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Young Students Exploring Measurement Through Problem Solving and Problem Posing

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The empirical data in this study are from a series of two lessons on measurement implemented in seven classes with 119 six-year-old students in Sweden. Both problem solving and problem posing were shown to be important in early mathematics when students in this study worked on one problem-solving task and one problem-posing task on measurement. As there are few studies specifically on problem posing in early mathematics and on young children's understanding of measurement, this study adds knowledge of value for both teachers and researchers. In the study, paper-and-pen work from the students was analysed together with interviews conducted after the students had worked on the two tasks. When solving the task on measurement, the students discerned shape, size, distance, and number as mathematical aspects of measurement. When asked to pose a similar task, only size and number reoccurred as mathematical aspects of measurement. However, other features from the problem-solving task reoccurred in the posed tasks: similar drawings were used in combination with questions on measurement as the mathematical content.

Introduction

When defining key competencies for lifelong learning, the European Community emphasizes as one competence the "ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations" (European Communities, 2007, p. 8). In line with this, problem solving as well as designing and formulating problem-solving tasks are

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emphasized in the Swedish primary school curriculum (National Agency for Education, 2017). The curricula for early mathematics in other Nordic and European countries contain similar emphasis (e.g., The Stationery Office, 1999; Utdanningsdirektoratet, 2013). As both problem solving and problem posing have gained territory in early mathematics curricula, research on problem solving in early mathematics has increased (Cai et al., 2015). Similarly, problem posing has long been studied in mathematics, but still few studies on problem solving in early mathematics exist, especially explorative studies on designing and testing problem posing within interventions (Cai & Hwang, 2019; Palmér & van Bommel, 2020; Singer et al., 2013). This is especially true when considering problem posing in early mathematics (Palmér & van Bommel, 2020).

According to Lesh and Zawojewski (2007), a task becomes a problem-solving task when the individual who is to solve the task has to develop strategies and/or knowledge not yet obtained in order to be able to solve it. The purpose of problem solving in mathematics education is twofold: to strengthen students' ability to learn mathematical content and to develop their ability to solve problems, an important skill in the 21st century (European Communities, 2007). However, even though problem solving in mathematics is emphasized in steering documents for teaching in many countries (Lesh & Zawojewski, 2007), few early childhood education programs provide mathematically challenging activities (Cross et al., 2009; Perry & Dockett, 2008). Further, problem posing is an alternative teaching approach to foster students' problem-solving abilities (Chen et al., 2013). The idea is the (mathematical) properties of a problem-solving task may become clearer when students are to not only solve the task, but are also asked to pose similar questions. However, the responsibility for problem posing has traditionally been on the teacher and the textbook authors (Silver, 1994). Even though there are few studies on problem posing in early mathematics, those existing have shown that by incorporating problem posing as part of problem solving, students' problem-posing as well as problem-solving skills may develop (e.g., Ellerton et al., 2015; Palmér & van Bommel, 2020).

According to Cai and Hwang (2019), one potential benefit with problem posing is its capacity to reveal useful insights about students' mathematical thinking. Problem posing can take place before, during, or after problem solving, where the students are to generate new problems or to reformulate a given problem (Silver, 1994). Stoyanova and Ellerton (1996) give names to these different situations. When students are asked to pose a problem without any restrictions, this is called *problem posing in a free situation*, for example, when asked to create a task for a friend to solve. When students are asked to pose a problem to a given stimulus or a given situation, this is called *problem posing in a semi-structured situation.* And when students are asked to reformulate a specific problem, this is called *problem posing in a structured situation* (Stoyanova & Ellerton, 1996). In this study the students were asked to pose a *similar* task to a friend after working on a problem-solving task on measurement. Thus, the problem posing came after problem solving, which is problem posing in a semi-structured situation (pose a problem to a given stimulus or a given situation).

