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Pre-Service Secondary Mathematics Teachers' Opportunities to Learn Reasoning and Proof in Algebra

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This study examined opportunities provided for preservice secondary mathematics teachers (PSMTs) to learn reasoning and proof in algebra from the perspective of college instructors. We analyzed interview transcripts of 15 course instructors recruited from three teacher education programs in the United States. We examined the reported opportunities provided for PSMTs to engage in proving-related activities, including making conjectures, investigating conjectures, developing arguments, evaluating arguments, and disproving by using counterexamples. We also analyzed instructional strategies reported by the instructors. We found the inconsistency between instructors' perceptions of the importance of reasoning and proof in algebra and instructor-reported opportunities to learn. Findings also indicated that developing arguments was reported the most frequently. In addition, instructors reported more pedagogy-focused general teaching strategies than proof-specific teaching strategies.

School algebra emphasizes the "relationship among quantities, including functions, ways of representing mathematical relationships, and the analysis of change" (National Council of Teachers of Mathematics [NCTM], 2000, p. 37). Algebra is a strand of the school mathematics curriculum and should be accessible to all pre-K-12 students (NCTM, 2014). Algebra is considered the gatekeeper for higher mathematics (National Mathematics Advisory Panel, 2008). Thus, having equal access to algebra learning is crucial in building social equity (Moses & Cobb, 2001). The importance

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of algebra has been stressed at both the national and state levels. The *Common Core State Standards for Mathematics* advocate for fostering students' algebraic thinking ability in grade levels as early as kindergarten (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Half of the U.S. states require students to complete at least Algebra 1 for their high school diplomas (Teuscher et al., 2008).

Despite the importance of algebra in school mathematics, research shows a variety of difficulties that both secondary students and teachers have encountered in algebraic reasoning and proof. A largescale nationwide survey about the conceptions of proof of high-attaining 14- and 15-year-old students discovered that the majority of the students were unable to construct valid (logically correct) proofs in the domain of number and algebra and preferred to use empirical everyday language rather than algebraic symbols for their proof constructions (Healy & Hoyles, 2000). Similarly, by studying 10 high school algebra students' justifications for simple statements about combining odd and even numbers, Edwards (1998) found that most of the students tended to reason empirically. When prompted to provide further explanations, only three students were able to provide relatively coherent arguments without using standard algebraic notation (two arguments were based on visual representations and one was informal and incomplete). In a similar context, Miyakawa (2002) analyzed proof constructions of 37 ninth graders and found that a lack of general mathematical knowledge (not only proof specific knowledge) can also contribute to students' difficulties with proofs.

Secondary teachers have had difficulties emphasizing the importance of proof in mathematics teaching and learning and have demonstrated insufficient knowledge about proof-related concepts. A survey of 78 secondary mathematics teachers about their attitudes and beliefs on proving in the mathematics classroom found that although most of the teachers agreed with the importance of proving in developing students' logical thinking skills, the majority of them believed that most rules and theorems used in a high school mathematics classroom should be proven by teachers, not students (Kotelawala, 2009). By analyzing the written responses of 95 pre-service elementary and secondary school mathematics teachers to specifically designed mathematical induction tasks, Stylianides et al. (2007) found both groups of pre-service teachers had difficulties with multiple aspects of mathematical induction, including the basic step, the inductive step, and the possible values in the truth set. Similarly, Bleiler et al. (2014) noticed that pre-service secondary mathematics teachers (PSMTs) had insufficient understanding of the proof by contradiction method while studying 31 teachers' validations of proof. In a case study investigating 22 PSMTs' understanding of proof through concept map building activities, Öçal and Güler (2010) found evidence indicating PSMTs' incomplete knowledge and insufficient mental construction of proof related concepts (e.g., not able to apply the theorem in the proof process in the constructed concept map). Their findings are consistent with Knuth's (2002) findings. In his study with 16 in-service teachers' conceptions of proof, that teachers did not view proof as a tool for learning mathematics and held insufficient understanding about the nature and elements of proof.

