The Mathematics Educator
2020 Vol. 29, No. 1, 85–111

Analysis of Elementary Preservice Teachers' Identification of Mathematical Problem-Solving Tasks and Anticipated Student Solutions

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According to the Association of Mathematics Teacher Educators (AMTE, 2017), teachers should first try to see problems through their students' eyes, anticipate, understand, and analyze students' varied ways of thinking, and respond appropriately. In this study, we engaged preservice teachers (PTs) in planning to implement problem-solving tasks; explored how they identified problem-solving tasks; and characterized their anticipated responses to those tasks. PTs' competencies and deficiencies in selecting problem-solving tasks and anticipating solutions were described. The results inform the design of more effective interventions in math methods courses to help PTs to plan for implementation of problem-solving in their future teaching.

The National Council of Teachers of Mathematics (NCTM, 2014) and Common Core State Standards for Mathematics (CCSSM; Common Core State Standards Initiative, 2010) recommended problem solving as part of effective classroom planning and instruction. The NCTM published *Principles to Actions* in 2014 with the goal "to fill the gap between the development and adoption of CCSSM and other standards, and the enactment of practices, policies, programs, and actions

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required for their widespread and successful implementation" (p. 4). Engaging students in challenging tasks that involve active meaning making and support meaningful learning is identified as one of the foundational principles of effective teaching by NCTM. The eight effective mathematics teaching practices stipulated in the *Principles to Actions* represent essential teaching skills and a core set of high-leverage practices that are necessary for deep learning of mathematics. One of the eight practices is implementing tasks that promote reasoning and problem solving by allowing students access to the mathematics through multiple entry points, including the use of different representations and tools, and fostering the solving of problems through varied solution strategies. Beyond the policy documents and standards, problem solving has always been a goal of mathematics instruction as a means to encourage high-level student thinking and reasoning, and hence maximize student learning (Boaler & Staples, 2008).

Nevertheless, problem-solving tasks are the most difficult to implement well, and are often transformed into procedural exercises during instruction, particularly in the U.S. classrooms (Stigler & Hiebert, 2004). In this respect, the Association of Mathematics Teacher Educators (AMTE, 2017) and NCTM (2014) recommended writing lesson plans that include anticipated student responses to assigned tasks along with the teacher's own responses. They explained that such practice allows teachers to tentatively plan follow-up questions and instructional moves, instead of supplying students with the answers to the problems. Anticipating student responses was included in the first-ever comprehensive Standards for Preparing Teachers of Mathematics put forth by the AMTE in 2017. According to the AMTE (2017) standard, instead of demonstrating their approaches to a problem or correcting error, teachers should try to see problems through their students' eyes, anticipate, understand, and analyze students' varied ways of thinking, and respond appropriately.

We have explored the importance of anticipating student answers in teaching and learning of mathematics, and found that preservice teachers (PTs) and beginning teachers have difficulty anticipating student solutions (e.g., Ball & Bass, 2000; Hill et al., 2008; Thompson et al., 2011). Hallman-Thrasher (2017) further reported that PTs experienced difficulty in helping and responding to students when they encounter unanticipated solutions to problem-solving tasks. Bruun (2013) specified that elementary teachers' preferred problem-solving strategies to teach were identifying key information (e.g., circle the numbers), looking for clue words (to choose an operation), and drawing a picture. Researchers, hence, emphasized the need to focus on a wide range of problem-solving strategies when educating mathematics teachers. For example, Bruun suggested that both in-service and PTs need to be trained on heuristic problem-solving methods. Hallman-Thrasher suggested that more research is needed to understand how teacher education can support PTs' enactment of problem-solving tasks. Although much research has been done on the importance of problem solving, best practices for teacher educators in enabling PTs to enact problem solving in their classrooms are still being investigated. Particularly, research focused on methods for the development of PTs' abilities to anticipate a variety of student solutions is needed as it is one of the important skills for preparing teachers of mathematics; and one of the difficulties for beginning teachers and PTs. The following research questions guided this study:

- How do preservice elementary teachers identify problem-solving tasks in the context of a mathematics methods course?
- In what ways do preservice elementary teachers anticipate student solutions when planning to implement a mathematical problem-solving task?

Literature Review

Quality of mathematics instruction begins with the quality of mathematical tasks. Student performance gains are greater where tasks are both set up and implemented to encourage the use of multiple solution strategies, multiple representations, and explanations (Stein & Lane, 1996). Students learn more, enjoy mathematics more, and progress to higher mathematics levels in classrooms in which teachers consistently implement tasks encouraging high-level student thinking and reasoning, as compared to classrooms in which the tasks are routinely procedural in nature (Boaler & Staples, 2008). Moreover, tasks that promote problem solving and reasoning are effective tools for identification of mathematical creativity by means of multiple solutions and multiple representations (Leikin & Lev, 2007). Particularly in the context of solving problems, students better comprehend, retain, and transfer knowledge (Jonassen, 2011).