This study is part of a research project on problem solving and problem posing with Swedish six-year-olds who have not begun formal schooling (in Sweden, formal schooling starts at the age of seven). As mentioned, there are few studies on problem solving in early mathematics and even fewer on problem posing. Therefore, the overall aim with the research project is to investigate possibilities and limitations when teaching problem solving and problem posing with young students who may not yet know how to read or write. The first years of the research project focused mainly on problem solving, which was considered both a purpose and a strategy (e.g., Lesh & Zawojewski, 2007). This implies that by working on problem solving the students simultaneously explore new mathematical content. The research project shows positive results regarding students' learning of mathematics (e.g., Palmér & van Bommel, 2018a; van Bommel & Palmér 2016), as well as students' feelings towards problem solving (Palmér & van Bommel 2018b). As both research and the Swedish curriculum pointed out possible applications of problem posing, we later expanded the research project to also include problem posing. In line with

Cai and Hwang (2019), problem posing in the research project involves activities where the students are to formulate or reformulate a task based on a particular context. In our study, this context is problem posing in direct connection to problem solving. In one initial study on problem posing, we found creating a similar task involved young students' reflections on the original task. When asked to pose a similar task, students used the discerned aspects of the original task in their posed tasks (see further section on Variation theory of learning regarding discernment and aspects), while interpreting the meaning of *similar* task. Thus, the analysis of the posed tasks seemed to inform us about the students' interpretation of the original problem-solving task. Furthermore, we found young students not only struggled with changing perspective from searching for information to providing information, but also from searching for a solution to searching for a question to pose (e.g., Palmér & van Bommel, 2020). The study presented here is a continuation of these findings to further develop our knowledge on students' learning of mathematics when working on problem solving and problem posing in early mathematics education. The aim is to more systematically investigate connections between the original problem-solving task and student-posed tasks. Such an investigation may provide information on how to work with problem posing in early grades, young students' reflections on original tasks in problem posing, as well as students' learning of the mathematical content of measurement. The study focuses on the following three questions:

- What aspects do young students discern when exploring a problem-solving task on measurement?
- After working on a problem-solving task on measurement, what tasks do young students pose when asked to pose a similar task to a friend?
- When asked to pose a similar task to a friend, what aspects of the initial problem-solving task are visible in the new tasks posed by the young students?

Theoretical Foundation: Variation Theory of Learning

The theoretical foundation for the study is the variation theory of learning developed by Ference Marton (e.g., Marton, 2015). In every situation several aspects of a phenomenon can be discerned, and those that are discerned are decisive for how the phenomenon at hand is experienced. The variation theory of learning directs attention to what aspects of a phenomenon a person discerns, as these discerned aspects indicate how this person experiences the meaning of the phenomenon. According to the theory, learning occurs when learners discern new and necessary aspects of a phenomenon.

Variation theory follows from the phenomenographic research tradition (Marton et al., 2004), and the theory has been widely used in educational studies in a range of subjects, though mostly in mathematics (e.g., Björklund et al., 2021; van Bommel, 2012). Based on this theory, learning is always directed to a phenomenon, a so-called object of learning. The object of learning is the content or skill to be learned during a lesson or a series of lessons. In this study the focus is on aspects of measurement discerned by the students when working on a problem-solving task, as well as on a problem-posing task. Thus, measurement, or more precisely, measurement through comparison, is the object of learning within the context of problem solving and problem posing. As mentioned, learning occurs when learners discern new and necessary aspects of the learning object. Cai and Hwang (2019) emphasize one potential benefit of problem posing is the capacity to reveal useful insights about students' mathematical thinking. Expressed in terms of the variation theory of learning, problem posing may reveal what aspects of a phenomenon, the object of learning, students discern. Working first on problem solving and then problem posing makes it possible to investigate similarities and differences in the aspects discerned by the students within and between these two connected lessons. The discerned aspects indicate how students experience measurement through comparison.