Teachers' knowledge and belief about proof can have an impact on their teaching practice and thus their students' learning of proof (Beswick, 2012). To begin to address secondary students' difficulties in mathematics learning, it is reasonable to examine how PSMTs are prepared for teaching in the collegiate setting (Tchoshanov, 2011). Research shows the extensiveness and rigor of PSMTs' opportunities to learn (OTLs) mathematics content and pedagogy can affect their readiness to teach mathematics; therefore, it is important to understand what OTLs have been provided for pre-service teachers in their teacher preparation programs (Schmidt et al., 2011). Since limited research has been done to investigate PSMTs' OTLs in algebraic reasoning and proof (Ko, 2010), this study aims to investigate OTLs provided in both mathematics and mathematics education courses offered at three teacher education programs. Our study was part of a larger research project, Preparing to Teach Algebra (PTA), which investigates PSMTs' OTLs about algebra and algebra teaching. In this paper, we focus on the *reasoning and proof* aspect of PSMTs' OTLs about algebra. This study sought answers to the following research questions: What opportunities do collegiate instructors think they provided for PSMTs to learn reasoning and proof in algebra in their mathematics or mathematics education course? What instructional strategies do collegiate instructors recall they employed in teaching reasoning and proof in algebra in those courses?

Literature Review

Opportunity to Learn (OTL)

OTL was initially defined as "the opportunity to study a particular topic or learn how to solve a particular type of problem presented" (Husén, 1967, pp. 162-163). OTL is also widely used in international studies to ensure the validity of the comparison of students' achievement as researchers realized the importance of considering curricular differences when comparing students' achievement across the nations (Liu, 2009). The concept of OTL has been evolving and expanded in multiple dimensions, including content coverage, content exposure (i.e., time spent on the content), content emphasis, and the quality of instructional delivery (Liu, 2009). Besides different aspects of measuring OTL, researchers also developed different methods to describe OTL. These methods include teacher and student surveys (Törnroos, 2005), teacher logs (Fisher et al., 1981), classroom observations (Hiebert et al., 2003), and analyses of instructional materials (Thompson et al., 2012). Though all three international mathematics studies (the First International Mathematics Study, the Second International Mathematics Study, and the Trends in International Mathematics and Science Study [TIMSS]) used teacher questionnaires (Floden, 2002), TIMSS's video-taping method addressed the possible bias of the self-reported survey method through collecting recordings of classroom observations. However, the video studies method had shortcomings and was limited due to its high cost and feasibility.

As stated by Schmidt et al. (2008), "OTL not only is important in understanding K-12 student learning, but it is also likely important in understanding the knowledge base of the teachers who teach them, which then has the potential to influence student learning as well" (p. 736). Studies have revealed the relationship between preservice teachers' opportunity to experience content in the course and their content knowledge and pedagogical content knowledge measured by international assessment (Schmidt et al., 2008; Schmidt et al., 2011). Research also found that preservice teachers' self-reported opportunities to learn general teaching methods were associated with high ratings of enactment of ambitious mathematics practices during their first year of teaching (Youngs et al., 2022). When studying the mathematics teacher preparation programs, OTL involves similar dimensions to those when studying K-12 education, including the content coverage in mathematics, mathematics pedagogy, and general pedagogy and learning experiences with the knowledge, skills, or instructional activities related directly to mathematics teaching (Schmidt et al., 2008). The ideas from the literature guided us to investigate OTL in two dimensions: content and instructional strategies.

The Teaching of Reasoning and Proof

According to the *Principles and Standards for School Mathematics* (PSSM; NCTM, 2000), mathematics education programs should cultivate students' mathematical reasoning ability consistently through K-12 levels and across many contexts beyond geometry. The PSSM K-12 reasoning and proof standards include "recognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; select and use various types of reasoning and methods of proof" (p. 56). These standards are also elaborated on by teacher preparation standards including the *Standards for Preparing Teachers of Mathematics* (Association of Mathematics Teacher Educators [AMTE], 2017) and the *Standards for the Preparation of Secondary Mathematics Teachers* (Rasch et al., 2020). The *Mathematical Education of Teachers II* (Conference Board of the Mathematical Sciences [CBMS], 2012) emphasized that reasoning and proof "should be integrated across the entire spectrum of undergraduate mathematics" (p. 56). The recommendations from the standards and the analytical framework used in studies investigating opportunities to learn reasoning and proof in high school mathematics textbooks (e.g., Thompson et al., 2012) informed the content framework.