Nevertheless, it is often challenging for teachers to implement high-quality tasks. Literature identified difficulties associated with the implementation of these tasks such as the amount of allocated time for the task (either too little or too much), classroom management, and teachers' experience, knowledge, and beliefs (e.g., Henningsen & Stein, 1997; Remillard, 2005; Stein et al., 1996; Watson & Mason, 2007). One difficulty is that teachers are comfortable with the processes they experienced as learners of math, and thus are challenged by giving up control and becoming a facilitator of student learning (Borko et al., 2000; Crespo & Featherstone, 2006; Kersaint & Chappell, 2001; Smith, 2000). Another difficulty is that teachers are asked to teach content they did not learn in school and to use pedagogy they did not experience as learners due to changes in the mathematics curriculum (Sakshaug & Wohlhuter, 2010). Teachers' knowledge and confidence in mathematics are important factors in whether they adopt a problem-solving approach to teaching the subject matter (Anderson, 2003). For example, in Sakshaug and Wohlhuter's (2010) study, teachers were not comfortable with the mathematics in problem-solving activities, wishing for an answer key or unsure if their work was correct. According to Guberman and Gorev (2015), teachers should have a deep understanding of mathematics to be able to choose or create suitable problem-solving tasks and effectively react to various solution strategies undertaken by their students. Only then, they can conduct meaningful mathematical conversations that help students connect new material to previously learned concepts (Guberman & Gorev). Stylianides and Stylianides (2008) observed a decline in cognitive demands during implementation of high-quality tasks in some seventh grade mathematics classrooms. They associated the decline with teachers' weak content knowledge, or teachers' use of textbooks that did not support them to understand the mathematical goals of tasks and to appreciate the different levels of mathematical appropriateness associated with possible student solutions. However, Stylianides and Stylianides reported that strong mathematical knowledge was not sufficient for successful implementation of those tasks. They advised that teacher preparation and professional development programs have a critical role in equipping teachers with the necessary mathematical and pedagogical knowledge to successfully implement such tasks.

As noted by Bailey and Taylor (2015), participating in problem-solving activities and reflecting on the experience is an important aspect of developing PTs' positive dispositions towards teaching through problem solving. They argued that this is a first step, and recognized a next step to be PTs' enactment of a problem-solving approach in the classroom. Yet, for addressing the issues related to enactment phase, it is crucial to initially investigate how PTs identify problem-solving tasks, because whether they can successfully enact a problem-solving approach depends on their selection of appropriate tasks. Although, problem solving and problem-solving tasks are well elaborated in the standards, teachers have difficulty identifying tasks that promote problem solving. For example, in Kartal's (2015) study, teachers considered any word/contextual problems, or application problems as problem-solving tasks in an effort to support CCSSM practice—make sense of problems and persevere in solving them. However, based on the most recent standards and recommendations by the NCTM and CCSSM, as well as on historical (Hiebert et al., 1997) and relatively recent (Van de Walle, 2007) views, for a task to lend to problem solving: it should not have a prescribed approach, rules, or methods to solve (i.e., problematic); it should allow for multiple entry and exit points; it must include high-level cognitive demand; and it must include a relevant context. Van de Walle (2007) defined *relevant context* as a context that reflects the cultures and interests of the students in the classroom or uses everyday situations. Relevant context can increase student participation and student's use of different problem-solving strategies, and help students develop a productive disposition toward mathematics (Tomaz & David, 2015). To this end, relevant context allows problem solvers to make connections with prior knowledge and engages and motivates students to a greater degree (e.g., Jacobs & Ambrose, 2008; Woodward et al., 2012).

To sum up, selecting and identifying problem-solving tasks and anticipating a variety of student solutions are critical for a successful implementation of problem solving (AMTE, 2017; Bailey & Taylor, 2015; Guberman & Gorev, 2015; NCTM, 2014). Therefore, in this study, we investigated how elementary PTs identify problem-solving tasks, and explored and characterized their anticipated student solutions in an attempt to equip PTs with the necessary mathematical and pedagogical knowledge to successfully implement such tasks. We provided suggestions for how to structure a mathematics methods course to position PTs to consider opportunities to learn how to plan for and use problem-solving tasks.

Theoretical Framework

We adopted Van de Walle's (2007) definition of problem-solving tasks as being problematic, allowing for multiple entry and exit points, high-level in cognitive demand, and involving a relevant context. A task being problematic is closely related to students' prior knowledge and prior exposure. For example, a simple addition/subtraction word problem can be problematic for students, if they were not exposed to such problem before. Therefore, the role of students' prior knowledge and experience was considered in determining whether a task is problematic or not for the purpose of this study. Multiple entry points include the use of different representations, tools (e.g., picture, table, graph, manipulatives, etc.), and varied solution strategies that reveal a range of mathematical sophistication (e.g., finding a pattern, working backwards, using representations or demonstrations with manipulatives, acting out a problem, or using algorithms, etc.). Relevant context is defined as a context that reflects the cultures and interests of the students in the classroom or uses everyday situations.