Problem Solving and Problem Posing in Early Mathematics Education

Recently, several journals and publications have focused on problem posing (see for instance, Cai & Leikin, 2020; Singer et al., 2015), which indicates an increasing interest in problem posing as a specific and crucial part of problem solving. Problem posing is not new. It was addressed a century ago as an essential part of mathematics. In their reflection on the past, present, and future of problem posing, Ellerton et al. (2015) refer to Henry Belfield, who in 1887 suggested letting students transform abstract examples into concrete problems. Even though posing problems is not new, engaging young students in this activity is not common. Previous research on problem solving and problem posing has mostly focused on secondary school, although some studies with younger students have been conducted. Engaging in and becoming familiar and comfortable with problem solving is pointed out as important by English (2004), Lesh and Zawojevski (2007), and Casey (2009). For instance, English (2004) found young students engage spontaneously in problem-solving activities outside school. These informal experiences can be built upon in problem-solving activities in formal schooling. More recently, Kalmpourtzis (2019) broadened the area of problem posing to game design for five- and six-year olds and stated problem-posing skills were developed while engaging in the context of game design. In another study, young students' problem-solving and problem-posing skills were found to be related to their beliefs and general mathematical abilities and strong correlations were shown (Chen et al., 2013). Frosse et al. (2020) focused instead on preschool teachers' views on preschoolers' spontaneous problem posing and found, amongst other things, the posed problems differed with respect to the nature of the problem (routine and non-routine).

Young Students Measuring Length

The mathematical content in the problem-solving task used in this study is measurement with a focus being on the aspects that students discern when a comparison of length is to be made.

Measurement is about comparing objects, either directly or indirectly, and about measuring with formal or informal units. Sometimes in measurement, the direct comparison of the lengths of two objects is not possible, which is the case in the problem-solving task used in this study. Thus, a reference object against which both objects can be compared is needed. Using a third object to compare two other objects calls for transitive reasoning, which means the comparison is made with an independent reference object (Kamii, 2006; Piaget, 1952). Students begin to estimate and measure lengths early by comparing and using objects or their own bodies as reference objects (Öhberg, 2004). According to Wright et al. (2007), young students experience concepts and tools in play and everyday activities can be used to reason about, compare, and measure objects. However, these experiences are not necessarily described in terms of, for example, length. Nevertheless, such experiences form the base for understanding principles that are necessary to follow in order to perform formal measurement with entities. Formal measurement with entities requires knowledge of properties, units, and scale: properties in the sense of identifying which aspect to measure, units in terms of which unit of measure to use, and finally, scale in terms of combining units (Wright et al., 2007). There are studies where students' ability to measure length is described as incremental (e.g., Sarama et al., 2011). First, the students need to distinguish length as an aspect of an object, then length can be measured by direct comparisons followed by comparisons of lengths using a third object (reference object). Thereafter, students begin to measure length by repeating a unit until finally measuring it with formal units. However, such a stepwise development does not imply later steps are better than previous ones; the most suitable measurement strategy depends on the situation (Sarama et al., 2011).

Method

As mentioned, the two problem-solving and problem-posing lessons in this study are part of a larger longitudinal research project. This research project is conducted through design research, which implies a cyclic process of designing and testing interventions situated within an educational context. The purpose of design research is to provide both practical and theoretical answers to questions that are relevant to both research and educational practice (McKenney & Reeves, 2012). In this study, we will not focus on the enirety of this intervention but on one design cycle involving one problem-solving and one problem-posing lesson on measurement followed by an interview with the students.