Next, we reviewed instructional strategies reported in the literature that could effectively support teaching proof construction. The reviewed strategies are limited to those relevant to our data and informed the development of our framework of instructional strategies. Dreyfus (1999) claimed that mathematics instructors tend to teach proving as polished formalism although the instructors are aware of the zig-zag and informal processes through which new mathematical knowledge is created. Weber (2004) argued that it is widely believed that advanced mathematics courses are taught in a way, referred to as a "definition-theorem-proof" format, which could prevent college students from experiencing authentic proving process as mathematicians experienced. Studies have reported teaching strategies used to engage students in proving activities at the college level (Smith et al., 2010; Weber, 2004). Weber (2004) reported a well-recognized mathematics professor's three styles of teaching proof construction: logico-structural, procedural, and semantic. A *logical-structural* teaching style is demonstrated when an instructor stresses the use of unpacked definitions to begin and end the proof. A *procedural* teaching style includes instructors' efforts to start a proof by writing an incomplete outline to illustrate the proof's structure and complete the proof by filling in the gaps missing from the argument. A *semantic* teaching style describes instructors' actions of presenting an intuitive description of the idea and their use of drawing to investigate the plausibility of the statement. A rigorous proof would follow later. Numerous studies have investigated the significant role examples play in proof construction. Aricha-Metzer and Zaslavsky (2019) investigated the nature of middle school, high school, and undergraduate students' productive and non-productive example-use for proving. Researchers distinguished empirical example-use from generic example-use. An *empirical* example focuses on the specifics without seeing it in a general way and can be used to support making a conjecture. A *generic* example is a specific example presented in a way that can be seen as representing a class of objects and supports the development of the key ideas of the complete proof. Aricha-Metzer and Zaslavsky (2019) reported that 57% of students used examples generically and productively for proving. Leron and Zaslavsky (2013) suggested that using carefully selected generic examples to first construct a generic proof could be an effective strategy to support undergraduates' construction of a formal proof.

Principles to Actions (NCTM, 2014) called on mathematics teachers to pose meaningful questions to access, probe, and advance students' thinking and reasoning. In their case study, Martino and Maher (1999) found that timely and purposeful questioning can stimulate fourth and fifth grade students' reorganization of a solution into a more global justification. The *Principles and Standards for School Mathematics* (NCTM, 2000) emphasized fostering the learning habit of "asking why" because it is "essential for students to develop sound mathematical reasoning" (p. 344). These policy recommendations and research indicate that posing meaningful questions that invite students to justify their claims could be a powerful tool to promote reasoning and proof activities in classrooms. The instructional strategies reviewed in this section are the foundation for our framework of instructional strategies.

Methods

Research Site and Data Collection

We examined the variations and unique opportunities provided in secondary teacher education programs (STEPs) at three contextually different public universities, located geographically convenient to the research team. University A is a Master's degree-granting institution that offers a four-year STEP. Both mathematics and mathematics education faculty members are housed in the mathematics department. University B is a Ph.D. granting institution that offers a five-year STEP. Mathematics and mathematics education faculty members are housed in different departments of two colleges. At University B, the internship includes a full academic year of study and an experience in a selected school. University C is also a Ph.D. granting institution, except it offers a four-year STEP. Similar to University B, the mathematics and mathematics education faculty members at University C are housed in different departments of two colleges. PSMTs at Universities A and C are certified to teach students in grades 5-12; whereas, PSMTs at University B are certified to teach students in grades 7-12. The total required credits for mathematics courses at the three universities are (from A to C): 45, 36, and 46, respectively. The total required credits for mathematics education courses are 15, 17, and 17, respectively.

