central angle AOB should correspondingly “flip” to its reflex angle with an angle region being above the segment pairs AO and BO (see the green region in Figure 13b).

Figure 12
Jack Moved Point C to Get “Really Close” to Point A

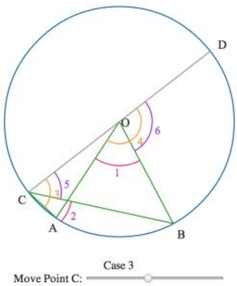
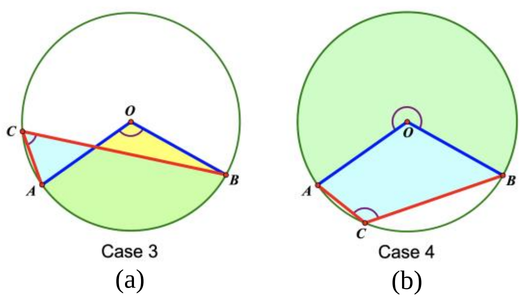


Figure 13
An Interpretation of Jack’s “Flipped Over” as the Central and Inscribed Angles’ Regions Changing From (a) Below the Segment Pairs to (b) Above the Segment Pairs

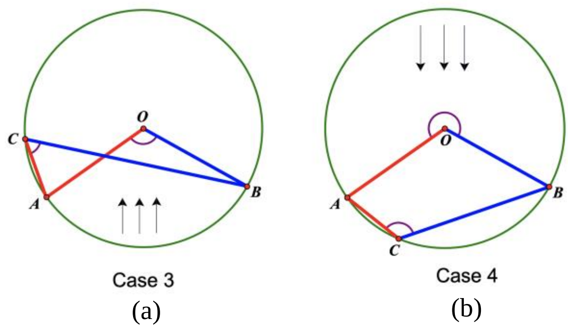


An alternative interpretation of Jack’s “flipped over” was that before C passed A, the inscribed angle ACB was constructed by AC on left and BC on the right (facing into ∠C from the bottom of the circle; see Figure 14a); after C passed A, the angle was constructed by BC on the angle’s left and AC on the right (facing into ∠C from the top of the circle; see Figure 14b). Since the ray pairs of the inscribed angle and its “flipped” angle had different orientations, the central angle AOB should correspondingly “flip” to its reflex angle with BO on the left and

AO on the right (viewing from the top of the circle; see Figure 14b). It is also possible that Jack considered CA and OA to be the initial sides of the angles and CB and OB to be the terminal sides, and they switched after the angles are “flipped over.” This meaning aligns with the angle-as-turn meaning in a way that it distinguishes the initial side from the terminal side, although Jack might not necessarily imagine a dynamic opening from one to another.

Figure 14

Another Interpretation of Jack’s “Flipped Over” as the Central and Inscribed Angles’ Ray Pairs Changing From (a) CA and OA Being on the Left to (b) CA and OA Being on the Right



Discussion

We summarize the three students’ meanings in terms of angles as regions, ray pairs, turns, and arcs in Table 1.

Our analysis suggests that, regardless of angle contexts or grade levels, students’ understandings of angles as ray pairs, angles as turns, and angles as regions persisted from elementary students (e.g., Clements & Battista, 1989; Foxman & Ruddock, 1984; Keiser et al., 2003; Mitchelmore & White, 1998) to late undergraduates. We contribute to the current literature by documenting how these meanings could manifest in different, creative ways as these students applied them to a circle geometry context and construct meanings of central and inscribed angles. However, we also observed that these advanced undergraduates, many of whom would become mathematics teachers, experienced different kinds of perturbations when leveraging multiple angle meanings to make sense of the given situation. For example, Joanna' perturbation originated from two incompatible meanings she held—that is, a central angle should be twice as big as its corresponding inscribed angle, and meanwhile, it should not be "too big" to enclose a region—leading to her doubts in her solution. Also, her "outside-inside" meaning was too specific to the particular situation to be a generalizable understanding of angles. We conjecture that this meaning might fail to account for other situations where no reference objects or shapes (e.g., a closed figure: a quadrilateral) can be identified, or where it is necessary to conceive of a single angle that changes between "inside" and "outside" among cases. For instance, we wonder: how would she perceive "outside-inside" for Case 1 (see Figure 5a) and Case 3 (see Figure 5c)?

Table 1
Students' Multifaceted Meanings for Central and Inscribed Angles

Category	Description of meaning	Student
Angles as regions	A student imagines that an inscribed angle is inside a region bounded by a quadrilateral within a circle and its corresponding central angle is outside the same quadrilateral.	Joanna (post-test)
	A student imagines that the measure of a central angle (as a region) is bigger than that of its corresponding inscribed angle.	Joanna (post-test)
	A student imagines that a central angle and its corresponding inscribed angle both bound a region either below or above a segment pair.	Jack (post-test)
Angles as ray pairs	A student imagines that a central angle is an angle constructed by two radii meeting at the center of a circle.	Hayley (pre- and post-test)
Angles as oriented ray pairs or turns	A student imagines that an angle is oriented by the relative position of one side (could be considered as an initial side) and another side (could be considered as a terminal side), and this orientation should be consistent between a central angle and its inscribed angle.	Jack (post-test)
Angles as arcs	A student imagines that an inscribed angle and its corresponding central angle share a subtended arc.	Hayley (after intervention)

Another contribution of our work is identifying several productive central-inscribed angle meanings. By "productive," we refer to meanings that allow students to assimilate various angle situations, especially situations beyond elementary mathematics. Specific to the present study, we speak of this productivity in terms of two aspects: (a) whether the meanings afford students' construction of a correspondence between a central and an inscribed angle, and (b) whether the meanings afford students' assimilation of a reflex angle.