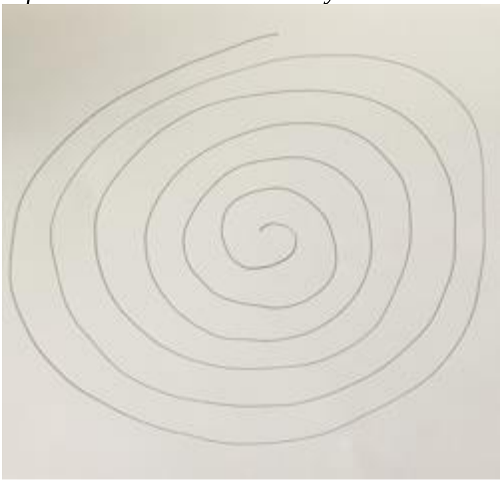
Participants: Selection of classes

The educational context for this study is the Swedish preschool class. Swedish preschool class is a year of schooling that 6-year old students attend the year before their formal schooling begins. The aim of preschool class is to facilitate a smooth transition between preschool and school and to prepare students for the next step of their education. The selection of the seven classes in the study was based on the teachers' interest in participating. Based on several years of collaboration, these teachers have occasionally been asked by the researchers to implement problem-solving and problem-posing lessons in their classes and to collect students' documentations from these lessons. The intention with this design is to increase internal *validity*, which is of importance in educational design research, even though the external *validity* may become lower (Palmér & van Bommel, 2021). These teachers are educated preschool class teachers, which implies they completed a minimum of a three-year university course in teacher education. Having participated in several of the previous design cycles, these teachers are familiar with problem solving and problem posing as well as educational design research and the aim of the study. However, the implemented problem-solving and problem-posing task on measurement was new for them and their students. Altogether, 119 students from these seven classes participated in the design cycle. In accordance with the Swedish Research Council's (2017) ethical guidelines, the students' guardians were informed about the study, and they approved their children's participation.

Design: The problem-solving and the problem-posing lessons

Two lessons were conducted within this design cycle. The two lessons were planned by the researchers and presented to the teachers verbally and in writing. In the first lesson, the focus was on problem solving. In this lesson, the teachers first gathered their students around a table. The students watched as the teacher drew a wheeled curve that started in the middle of a paper and spiraled outwards (Figure 1).

Figure 1
An Example of a Wheeled Curve Drawn by the Teachers



When this was done, the students were asked how long the drawn curve would be if straightened. The teacher measured with her arms and asked the students if they thought the curve would be shorter or longer than specific distances. After that, the teacher and the students together measured the drawn curve by covering it with a string. This string was then compared with the length of the teacher's arms. In this way transitive reasoning was introduced, implying comparison of measurement to be done with a reference object (Kamii, 2006). The students were then given the task to make a similar drawing in pairs, with a curve spiraled in the same way on a paper. The task was to try to draw a curve that was the same length as the curve drawn by the teacher. When the students had drawn their curves, the curves were measured with a string as well (Figure 2). All students used strings of different colours, making it possible to know which string belonged to which drawing. Finally, all drawings and the lengths of all strings were compared to the teacher's drawing and string. Paper-and-pen work as well as photos taken during the lesson were collected by the teachers and sent to the researchers by post.

Figure 2

Example of a Student Measuring Their Own Created Curve with a String

In the second lesson the focus was on problem posing. The students were reminded of the task they had worked on in the previous lesson and were asked to pose a *similar* task to a friend. Solving the posed tasks was not part of the design cycle. The majority of the students worked with the same classmate as in the first lesson, although some students chose to work individually. The students were free to use any material they wanted to, and no specific instructions were given about how to design the task they would pose. All students used a white A4-size paper to document their posed task, and the majority of the tasks included both a picture and words. If needed, the teachers helped students document the wording of the tasks they posed. In those cases, the students expressed their tasks verbally to the teacher, who wrote the students' wording on the students' papers. In total, 66 tasks were posed by the students. Paper-and-pen work as well as photos from the lesson were collected by the teachers and sent to the researchers.

Interviews

Interviews were used to explore what aspects the students discerned when working on the problem-solving task. The interviews were conducted by the teachers. As the students were young, the teachers were considered best qualified to be sensitive to the students and thus to give each student the best conditions for answering the questions (Palmér & van Bommel, 2021). To obtain comparability with the teachers involved, a structured interview guide was used, and the teachers were instructed, in writing and verbally by one of the researchers, on how to carry out the interviews and how to take notes. The interviews followed a questionnaire developed by the researchers, where the teachers wrote down their students' answers to the interview questions. The questions included both multiple choice and open-ended questions. The students were asked three questions connected to the problem-solving lesson: 1) Did you find the task *really easy/easy/quite hard/hard*? (the students were to choose one option); 2) What made the task *really easy/easy/quite hard/hard*? (open question); 3) How did you go about drawing a curve with a length similar to mine? (open question). During the interview, the students were shown their own documentation from the problem-solving lesson as well as the teacher's drawing to increase the likelihood of the interviewer and the student talking about the same experience.