At each university, the research team worked with a site coordinator to select the required courses. All mathematics courses selected were related to algebra because we take a broader view of algebra as generalized arithmetic, as a study of pattern and function, as a tool for problem solving, and as a way of modeling and understanding physical situations (CBMS, 2001). We interviewed five mathematics or mathematics education instructors at each site (see Appendix 1 for instructors' backgrounds). The algebra-related courses included in our study were seleted to maximize the variety of the algebraic content coverage. The instructors were selected based on their availability. To gain perspectives from various instructors, no instructor was interviewed more than once. Course titles have been standardized (see Appendix 1) to protect universities' anonymity. Each semi-structured interview took 60-75 minutes. In addition to asking general questions about collegiate instructors' backgrounds and the course, we asked groups of questions relevant to the larger research project's themes. We asked the following interview questions pertaining to algebraic reasoning and proof: (1) "How important is reasoning or proof in your course?" (2) "Which aspects of reasoning or proof do you emphasize in this course?" (3) "Which specific problems or activities do you use to teach reasoning and proof related to algebra?" We asked the general questions first and probed with specific questions. After asking Question 2, we presented instructors with the five types of proving-related activities (see analytical framework section for details) and asked if the instructor emphasized each. The instructor was then asked to describe the activities using specific teaching examples. However, we did not provide a similar list of the instructional strategies after asking Question 3 due to the comprehensiveness of teaching strategies and the time constraint of the interview.

Analytical Framework and Data Analysis

Three members of the research team developed two analytical frameworks to answer each research question. The content framework was informed by research literature and constructed before data collection. The content framework aims to capture OTLs for engaging PSMTs in various proving-related activities, including *making conjectures*, *investigating conjectures, developing arguments*, *evaluating arguments*, and *disproving using counter-examples* (NCTM, 2000). The framework of instructional strategies intends to capture the instructional strategies reported by collegiate instructors to promote PSMTs' reasoning and proof. This framework was informed by the literature and the data. We started with the initial instructional strategy categories suggested in the literature and alternated the deductive (using initial categories to code data) and inductive methods (summarizing new categories from data) when analyzing the data (Hatch, 2002). Through an iterative process of comparing the initial and new categories, the initial categories were refined and completed. Therefore, the framework of instructional strategies presented in this paper does not include a comprehensive list of all effective teaching strategies suggested by the literature, but only those mentioned by the instructors. The final framework for instructional strategies has two categories: proof specific teaching strategies and general teaching strategies. We define *proof specific teaching strategies* as the strategies that have been suggested in the literature as effective ways to teach reasoning and proof. The framework for proof specific teaching strategies includes using examples to guide proof construction (Leron & Zaslavsky, 2013), purposefully allowing mistakes (Dreyfus, 1999), asking students to explain why (Martino & Maher, 1999), structuring the proof then filling in details (the procedural style defined by Weber [2004]), and using assumption/definition to guide proof construction (the logico-structural style defined by Weber [2004]). The *general teaching strategy* is general pedagogy that can support teaching any content. The purpose of this paper is not to report general teaching strategies but to contrast them with proof specific teaching strategies. The framework for general teaching strategies includes encouraging students to present and articulate ideas, having students imitate a proof, asking students questions, adjusting lectures based on students' needs, gradually increasing difficulty level, breaking down into steps, peer teaching, presenting materials in an organized fashion, and using puzzles.

Three members of the research team were involved in the coding process to secure the intercoder reliability. Each transcript was coded independently by two researchers. Discrepancies were settled through paired discussions. Through course selection, interview probes, and examination of course materials, our analysis was aimed at excluding non-algebraic opportunities. First, we ensured all the courses selected were related to algebra. In addition, the interviewers asked instructors to only report OTLs related to algebra. Interviewers also probed for detail when instructors mentioned a general opportunity. Furthermore, we examined course materials before interviews to understand the nature of each reported opportunity. The examination of the course materials also reduced the bias of the self-reported data. However, due to the lack of classroom observation data, our analysis is limited in providing a comprehensive description of the quantity and variety of OTLs.