We adopted Smith and Stein's (1998) taxonomy of mathematical tasks to classify mathematical tasks into lower- and higher-level cognitive demands, and each level is further categorized into two groups as shown in Figure 1. For example:

> Billy is making pies for a picnic at school. He wants to make 12 pies. According to his recipe, Billy needs 2 cans of cherry filling for one pie. How many cans of filling does Billy need to buy from the store?

This allows for multiple entry and exit points (e.g., can draw a picture to represent the situation, create a table, write an equation), and has relevant context (i.e., school picnic). Nevertheless, the level of cognitive demand depends on students' prior knowledge and exposure to multiplication. It requires a lower-level demand, procedures without connections, based on Figure 1, at the third-grade level; whereas it is a higher-level demand task at lower grade levels.

Methods of Inquiry

Sample and Context

The sample of the study consisted of 88 PTs enrolled in a math methods course. The setting for this study was a mid-sized, regional, Midwestern university teacher education program in the United States. Participating PTs were members of the elementary/middle (K–8) or special education (K–12) programs. Each PT was enrolled in a math methods course taught by one of the researchers. The math methods course focused on effective math teaching practices, inductive and developmental ways of teaching math, as well as inclusive methods. This course was the only math methods course the PTs took in their program, typically the semester before their student teaching experience.

During the course, PTs were presented with the definition of a problem-solving task: It involves higher-level cognitive demands—procedures with connections and/or doing mathematics as characterized by Smith and Stein (1998)—has multiple solution pathways that are not immediately known by the students, and has relevant contexts. PTs learned how to distinguish a problem-solving task from an exercise, identified given tasks as problem-solving tasks or exercises, and modified given exercises into problem-solving tasks. PTs were shown and discussed lesson plans for a planned problem-solving task that included an array of anticipated student responses. For example, they were engaged in the "candy jar" task (Smith et al., 2005):

> A candy jar contains 5 Jolly Ranchers and 13 Jawbreakers. Suppose that you have a new candy jar with the same ratio of Jolly Ranchers to Jawbreakers that Ms. Pascal had but it contains 100 Jolly Ranchers. How many Jawbreakers would you have?

Figure 1
The Task Analysis Guide

Lower Level Demands	**Memorization** • Involve either reproducing previously learned facts, rules, formulae or definitions OR committing facts, rules, formulae or definitions to memory. • Cannot be solved using procedures because a procedure does not exist or the time frame in which the task is being completed is too short to use a procedure. • Are not ambiguous. Such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated. • Have no connection to the concepts or meaning that underlie the facts, rules, formulae or definitions being learned or reproduced.
	Procedures Without Connections • Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task. • Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • Have no connection to the concepts or meaning that underlie the procedure being used. • Are focused on producing correct answers rather than developing mathematical understanding. • Require no explanations or explanations that focus solely on describing the procedure that was used.

Higher Level Demands	**Procedures With Connections** • Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • Suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps develop meaning. • Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. • Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.
	Doing Mathematics • Require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example). • Require students to explore and understand the nature of mathematical concepts, processes, or relationships. • Demand self-monitoring or self-regulation of one's own cognitive processes. • Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • Require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

Note. Adapted from "Selecting and Creating Mathematical Tasks: From Research to Practice," by M. S. Smith and M. K. Stein, 1998, *Mathematics Teaching in the Middle School*, *3*(5), p. 348. Copyright 1998 by the National Council of Teachers of Mathematics, Inc. Reprinted with permission.

First, PTs were asked to work on the problem in groups and anticipate as many different student solutions as possible. Second, one group shared their anticipated solutions on the board, and other groups added their solutions if different than the presented ones. In this way, PTs discussed an array of possible solutions that emerged from the class, such as cross multiplication, factor of change, and scaling up in tables. Then, the instructor presented all other possible solutions that were not anticipated by any of the groups, such as unit rate and incorrect additive. Hence, for the candy jar problem, a total of five different anticipated solutions were discussed, which were incorrect additive, factor of change, unit rate, cross multiplication, and scaling up using a table of values. All anticipated solutions are based on “The Case of Mr. Donnelly and the Candy Jar Task” (NCTM, n.d.). The PTs were then given other problem-solving tasks and asked to anticipate an array of solution strategies. This was followed by an assignment entitled Problem-Solving Task (PSTask) in which PTs were asked to select a problem-solving task as defined in the course, anticipate at least two different student solution approaches, identify difficulties students may have, and plan questions to help students overcome those difficulties. PSTask also asked for other information such as instructional objectives, related CCSSM standards, grade level, and prior knowledge. The information on grade level and prior knowledge was used in determining whether the task is problematic for the targeted student group. For example, PTs were told that if they claim a simple fraction subtraction/addition word problem to be a problem-solving task given that students do not have prior knowledge or experience, then they cannot anticipate any algorithmic and standard solution approaches to the problem (i.e., common denominator algorithm); rather, they should anticipate approaches that use demonstrations with manipulatives or pictures.