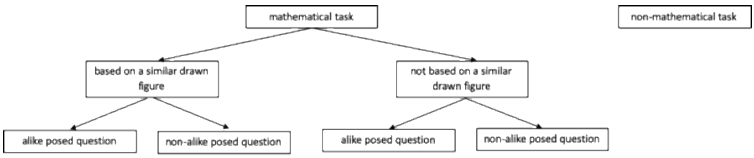
Analysis

The first part of the analysis focused on the first research question, i.e., on the aspects the students discerned when working on the problem-solving task on measurement.

As the students' paper-and-pen work only depicts the final drawings of the students and not their process, students' discerned aspects might not be visible in the final drawings. Therefore we complemented our data with student interviews where we tried to find out what aspects they had discerned. Thus, in this part of the analysis the paper-and-pen work was complemented with the students' answers to the three interview questions presented above, and in particular their answers to the question "*How did you go about drawing a curve with a length similar to mine*?" Examples of answers from the students were: "*We counted as many twirls as there were on yours"* and "*We tried to make our curve as big as yours."* In the analysis, we identified the mathematical aspects expressed and thus discerned by the students. The identified discerned aspects were not pre-conceived but were inductively based on similarities and differences in students' answers. After identifying discerned aspects in all students' answers, the frequency of each aspect was counted. In the examples above, the identified discerned aspects are *numbers* of twirls in the first example and the *size* of the curve in the second example.

The second part of the analysis focused on the tasks posed by the students and thus on the second research question. When analysing tasks posed by students, different kinds of classifications are often used (for example Cai & Hwang, 2003; Leung, 1997). To analyse the tasks posed by the students in all design cycles within this design study, a classification scheme for dealing with reformulated tasks developed by Carrillo and Cruz (2016) has been used as a starting point (see Palmér & van Bommel, 2020; van Bommel & Palmér, 2022). Carrillo and Cruz's classification scheme was selected as it focuses on the structure of tasks which implies analysing if the question posed is alike or different from the original question. Further, the scheme focuses on the connection between tasks and mathematics, which implies analysing if the posed question is a mathematical question or not. Thus, like our study, Carrillo and Cruz's classification scheme is about the similarities and differences between an original task and a posed task. Based on this, the classification scheme used in this analysis (Figure 3) included (1) the connection between the posed task and mathematics (mathematical or non-mathematical question), and (2) the connection between the posed task and the task in the problem-solving lesson (alike or different question). This second connection included whether or not the posed task was based on a drawing similar to the initial problem-solving task, as well as whether the question posed by the students was alike or different than the question in the initial problem-solving task and thus whether or not the task focused on measurement.

Figure 3
Classification Scheme for Analysing the Tasks Posed by Students



Finally, to answer the third research question (What aspects of the initial problem-solving task are visible in the new tasks posed by the students?) the posed tasks were deductively analysed based on the aspects found in the first analysis described above. Thus, the documentation of the student posed tasks was deductively analysed for the aspects expressed by the students in the interview on the problem-solving task. Other similarities were also analysed, for example, similarities between the original task and the posed tasks regarding the mathematical content, the question posed, and the drawings used.

Results

In the results, the intention is not to generalize, but to explore discerned aspects, strategies, and posed tasks. Thus the number of students will not be the main focus, even though numbers are mentioned in relation to the first research question.

What aspects do young students discern when exploring a problem-solving task on measurement?

The first analysis focused on the aspects discerned by the students when working on the problem-solving task. All but two students referred to the curve drawn by the teacher when asked what they did to draw a curve with a length similar to that in the teacher's drawing. Those two students instead indicated that they had referred to the string used to measure the curve drawn by the teacher. At the group level, when asked what they did to draw a curve with a length similar to that of the curve drawn by the teacher, the students discerned the following aspects: *shape, size, distance, and number* (Table 1). Thus, even though all but two students talked about the curve drawn by the teacher, there were differences regarding which aspects of the drawing they referred to.