Findings

We first report collegiate instructors' perceptions of the importance of reasoning and proof in algebra in their courses. We examined collegiate instructors' responses to interview question 1, and the numbers in Table 1 represent the number of instructors in each category. Most instructors (11 out of 15) reported that reasoning and proof in algebra play an *extremely important* role in their courses; fewer instructors (4 out of 15) reported that reasoning and proof in algebra was either *moderately important* or *not important* in their courses. Mathematics and mathematics education instructors perceived the role of reasoning and proof in algebra differently. All eight mathematics instructors reported that reasoning and proof in algebra were *extremely important*, compared to four out of seven mathematics education instructors with the same responses.

Table 1
Instructors' Perceptions of Importance of Algebraic Reasoning and Proof in Their Courses

	University A		University B		University C	
Level of importance	Math	Math Ed	Math	Math Ed	Math	Math Ed
Extremely important	3	0	3	1	2	2
Moderately important	0	1	0	0	0	0
Not important	0	1	0	1	0	1

Proof Related Activities

This section presents the findings of the algebraic proof related activities that collegiate instructors reported, captured by the five processes in the content framework. Table 2 presents the number of instructors who reported engaging PSMTs in each reasoning and proof process. The majority of instructors (13 out of 15) reported that they provided some opportunities to engage PSMTs in algebraic reasoning and proof processes. Roughly one third of the instructors reported providing opportunities in each of four processes: making conjectures, investigating conjectures, evaluating arguments, and disproving by using counter-examples. Developing arguments was reported the most frequently by about two-thirds of the instructors and was reported the most frequently in all universities. More instructors at University A and B reported this process than instructors at University C. For the non-proving/disproving processes, including making conjectures, investigating conjectures, and evaluating arguments, similar numbers of mathematics and mathematics education instructors reported providing such opportunities. But for proving/disproving focused processes, such as developing arguments and disproving by using counter-examples, more mathematics instructors reported providing such opportunities. In fact, all mathematics instructors reported engaging PSMTs in developing arguments. The difference between the OTLs provided by different course instructors could be due to their different course foci. For example, mathematics courses usually place a heavier emphasis on constructing proofs than mathematics education courses. We also found mathematics education instructors reported fewer OTLs in algebraic reasoning and proof than mathematics instructors.

Table 2
Opportunities to Engage in Algebraic Reasoning and Proof Processes Reported by Instructors

Reasoning and proof processes	A	B	C	Math	Math Ed	Total
Make conjectures	2	2	1	3	2	5
Investigate conjectures	2	1	1	2	2	4
Develop arguments	3	4	4	8	3	11
Evaluate arguments	2	1	1	2	2	4
Disprove by using counter-examples	2	2	1	4	1	5
Total	11	10	8	19	10	29

We end the section by relating collegiate instructors' reported OTLs with the instructors' perception of the importance of reasoning and proof in algebra. Table 3 shows the number of instructors in each of the OTLs quantity categories (from 5 to 0). These categories represent the number of the algebraic reasoning and proof processes the instructors' OTLs addressed. Five out of 15 instructors reported at least three processes. Only two instructors reported engaging PSMTs in all five processes. Even though 11 instructors indicated that reasoning and proof in algebra were extremely important, seven of them reported only implementing two or fewer processes. Two instructors reported implementing none of the five processes.

Table 3
Instructor's Reported OTLs Addressed in the Five Processes and Perception of Importance of Algebraic Reasoning and Proof

	Number of five reasoning and proof processes reported by each instructor					
	5	4	3	2	1	0
Extremely important/a big focus of the course	2[a]	1	1	1	6	
Moderately important/addressed to some extent			1		1	
Not important/addressed little						2

[a] Two instructors each reported engaging their PSMTs in all five reasoning and proof processes and deemed reasoning and proof as extremely important

Instructional Strategies

In this section, we present instructional strategies collegiate instructors reported. Most teaching strategies reported were general teaching strategies (31 instances) instead of *proof specific teaching strategies* (14 instances), even though during the interviews, instructors were prompted to recall proof specific teaching strategies. Some of the proof-focused teaching strategies, such as using examples to guide proof construction (Leron & Zaslavsky, 2013) were only reported by two mathematics instructors. This finding raises concern about how reasoning and proof in algebra could be effectively taught in these courses. Reporting general teaching strategies is not the focus of our study; therefore, we do not give a detailed account.