Data Collection and Analysis

In this study, we explored the tasks that were identified by PTs as problem-solving tasks; and the ways they anticipated student responses to the tasks in the PSTask assignment. First, whether each selected task was qualified for being a problem-solving task was determined by referring to Van de Walle’s

(2007) definition of the *problem-solving task*—as being problematic, allowing for multiple entry and exit points, high-level in cognitive demand, and involving a relative context. Then, using constant comparative analysis (Corbin & Strauss, 1990), we categorized problem-solving tasks. In order to establish interrater reliability, two researchers coded the data collaboratively, and the third researcher coded the data independently, with an agreement of 95%.

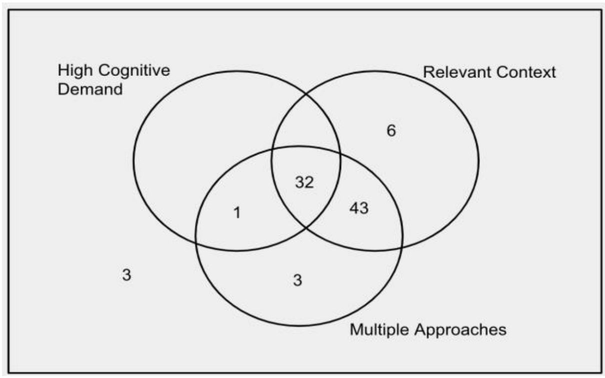
Each PT's anticipated student responses were first coded as representing related, contrasting, or the same strategies as recommended by Stein et al. (2008). Then, characteristics of anticipated student responses were further explored using constant comparative analysis. One researcher analyzed the data on anticipated student responses, developed codes while examining the work of the PTs, and adjusted codes accordingly as new characteristics were noticed. This iterative process continued until all data were coded consistently by all three researchers.

Results

Each of the 88 PTs selected or designed one problem solving task, and anticipated two student solutions for their task. As displayed in Figure 2, 36% (32/88) of the PTs selected or designed tasks that involve high cognitive demands, have relevant context, and allow for multiple approaches, and hence are problem-solving tasks. However, 64% (56/88) of the PTs' tasks were not problem-solving tasks because they lacked one or more of the required features.

Six of the 56 tasks—that were not problem-solving—had relevant context, but failed to involve high cognitive demands according to Smith and Stein's (1998) task analysis guide. Also, these six tasks did not allow for multiple approaches by requiring students to use a specific representation and/or strategy (e.g., create an equation, use your making 10 strategies, etc.).

Figure 2
Venn Diagram for PTs' Tasks in Relation to Three Components of Problem-Solving Tasks



There were 43 tasks that had relevant context and allowed for multiple approaches, yet required lower cognitive demands. In 40 of those low-cognitive-demand tasks, the required level was procedures without connections, which required an algorithmic thinking and/or use of procedures that were evident from prior instruction or experience. In 29 cases, these low-cognitive-demand tasks required students to practice basic addition, subtraction, multiplication, or division facts. For example, one PT proposed Item 3 in Table 1, which is a typical word problem requiring a known procedure (multiplication) to solve at the targeted grade level. Three of the 43 low-cognitive tasks involved memorization level.

Table 1
Examples of Tasks with Various Components of Problem-Solving Tasks

Item	Components of problem-solving tasks evidenced	PTs[a]	Example
1	High cognitive demand, multiple approaches and relevant context	32	"I have two types of boxes one hold 10 cookies and one box holds 5 cookies. If I have 75 cookies total how many boxes of 10 cookies do I have and how many boxes of 5 cookies do I have?" (Grade 2)

2	High cognitive demand and multiple approaches	1	"1. Group these shapes together in like categories. 2. Visually recognize which shapes are the same." (Students are given a diagram of various shapes: 2 triangles, a rhombus, a trapezoid, a square, a regular pentagon, a concave pentagon, a regular hexagon, a hexagon, and two cubes; Grade 2)
3	Relevant context and multiple approaches	43	"Billy is making pies for a picnic at school. He wants to make 12 pies. According to his recipe, Billy needs 2 cans of cherry filling for one pie. How many cans of filling does Billy need to buy from the store?" (Grade 3)
4	Multiple approaches only	3	"What factors make up the number 56? Show your work and explain your answer." (Grade 4)
5	Relevant context only	6	"The school library started with 63 books. In the morning, 28 books were checked out and, in the afternoon, 10 more were checked out. How many books (b) did the library have at the end of the day? Write the equation as shown in the first two problems and write your answer." (Grade 2)
6	None	3	"Draw two different shapes that have areas of 8. Label the measurement of each side. Draw two different shapes that have perimeters of 8. Label the measurement of each side." (Students are given gridlines to guide their drawings; Grade 3)