Table 1
Number of Students Referring to Each Aspect

	Shape	Size	Distance	Number
Number of students	35	23	21	43

The most common aspect referred to was *number* in the sense of the number of twirls, for example: *"We counted your twirls and did as many"* and *"We counted as many twirls as there were on yours."* Students referring to *distance* focused on the distance between the twirls, for example: *"Not too wide and not too narrow;" "Not too thick and not too thin;" "A little in the middle."* Sometimes when referring to the aspect of *number* of twirls, the students also referred to other aspects, for example: *"If it becomes bigger in the middle then there have to be more twirls outside,"* indicating distinguishing the aspects of both *size* and *number*, or *"It [their curve] became longer because we did more twirls;" "We did six twirls while you did five;" "We had small spaces between the twirls,"* indicating distinguishing the aspects of both number and distance.

Some students referred to the *shape* of the curve drawn by the teacher, for example: *"Tried to draw rounded and make it similar." Shape* was sometimes referred to in combination with another aspect, for example: *"It ought to be round and of similar size,"* distinguishing the aspects of both *shape* and *size*. Students only referring to *size*, often talked about whether the teacher's drawing was big or small, for example: *"We did it quite big like you did."* As presented above, *size* was also referred to in combination with *number*.

The students most often referred to one or two aspects; there were only three students who referred to three aspects, for example *“Smaller whirligig but more rounds”; “If you have less space between the curves it becomes long even though the whirligig is smaller.”* In this example, the student has discerned the aspects of *size*, *number*, and *distance*.

After working on a problem-solving task on measurement, what tasks do young students pose when asked to pose a similar task to a friend?

The second analysis focused on the tasks posed by the students. These tasks were analysed using the classification scheme presented in Figure 3. Based on this scheme, the posed tasks could be classified as one of the following: *non-mathematical task*, *mathematical tasks based on a similar drawn figure with an alike question*, *mathematical task based on a similar drawn figure with a non-alike question*, *mathematical task not based on a similar drawn figure with an alike question*, and *mathematical task not based on a similar drawn figure with a non-alike question*.

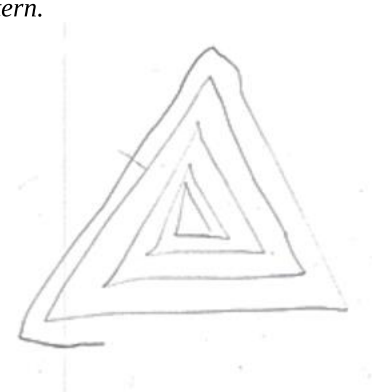
Tasks classified as *non-mathematical tasks* would be those where the students did not pose any mathematical question. In this design cycle, no such tasks were posed, meaning that all tasks that were posed by the students had content connected to mathematics.

An example of a *mathematical task based on a similar drawn figure with an alike question* is the task in Figure 4. The question posed was *“How long is the curve?”* The drawings in this category resembled, more or less, the drawing in the initial problem-solving task and were all based on a wheeled curve that started in the middle of a paper continuing outwards. This “wheel” could be curved as in the original drawing, but could also consist of straight lines, as in Figure 4.

An example of a *mathematical task based on a similar drawn figure with a non-alike question* is the task in Figure 5. The question posed was “*Continue the pattern”* (Figure 5). In this category there were also questions that focused on the number of twirls in a drawn figure, for example “*How many laps are there?”* These questions are mathematical, although they address mathematical topics other than the one in the original task.

Figure 4
How Long is the Curve?

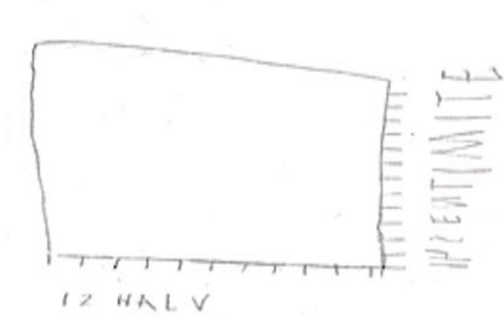
Figure 5
Continue the Pattern.