Next, we present findings relevant to collegiate instructors' reports of proof specific teaching strategies. Table 4 presents the number of instructors who reported a particular strategy. When comparing across universities, instructors at University B reported the highest instances of proof specific teaching strategies compared to University A and C. We also noticed that the majority of proof specific teaching strategies were reported by mathematics instructors; the only proof specific teaching strategy reported by mathematics education instructors was asking students to explain why. Among all proof specific teaching strategies, asking students to explain why was reported the most frequently by about one third of instructors. Though asking PSMTs to explain their thinking could be considered a general teaching strategy, when instructors focused on explaining “why” rather than explaining “how,” PSMTs were encouraged to provide reasons to support their claims in addition to telling the procedural process. Therefore, we include asking students to explain why in the proof specific teaching strategies.

Table 4
Proof Specific Teaching Strategies Reported by Instructors

	A	B	C	Math	Math Ed	Total
Purposefully allow mistakes	1	2		3		3
Ask students to explain why	1	2	3	4	2	6
Structure the proof and then fill in details	1		1	2		2
Use examples to guide proof construction		2		2		2
Use assumption/definition and conclusion to guide proof construction		1		1		1
Total	3	7	4	12	2	14

Three instructors reported purposefully allowing mistakes as a teaching strategy; however, they used them differently, depending on the extent to which an instructor or a PSMT was involved. We include some examples here to demonstrate instructors’ implementation of (or plan to implement) this teaching strategy. Instead of providing PSMTs with a polished proof as some instructors would do, the instructor of Mathematics Capstone for Secondary Teachers at University B wanted to show the PSMTs the struggles they might encounter when proving. His teaching strategy involved little PSMTs’ participation. He stated, “I start the proof by being the direct proof. Then I find myself against the wall and I tell them: what can we do?…Let us use the reasoning by contradiction.” Instructor of Real Analysis course at University B invited PSMTs into co-constructing proofs. He stated:

> By showing them the common way of thinking which is wrong and doesn’t work now, I show them why. Then they are interested in listening to the answer…I play a lot on their

> psychology in order to get them really motivated for the course and to understand that they are learning something new because what they thought before is not correct.

Similarly, the instructor of the Structure of Algebra course at University A also invited PSMTs to be independent thinkers and to construct proofs with greater participation. He reported:

> I'm having a conversation with them and a person has an idea. I want to follow that idea as faithfully as I can…I might write something down from their ideas that maybe not quite kosher, and I'll ask people: is this agreeable to everybody? And if it's not agreeable, I'm hoping someone will catch it and then we'll fix it. If they don't catch then I will catch it and will fix it…Eventually they have to go home and they're on their own.

Table 5

Instructor Reported Proof Specific Teaching Strategies and Perception of Algebraic Reasoning and Proof

	Number of five proof specific teaching strategies reported by each instructor					
	5	4	3	2	1	0
Extremely important/a big focus of the course			1[a]	3	3	4
Moderately important/addressed to some extent					1	1
Not important/addressed little					1	1

[a] One instructor reported three proof specific teaching strategies and deemed reasoning and proof as extremely important

We end this section by relating collegiate instructors' reported instructional strategies to their perception of the importance of algebraic reasoning and proof in their courses. Table 5 shows the numbers of instructors in each of the teaching strategy quantity categories (from 5 to 0). These categories represent the number of proof specific teaching strategies reported by each instructor. Even for the instructors who reported reasoning and proof in algebra as extremely important in their courses, only four out of 11 reported using two or three proof specific teaching strategies; most of them reported only one or no proof specific teaching strategy. Four instructors who deemed proof and reasoning in algebra as extremely important reported no proof specific teaching strategies, but all reported some form of general teaching strategies.