[a]Number of PTs with these tasks

As depicted in Figure 2, three of the PTs selected tasks that lacked both context and high cognitive demand. These tasks were simply exercises, such as "15 – 11 = ?" which can be answered in a variety of ways (i.e., there was not any specified strategy). A student could use mental math, counting on, or use manipulatives to model decomposing. There was only one task (see Item 2 in Table 1) that involved high cognitive demand, and allowed for multiple approaches, yet identified as “not a problem-solving task” for lack of relevant context.

As shown in Figure 2, three tasks failed to satisfy any of the three components of a problem-solving task. Figure 2 also indicated that 60% (52/88) of the PTs selected tasks that involved relevant context and or allowed for multiple approaches, yet failed to be a problem-solving task, because of lack of high cognitive demand. These results show that PTs had difficulty with selecting tasks that involve higher levels of cognitive demands—procedures with connections or doing mathematics.

Another pattern that emerged from coding of the tasks was that 36% (32/88) of the PTs’ selected tasks were typical word problems (i.e., exercises presented in context, which required using basic skills, procedures, or rules). Furthermore, four major categories emerged for the types of tasks: (a) calculating, (b) making decisions, (c) identifying, and (d) generating. Calculating tasks represented 68% (60/88) of the total; students were required to perform calculation(s) from a given situation, or symbolic or pictorial representation. The following is an example of such a problem-solving task:

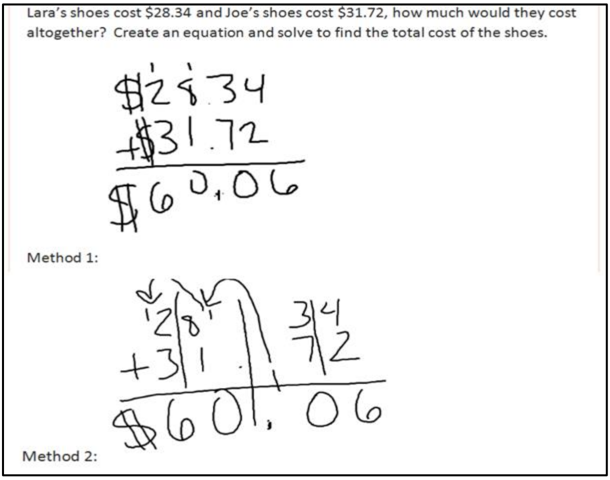
> Sarah is planning her day. This morning, she can meet with a teacher, clean the gutters, or get a haircut. For lunch, she can have a sandwich or pizza. This afternoon, she can shop for groceries or volunteer at the library. Given these choices, how many different combinations does Sarah have to choose from?

Only 15% (13/88) of the tasks presented options and asked students to make a decision. For example, one problem asked students to decide which of the two strings of lights of the same length was untangled for the longest length, deciding between one with a knot 11/12 of the way and another with a knot 7/8 of the way. Another problem asked students to decide “which pair of jeans is the better deal? How do you know?” based on a given price and sale information. Identifying tasks represented 13% (11/88) of the total; students were asked to identify a pattern and then calculate a value for the pattern, to list possible combinations of numbers or objects to reach a certain sum, or to categorize shapes. Only 3% (3/88) of the tasks were generating tasks. For example, one task asked students to create a word problem, as well as a solution method, with a real-life context for a given equation. Another task asked students to generate a method for finding volume in the given problem context.

Anticipated responses were coded in two different ways. First, each PT's anticipated responses were coded in relation to each other into three categories: same, related, and contrasting strategies (Stein et al., 2008). Second, all anticipated solutions were coded by the type of required understanding into two categories: procedural and conceptual. Figure 3 shows an example of anticipated responses that were coded as the same strategies. Both anticipated responses used the process of carrying the ones and tens values to the next column to the left.

Figure 3

Anticipated Responses Coded as the Same Strategies



For a task that required addition/subtraction, the two anticipated responses given in Figure 4 were coded as related strategies. The two strategies were somewhat different, as one utilized symbolic representation and the other one utilized pictorial representation along with counting, yet the underlying mathematical idea was subtracting 18 and 3 from 24 in both.

Figure 4
Anticipated Responses Coded as Related Strategies

a. Students could solve the problem by making it into an equation. The equation should be 24-18-3=? The question mark indicates that the unknown is in the results position. The other way students could solve the equation would be by drawing pictures. Students could draw 24 pictures, first crossing out 18 and then crossing off 3 more, leaving the remaining pictures as their answer. Both approaches should result in the answer being 3.