First, some meanings afford students' construction of a correspondence between a central and an inscribed angle. In particular, these are meanings that allow students to differentiate angles' orientation. For example, we can speak of an angle's orientation in terms of which arc it is pointing to. Haley's meaning for angle measures as arcs was foundational for her construction of central and inscribed angles as subtending or pointing to a common arc. We can also speak of an angle's orientation in terms of the relative position to some other geometric objects. Joanna conceived of a region enclosed by a central angle as located outside the interior of a quadrilateral and an inscribed angle being inside. Jack considered the region bounded by a central and inscribed angle should be either under or above a ray pair at the same time. We can also speak of an angle's orientation in terms of the relative position of the two rays (e.g., Jack). Collectively, these meanings enable students to relate and differentiate a central angle and its inscribed angle.

In contrast, a ray pair meaning, especially when the two rays are considered as equivalent rather than distinguishable, does not inherently support a distinction of angle orientation. A ray pair itself cannot point to a certain angle or indicate which angle is under consideration until a student conceives of a region, an arc, or a turn associated with that ray pair. For instance, Hayley initially conceived of a central angle as the angle constituted by two radii rather than being relative to a given inscribed angle.

Second, in the context of reasoning with static diagrams, conceiving angles as arcs is productive for students to assimilate reflex central angles, and angles as ray pairs and regions may not support such assimilation. Haley initially did not conceive of the reflex angle constructed by a ray pair. A student's ray pair meaning may constrain them from conceiving the single structure of two segments as having two measures or construct two angles corresponding to the segment pair. Joanna's angle-as-region meaning constrained her from perceiving the reflex angle as enclosing a region, leading to her perturbation and doubt in her solution. These findings support Hardison (2018) and de Matos's (1999) results that students do not spontaneously assimilate reflex angles, especially in non-rotational angle contexts. In contrast, a student holding an arc meaning of angles can conceive of an angle that subtends an arc longer than a semicircle (e.g., Hayley).

Implications

While we do not intend to generalize the students' angle meanings evident in the current study to other students or other contexts, we do need to highlight the contributions of our work here. We have not merely characterized these undergraduate students' struggles and successes when solving a geometry task; more importantly, we have explained their successes and difficulties by making inferences of their angle meanings. We have gone beyond categorizing the students' angle meanings to provide a fine-grained analysis of how these meanings they have constructed in prior school experiences have been consequential to their mathematical perception and activities in the present context of central and inscribed angles.

Moreover, these findings provide several implications for the instruction of relevant topics. Researchers (e.g., Mitchelmore & White, 1998) have suggested that angles as ray pairs and angles as regions were more prevalent in students when compared to other angle meanings. We highlight the need for mathematics teachers to support students in constructing angle meanings in relation to circles, arcs, and turns when teaching advanced angle topics. Ultimately making use of the circle context was critical to the success of both Hayley, who found a common subtended arc, and Jack, who imagined the angle changing as the vertex moved around the circle. In contrast, the presence of the circle context of the tasks did not inherently lead

Joanna to incorporate circles or arcs into her identification of central angle.

Additionally, we suggest that mathematics educators put efforts into supporting teachers in developing more robust meanings for angles, especially how different meanings are related, different, compatible, and not compatible with each other in different contexts. We should not take it as a given that teachers are aware of how these multifaceted angle meanings (e.g., ray pair, region, turn, circle, and arc) interact or which meanings are productive in what situations. We argue that such awareness should be critical in teachers' decision-making during mathematics teaching and essential for teachers to recognize, explain, and respond to their students' mathematical thinking when interacting with students. It will be challenging for teachers who possess single or disconnected angle meanings to be sensitive to the nuances between their students' angle meanings or to be sensitive to the students' awareness (or lack of awareness) of the relationships between these meanings.

Limitations and Future Work

One limitation of the present study lies in the small amounts of interactions we were able to engage in with those students. Our goal of the current study is to generate preliminary results of undergraduate students' inscribed–central angle meanings. We call for future research to extend and refine our characterizations either by identifying new models or adding nuances that we may have missed to the current models. This requires the researcher to engage in extensive interactions with the students and develop working models to document students' characteristic ways of operating. We iterate that we do not achieve the goal of characterizing these students' inscribed–central angle schemes or angle schemes in general, and our work here is merely a building block for this line of inquiry. One way to develop finer models of students' angle schemes is to engage them in reasoning with a variety of problem contexts and topical areas and to investigate their assimilation and accommodation activities among different situations. As we have learned from Hayley, a student who constructs an arc image of one angle does not necessarily imply that they can assimilate a shared arc subtended by two angles.