The mathematical tasks that were not based on a similar drawn figure were sometimes related to measurement and sometimes not. An example of a *mathematical task not based on a similar drawn figure with an alike question* is the task in Figure 6. The question posed was "*Measure how long the unicorn is*." Similarly, the task in Figure 7 is an example of a *mathematical task not based on a similar drawn figure but with an alike question*. These students elaborate with clues in their task. For two of the sides the length is given: 14 centimetres, 12.5 centimetres. For one of the other sides a clue was given *"This side is probably a little longer than the opposite."* The question posed was "*Since one side is a little longer than the other side, you are to figure out how long the square is*." The students further state that "*there are only clues given to two sides as the task otherwise would become too easy*."

Figure 6
Measure How Long the Unicorn Is.

Figure 7
Since One Side is a Little Longer Than the Other Side, You Are to Figure Out How Long the Square Is.



Finally, an example of a *mathematical task not based on a similar drawn figure with a non-alike question* is the task in Figure 8. The question posed was *"Split the candy evenly between the men."* The question is mathematical; however, it addresses a different mathematical topic than addressed in the original task.

Figure 8
Split the Candy Evenly Between the Men

When asked to pose a similar task to a friend, what aspects of the initial problem-solving task are visible in the new tasks posed by the young students?

When looking for similarities between the tasks posed by the students and the initial problem-solving task, we identified whether the aspects mentioned by the students in the interview (the first research question) were expressed in the posed tasks. In the tasks posed by the students, the aspects of *number* and *size* reoccurred while the aspects of *shape* or *distance* did not. Tasks where number reoccurred were those where the question posed focused on the number of twirls in a drawn figure, for example, "*How many laps are there?"* Tasks where size reoccurred were those where the question posed focused on the size of the drawn figure, for example, "*Draw a picture of similar size."* There were nevertheless other features in the problem-solving task that reoccurred in the tasks posed by the students. Most common in the posed tasks was a drawing of a wheeled curve that started in the middle of a paper continuing outwards. And, the most common question was to find out the length of the curve. However, as shown, there were also mathematical tasks that included neither a similar drawing nor a similar question (Figure 8 is one example). Regarding the mathematical content in the two lessons, the initial mathematical content of measurement was often visible in the posed tasks. However, other mathematical content, such as patterns or division, were also addressed in the posed tasks (Figures 5 and 8 are two such examples). There were no clear links between the aspects mentioned by the students in the interviews and their posed tasks. For example, some students expressed during the interview a discernment of the size of the teacher's initial drawing, and then posed a task focusing on size. Other students expressed discernment of one or two aspects during the interview but neither one of these were visible in their posed tasks.

Discussion and implications

This study is an example of a study within the limited studies on designing and testing interventions for the teaching and learning of problem posing (Cai & Hwang, 2019; Singer et al., 2013). A first conclusion drawn is the presented results confirm our previous positive results showing that young students learn mathematics through working with problem solving and problem posing. Thus, the study contributes to the increasing but still limited research on early problem solving and problem posing (Cai et al., 2015).

The theoretical foundation for this study is the variation theory of learning (Marton, 2015), and based on this theory, learning implies discerning, differentiating, and finally merging necessary aspects of a learning object. The design of the problem-solving lesson was intended to make some necessary aspects possible to discern: size (the size of the drawn curve), shape (the spiral-formed shape of the drawn curve), distance (the distance between the twirls in the drawn curve), and number (the number of twirls in the drawn curve). Even though these aspects were not explicitly put forward by the teachers in their enactment during the lessons, they can be considered as *potential learning outcomes* (Marton, 2015).