On the other hand, Figure 5 shows contrasting strategies to a problem-solving task. One strategy used equations to represent the situation (i.e., algebraic thinking); another strategy used table of values and counting. The results showed that 23% (20/88) of the PTs provided strategies that were essentially the same, 48% (42/88) provided related strategies, and only 29% (26/88) provided contrasting strategies.

Another characteristic of anticipated solutions was that a solution could exhibit procedural fluency or conceptual understanding, or neither, or both. An anticipated procedural solution makes use of general procedures, prescribed rules and or formulas, and standard algorithms with no connections to the meanings in context that underlie the procedure that was used. An anticipated solution that exhibits conceptual understanding involves representations of the problem situation and/or non-algorithmic thinking. These definitions were based on NCTM's (2014) assertion of procedural fluency as "the meaningful and flexible use of procedures to solve problems" and conceptual understanding as "the comprehension and connection of concepts, operations, and relations" (p. 7). For example, anticipated responses in Figure 3 were coded as procedural only, as both solutions only make use of standard algorithms.

Figure 5
Anticipated Responses Coded as Contrasting Strategies

Erin and Conner want to raise money for the local animal shelter. To do this, they decide to sell homemade puppy chow. Erin's dad gives her $11 for supplies and ingredients, and Conner's mom gives him $9. Conner decides to sell his puppy chow for $2.50 per bag. Erin sells her puppy chow mix for $4 a bag. They each want to raise $25. How many more bags of puppy chow does Conner need to sell to raise the same amount of money as Erin?

Approach #1: Set up equations....

Erin: $4x - 11 = 25$ Conner: $2.5x - 9 = 25$

....And solve for *x*:

Erin: $4x - 11 = 25$ Conner: $2.5x - 9 = 25$
+11 +11 **+9 +9** *(add expenses to anticipated profit)*

$4x = 36$ $2.5x = 34$
/4 /4 **/2.5 /2.5** *(divide by the cost of each bag)*

$\boldsymbol{x = 9}$ $\boldsymbol{x = 13.6}$ (round up to **14**) *(x is the amount sold)*

14 - 9 = 5

Approach #2: Scaling up:

~~28-11=17~~ ~~32-11=21~~ 36-11=25 (subtract the expense amt)

E	4	8	12	16	20	24	28	32	**36**					
C	2.5	5	7.5	10	12.5	15	17.5	20	22.5	25	27.5	30	32.5	**35**

~~25-9=16~~ ~~27.5-9=18.5~~ ~~32.5-9=23.5~~
~~30-9=21~~ 35-9=26

E: 36 - 11 = 25

C: 35 - 9 = 26

Then, students physically count how many spaces are filled.

Figure 6 contains an example of a task and anticipated solutions that show conceptual understanding with no procedural fluency. Both counting fingers or pictures involve use of representations, and hence connections to the meanings in the M&M context. Using representations along with the procedure of counting is not considered as a procedural solution in this problem, because PTs' emphasis was on the procedure of addition operation as evident in the Task 2 statement which includes words such as "plus," "plus another," and "sum."

Figure 6
Anticipated Responses Coded as Conceptual Understanding

Problem-Solving Task:
Task 1: Carter had 7 M&Ms, Mandy had 6 M&Ms, and Carly had 4 M&Ms. How many M&Ms did they have all together? Can you show me using the M&Ms on your table?
Task 2: Can you add 3 green M&Ms plus 6 red M&Ms plus another 5 blue M&Ms, and read the sum for me?
Task 3: Can you write the number of M&Ms you see here?
Anticipated Solutions:
One approach children could use to solve this problem is just by using their fingers to count. Another approach would be to draw pictures. These approaches are both similar in the way that the student can have a visual of something being added.

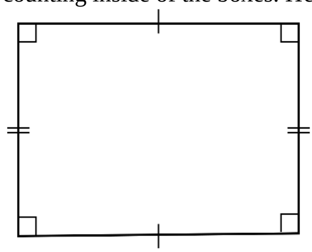
The task shown in Figure 7 included an anticipated solution using the formulas that displayed both procedural fluency and conceptual understanding, and a solution of drawing on grid paper and counting the squares and borders that displayed conceptual understanding only. The majority (76%) of the PTs were able to anticipate solutions that involved both conceptual and procedural understanding, 19% of the anticipated solutions involved only conceptual, and 5% involved only procedural understanding.