Discussion and Conclusion

Though several studies have reported on secondary mathematics teachers' engagement in proof related reasoning, most studies examined activities in one particular course (e.g., Bleiler-Baxter & Pair, 2017). This study aims to provide a broader picture of opportunities offered at the level of teacher education programs. We investigated 15 collegiate instructors' recollections of opportunities to engage PSMTs in reasoning and proof in algebra in both mathematics and mathematics education courses in three teacher education programs. We noticed that developing arguments was reported the most frequently by the instructors. The other four processes, including making and investigating conjectures, evaluating deductive arguments, and disproving by using counterexamples, were reported less frequently. Professional organizations (e.g., AMTE, 2017) call for greater attention to providing future teachers opportunities to engage in all five processes. Our data analysis implies that some of the courses in our study may have missed some of the important reasoning and proof processes. We also noticed inconsistency between instructors' perceptions of the importance of reasoning and proof in algebra and reported OTLs. Even for the 11 instructors who claimed that reasoning and proof in algebra were extremely important, only four reported at least three reasoning and proof processes. This finding suggests that support needs to be provided to instructors who are unaware of the importance of engaging PSMTs in all five processes. We acknowledge the limitation of self-reported data but want to emphasize that instructors were shown the five processes on a handout during the interviews, which helped minimize the chance of forgetting to report some of the processes.

We also noticed that most of the reported instructional strategies are general teaching strategies instead of proof specific instructional strategies. Instructors' demonstration of the proof specific instructional strategies may help PSMTs learn the metacognitive strategies that could guide PSMTs' progress when they construct proofs (Harel & Sowder, 2007). Researchers (e.g, Aricha-Metzer & Zaslavsky, 2019; Leron & Zaslavsky, 2013) have documented the importance of exploring examples in proof constructions. However, only two instructors mentioned using examples to guide proof construction. Prior literature suggests engaging students in reasoning and proof as a process, rather than as a completed product (Harel & Sowder, 2007; NCTM, 2000). But purposefully allowing mistakes was reported by only three instructors. The lack of opportunities to experience proving as a zig-zag process may lead to PSMTs' narrower view of mathematical proof (Hanna, 2000). Studies (e.g., Ball, 1988) have found that new teachers are likely to teach mathematics in the way they were taught despite their training to reform their teaching practices. We also noticed inconsistency between instructors' perceptions of the importance of reasoning and proof and their instructional strategies. Even though all 11 instructors claimed that reasoning and proof in algebra were extremely important, seven reported using only one or no proof specific teaching strategies. Note that we did not provide a handout for the instructional strategies to facilitate instructors' recall. Therefore, instructors may have forgotten to report certain OTLs. If the instructor did not report a strategy, it does not necessarily mean the instructor did not implement it. However, opportunities reported from the instructors' perspectives most likely reflect instructors' emphasis on teaching and may play a significant role in PSMTs' learning. Future research should expand the scope of our study by including classroom observation data or student interviews.

We summarize the limitations of the study. First, due to the limited number of participating universities, our observations of the differences among the teaching strategies at different universities might be limited. Future studies could look into the role of the department or program setting in instructors' conceptualization of reasoning and proof in algebra with a larger sample size. Second, the selected courses did not include all the required courses for PSMTs and were not uniform across universities. With limited time and resources to collect data, we decided to include a variety of mathematics and mathematics education courses. We attempted to provide a broader picture of the OTLs provided in various teacher preparation programs. Furthermore, due to the lack of classroom observation and student data, the instructor interview data analyzed in this study may have reflected incomplete knowledge of the OTLs to a certain degree. We also acknowledge that even with the efforts to only include OTLs for algebraic reasoning during the data collection and analysis processes, we may not have completely eliminated OTLs for general reasoning. Future studies are encouraged to expand our research and investigate how teacher preparation programs can prepare PSMTs to teach reasoning and proof in algebra.

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