During the two lessons, different aspects were discerned by the students. The students expressed in the interview that they had discerned the aspects of *shape, size, distance,* and *number*, when solving the first task. When posing a similar task, the students' reflections on the original task became visible in that the posed tasks indicate the students' interpretation of the original problem-solving task (Carrillo & Cruz, 2016). When posing, only size and number were expressed in the posed tasks as aspects of measurement. Thus, in this study, the *lived object of learning* slightly differed between the two lessons. In the problem-solving lesson, the lived object of learning related to the aspects of *shape*, *size*, *distance*, and *number* while the lived object of learning in the problem-posing lesson only focussed on the aspects of *size* and *number*. This may be due to the open request to pose a similar task without clarifying in what sense the task ought to be similar. For example, as seen in the results, *similar* can be interpreted as a task on mathematics in general, a task on measurement in specific, or as a task based on a similar drawing. This implies if a teacher or a researcher wants to investigate explicitly if and how students would pose a task on measurement (which was not the intention of this study), this must be made explicit when students are to pose a similar task, e.g., "*Pose a similar task on measurement to a friend*." Further, the step to generalization was not expressed in the problem-posing tasks, and the design of the lesson is to be developed within the design research study to address fusion of aspects more explicitly and to make generalization possible.

In forthcoming design cycles, an additional part of the lesson sequence will be developed where the students jointly explore possible aspects to discern before posing their tasks. This exploration will be conducted in a whole class discussion. Also, a second interview focusing on the mathematical content of measurement will be added after the problem-posing lesson. As the aim of the overall design study is that students can become problem solvers *and* learn mathematical content, content questions need to be added in a second interview to see how well this is accomplished. Considering that problem solving and problem posing are both a purpose and a strategy (e.g., Lesh & Zawojewski, 2007), and provided both practical and theoretical design principles (McKenney & Reeves, 2012) for teaching early mathematics through problem solving and problem posing, these results need to be connected to specific mathematics content to finally result in general (connected to problem solving and problem posing) and specific (connected to different mathematical content) design principles.

Concerning the posed tasks, all students produced mathematical tasks, which is not always the case in similar situations (Palmér & van Bommel, 2020; van Bommel & Palmér, 2022). This might be because of a low threshold for the creation of a new drawing but also because these students are used to problem solving and problem posing, recognising these as a mathematical activity. The questions posed with the new drawings varied, and as mentioned before, the aspects of measurement addressed differed in the two lessons. By incorporating problem posing as part of problem solving, students' knowledge on measurement may have developed. Opening up students' own tasks also opens up creativity and differentiation. For instance, some students used the concept of centimetres in their posed task, a concept not included in the initial problem-solving task nor in the curriculum for this grade but obviously present in some of the students' curriculum. Sarama et al. (2011) describe students' ability to measure length as incremental. In our study all students distinguished length as an aspect of an object, measured by direct comparisons, and made comparisons of lengths using a reference object. The use of a reference object (e.g., Kamii, 2006; Piaget, 1952) requires transitive reasoning, where students compare using this independent third object. The use of units or scale, as mentioned earlier when referring to Wright et al. (2007), made it possible for some students to measure length by repeating a unit and for other students to even measure with formal units. In order to obtain a common understanding of the object of learning, this variety can be used in class to point out differences and similarities with a focus on the aspects of the object of learning. Thus, these two lessons may have strengthened the students' knowledge of measurement and strengthened their emergence as problem solvers.

A limitation of this study is that the analysis of the first research question is based on what the students expressed verbally, while the analysis of the second research question is based on students' paper-and-pen work. The students may have discerned more aspects in the lessons but not verbalized them in the interview or expressed them in the paper-and-pen work. However, our conclusion tells us that critical aspects regarding measurement (size, shape, distance, and number) were discerned by the students as they worked on these problem-solving and problem-posing tasks on measurement. These aspects were possible to explore further due to the diversity of the tasks, where the combination of problem solving and problem posing offered students opportunities to reflect on measurement in different but connected ways.

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Author contributions

The two authors collaborated in this intervention over several years. The design cycle presented in this article was jointly planned by the two authors and then implemented by the preschool class teachers. The empirical data were initially analysed by the first author, and the article was jointly written by the two authors.

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