Figure 7
Anticipated Responses Coded as both Procedural Fluency and Conceptual Understanding

Problem-Solving Task:
Part A. Ms. Kressin wants to build a pool in her back yard. She has three different options to choose from, but does not know which is the biggest one to choose from. The options are: 20 feet by 12 feet (Option A), 10 feet by 2 feet (Option B), and 20 feet by 6 feet (Option C). Find the area of each option using your own method. Show your work. Which would be the best option and why?
Part B. Ms. Kressin forgot that she needs a fence to go around the pool. The fence will cost $1.50 per foot. Which option would be the cheapest? Show your own using your own method.
Anticipated solutions:
Solution 1: Student 1 uses the area formula length times width using drawings of each and labeling each side of the square as what it was in the

problem, which is a = l x w. They labeled each length and width as needed. They found that Option A was 240 feet, Option B was 20 feet, and Option C was 120 feet. The best option would be Option A. For the next part, they used the formula for perimeter, which is p = 2(l + w). The perimeter was 64 feet for Option A, 24 feet for Option B, and 52 feet for Option C. After, they multiplied each of the feet by $1.50 to see how much it would cost. Option A would cost $96, Option B would cost $36, and Option C would cost $78. They wrote that the least expensive option would be $36 dollars. They used the picture below each time to write out the length and width.

Solution 2: Student 2 is given grid paper. The student knows that anything inside the rectangle is the area. He counts however many boxes needed to for each and finds the area by counting inside of the boxes. He comes to the same answers as Student 1. For the perimeter, the student knows that the sides around the rectangle are the same. For Option A, he decides to add 20 + 20 + 12 + 12. He can do this with each option. He then takes each perimeter and multiples it by $1.50 to get the same answer.



Finally, we noticed that anticipated solutions may correspond to processes recommended in the related standard cited by the PTs for the PSTask. For example, the task in Figure 6 had anticipated solutions that aligned with processes suggested in the relevant standard in the operations and algebraic thinking domain (see Figure 8). Note that the standard suggested using objects (the student’s fingers) or drawings to solve the problem; therefore, such actions are related to the standard. The area and perimeter problem presented in Figure 7 referenced the standard in the measurement and data domain that is provided in Figure 8. This standard in Figure 8 requires a process, namely the application of formulas. The application of formulas is seen in the first anticipated student solution (using perimeter and area formulas) but not in the second solution (drawing on a grid).

A particular process or a selection of processes were recommended by the standards for 56% (49/88) of the PTs’ tasks. The process from the relevant standard was used by 51% (45/88) of the PTs; 30 of those 45 PTs submitted anticipated solutions that displayed both procedural fluency and conceptual understanding. Responses that were related strategies were anticipated by 20 PTs who used a suggested process from the standards, 12 PTs anticipated responses that were the same strategy, and 13 PTs anticipated responses that were of contrasting strategies. In summary, the resulted types of tasks and anticipated solutions are illustrated in Figure 9.

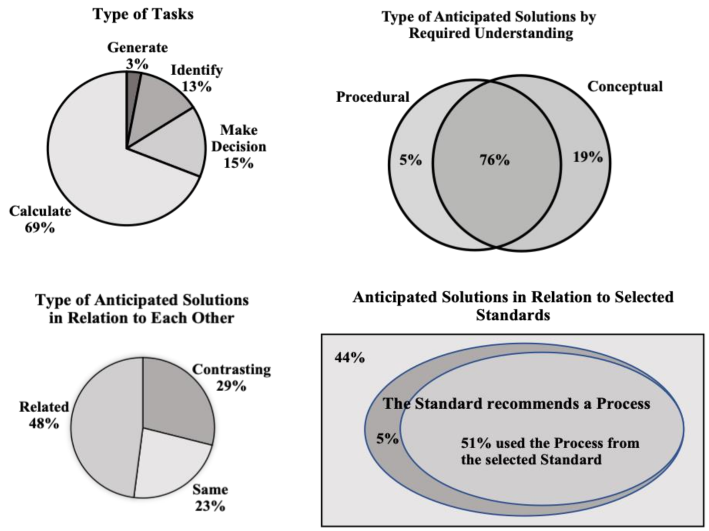
Figure 8

Sample CCSSM Standards Guided PTs' Anticipated Solution Strategies

CCSS.MATH.CONTENT.1.OA.A.2. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem. CCSS.MATH.CONTENT.4.MD.A.3. Apply the area and perimeter formulas for rectangles in real world and mathematical problems. For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor. (Common Core State Standards Initiative, 2010)

Figure 9

Summary of the Types of Tasks and Anticipated Solutions that Emerged from PTs



Discussion

The majority (60%) of the PTs had difficulty with selecting tasks that involve higher cognitive demands and they struggled to move beyond the familiar concept of "problem" as a word problem. Word problems often require limited cognitive demand with little ambiguity, and hence are at the procedures-without-connections level (i.e., one of the lower-level demands). In contrast, it seems that the PTs conceived word problems as real-life problems, and hence believed that they required deeper levels of understanding and higher cognitive demands. Their difficulty with selecting tasks that involve higher cognitive demands, as well as their vision of problem-solving tasks as typical word problems explains the high frequency (68%) of calculating problems rather than those involving making decisions, identifying, or generating. Especially, the scarcity (15%) of decision-making problems is concerning, as decision making in everyday life, in the workplace, and in our democratic society have been championed in recent mathematics education reform documents as a core component of teaching and learning mathematics.