Another way to extend our study is to explore students' inscribed–central angle meanings with a broader population. Our findings here are limited to three undergraduate students. We have identified a large amount of angle research at the elementary level, but little at the secondary or undergraduate level. The lack of research at these advanced levels constrains us from understanding in what ways students' early angle meanings are consequential to their learning of more advanced, complex angle-related topics, such as radian measure, polar coordinates, complex numbers, trigonometric functions, and circumscribed angle.

We also consider it is worth investigating the effects of different kinds of interventions, illustrations, and technology (e.g., dynamic images, static images, and arc-based instruction) on developing students' angle meanings. For example, by interacting with the dynamic images, Jack modified his solution by reasoning about the changes of angles and gaining insights into the details between cases. We are curious about the affordances of technology in terms of supporting students' construction of dynamic images of angles. We wonder whether showing students continuous, dynamic cases, as opposed to separate, static cases, impacts their angle meanings. We also call for future researchers to explore how dynamic diagrams, compared to static diagrams, draw students' attention to different components of angles (e.g., vertex, ray pair, and arc), leading to their construction of different angle meanings. Incorporating an eye-tracking technique may be helpful to capture students' trajectories of attention in these contexts.

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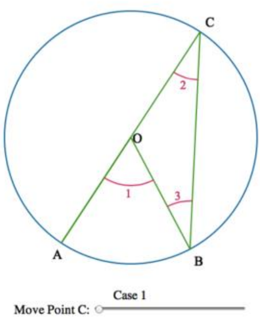
Appendix

The following four figures are snapshots of the full presentation for the dynamic group. The presentation for the static group will be similar but without the slider.

Figure A1

Dynamic Presentation: Case 1

Theorem: **Inscribed Angle Theorem**
Statement: A central angle subtended by an arc is twice the inscribed angle substended by the arc.
Given: Circle O, central angle ∠1, inscribed angle ∠2, substened arc AB
To show: ∠1=2∠2

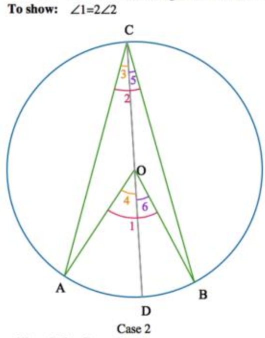


Statements	Reasons
1. ∠1 is an exterior angle of ΔBOC	1. Definition of exterior angle
2. ∠1=∠2+∠3	2. Exterior angle theorem
3. OB=OC	3. Radius of Circle O
4. ∠2=∠3	4. Isosceles triangle theorem
5. ∠1=∠2+∠3 =2∠2	5. By algebra

Figure A2

Dynamic Presentation: Case 2

Theorem: **Inscribed Angle Theorem**
Statement: A central angle subtended by an arc is twice the inscribed angle substended by the arc.
Given: Circle O, central angle ∠1, inscribed angle ∠2, substened arc AB
To show: ∠1=2∠2

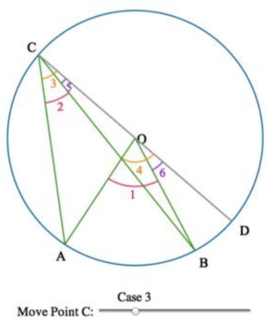


Statements	Reasons
1. Draw a diameter through C and O	1. By construction
2. ∠4=2∠3	2. By case 1 (arc AD)
3. ∠6=2∠5	3. By case 1 (arc BD)
4. ∠1=∠4+∠6 =2(∠3+∠5) =2∠2	4. By algebra

Figure A3

Dynamic Presentation: Case 3

Theorem: **Inscribed Angle Theorem**
Statement: A central angle subtended by an arc is twice the inscribed angle substended by the arc.
Given: Circle O, central angle ∠1, inscribed angle ∠2, substened arc AB
To show: ∠1=2∠2

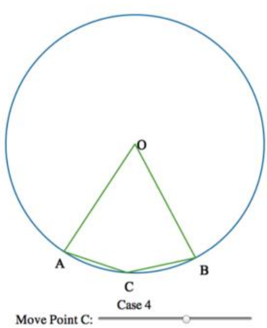


Statements	Reasons
1. Draw a diameter through C and O	1. By construction
2. ∠4=2∠3	2. By case 1 (arc AD)
3. ∠6=2∠5	3. By case 1 (arc BD)
4. ∠1=∠4-∠6 =2(∠3-∠5) =2∠2	4. By algebra

Figure A4

Dynamic Presentation: Case 4

Theorem: **Inscribed Angle Theorem**
Statement: A central angle subtended by an arc is twice the inscribed angle substended by the arc.
Given: Circle O, central angle ∠1, inscribed angle ∠2, substened arc AB
To show: ∠1=2∠2



Statements	Reasons