As evidenced by PTs' proposed problem-solving tasks, many PTs in this study believed that word problems should be used to practice procedures that have been learned previously, rather than seeing word problems as potential problem-solving tasks in which the students do not already know, but determine themselves, the solution pathways that will lead to success. For example, one student proposed for the problem-solving task at the second-grade level:

> The school library started with 63 books. In the morning, 28 books were checked out and, in the afternoon, 10 more were checked out. How many books (b) did the library have at the end of the day? Write the equation as shown in the first two problems and write your answer.

This problem could be modified to be a problem-solving task with two changes: removal of the directions to write an equation, and use of the problem before students had developed subtraction skills. If the students had not encountered subtraction, this could be seen as a higher-level cognitive task with an unknown solution path, and students could have solved this using manipulatives, drawing diagrams, or any other solution path they could envision. But, because this was intended for students who know how to subtract, and refers to two previous problems that model equation writing, students were applying a known procedure to the problem, which is a lower cognitive demand. Many of the PTs (58% or 51/88) provided tasks that required students to practice previously learned procedures. And 12 of the 51 cases included the solution path in the problem instructions. This result might suggest that the PTs were reluctant to or did not know how to engage their students in productive struggle, so they made sure their students had a clear path to the problem solution.

It is important to note that 23% of the PTs provided anticipated strategies that were essentially the same strategies. More specifically, the PTs considered that carrying out the same procedures in different order or in a somewhat different way made up different strategies. The majority of the PTs came up with related strategies, which involved the same mathematical idea using different representations. Only 30% of the PTs provided contrasting strategies for their task. These results indicate that PTs need to engage in activities, in methods courses, in which they discuss what makes two strategies different, the underlying mathematical idea that is depicted in a given strategy, and what other mathematical ideas could be used for the task being discussed.

Our results showed that only four PTs anticipated solutions that did not display conceptual understanding. Solutions that displayed only conceptual understanding were anticipated by 17 PTs, while the remaining PTs provided at least one anticipated solution that displayed both procedural fluency and conceptual understanding. PTs who attempted to create problem-solving tasks, whether they were successful or not, realized that developing a solution strategy relied on having a conceptual understanding of the problem being posed. This conclusion is supported by Jonassen (2000), who surveyed the literature and found that knowledge of how concepts in a domain are interrelated is essential to success in problem solving.

CCSSM content standards that included the suggestion for a process or processes were used by 56% (49/88) of the PTs. Interestingly, in most cases (45/49), the selected standards suggestions influenced the PTs and hence anticipated solutions that are outlined in the standard. Given that standards require both procedural and conceptual ways by means of variety of representations, as exemplified in Figure 8, PTs who used such standards anticipated both procedural and conceptual solutions. The results suggest that discussion about a process or processes outlined in the CCSSM content standard corresponding to a particular problem-solving task might provide a foundation for developing elementary school PTs' ability to anticipate student responses that display both procedural fluency and conceptual understanding. Therefore, we recommend that mathematics educators include such practice into mathematics methods courses.

Future Directions

Results seen in this study will orient future data gathering and analysis. Noting the weaknesses of problem-solving tasks proposed by PTs guides future research on the current project, as well as other researchers and teacher educators. Realizing what PTs understand and where they struggle to identify appropriate problem-solving tasks allows researchers to plan next steps more purposefully. Exploring PTs' competencies and deficiencies in anticipating solutions may inform the design of more effective interventions in methods courses to better help PTs improve their collection of anticipated solutions.

In the next phase of the larger study, PTs in this study will receive peer feedback on their selected PSTask and they will undertake the problem-solving task assignment a second time during each semester. The results of those assignments will be analyzed to determine how the PTs' identification of problem-solving tasks changes over time with re-designed interventions based on the findings of the current study, and after feedback from the instructor. In addition to collected assignments, PTs will be interviewed about how they see their chosen task as fitting the definition of a problem-solving task. Researchers will attempt to relate instruction and feedback received by the PTs to changes in problem-solving task identification and array of anticipated solutions. In addition, PTs' anticipated erroneous solutions and their ways to plan for questions to help students overcome those faulty reasonings will be investigated. Ongoing analysis of PTs' attempts at anticipating a variety of student solutions as well as difficulties will provide a deeper understanding of PTs' ability to anticipate what students know and what misconceptions students have. Researchers will also make plans to integrate the effective mathematics teaching practice "support productive struggle" into the problem-solving module of the methods course, to address PTs' tendency to provide practice problems and direct instructions that leads to a decrease in the required level of the cognitive demand.

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