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Contemporary, Emergent Mathematics for Teachers: A Case Study on an Online Graduate Program

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In this paper, we present a qualitative case study on an online graduate program for practicing teachers and educators that explores contemporary mathematics and its integration into K to 12 education. Data for the study comprised students' work in the courses, student feedback, and notes from instructor debriefings for two cohorts of the program. The findings are organized in terms of (1) specialized knowledge for teaching mathematics, (2) online professional learning communities, and (3) online teaching in general. We conclude the paper with suggestions for online mathematics teacher education and the identification of venues for future research.

In this paper, we present a qualitative case study on the online graduate program, *Contemporary, Emergent Mathematics for Teaching,* offered to teachers and educators. The program was intended to address contemporary mathematics that is not traditionally included in programs of studies but can be incorporated into K to 12 education. Such content is relevant for several reasons. For example, teachers could incorporate mathematical applications into their class to support or enrich students' learning. Furthermore, teachers may not be familiar with the new mathematical content — for example, cryptography, graph theory, and coding — that is now being included in some programs of studies around the world. Such mathematical content as presented in the case study would also extend teachers' mathematical knowledge at the horizon, as

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described by Jakobsen et al. (2012) and Zazkis and Mamolo (2011)—we elaborate on this kind of knowledge in the theoretical framework section—and benefit both students and teachers. The contributions of the paper include the conceptualization of online professional learning communities in the context of in-service teacher education with practical suggestions for other online mathematics teacher education programs aimed at teachers as professionals.

Online Mathematics Teacher Education

Research on online education has already identified suggestions for online teaching. Muir et al. (2019), for instance, studying online student engagement and how such engagement fluctuated over the course of a semester, found that fluctuating engagement was influenced by a variety of factors, including “assessment, unit/s workloads, nature of the units (including relevance), lecturer presence, and work/life commitments” (p. 266). They suggested specific practices for instructors to improve their students’ online engagement such as providing them with full access to learning resources, being responsive, and providing enough time for students to do the compulsory activities.

Research specific to teacher education has also provided suggestions for online programs. Dyment and colleagues (2019) conducted a systematic literature review on online initial teacher education and interviewed instructors and teachers over several years. They suggested several principles for effective online teacher education, including being present and involved; providing opportunities for students to develop relationships and contribute meaningfully to the online learning environment; building the course with flexibility in mind; ensuring assessment tasks are authentic, applied, and constructively aligned with course aims and learning activities; and facilitating meaningful connections between university study and learners’ professional contexts and experience.

There has been debate on the relation between the structure of online programs and students’ autonomy. For example, Moore (1993) suggested three interdependent factors/elements to be considered when thinking of designing an online learning environment, including (a) dialogue, (b) structure, and (c) autonomy. Moore argued that less structure and more dialogue would allow learners to experience more autonomy in an online learning environment. However, when Larkin and Jamieson-Proctor (2015) used Moore’s model to design an online mathematics teacher education program, they indicated that learners had more autonomy when more structure and more dialogue were provided. Such a conclusion suggests that *teaching presence* is critical in any engaging online program (Anderson et al., 2001). Such a teaching presence involves “devising and implementing activities to encourage discourse between and among students, between the teacher and the student, and between individual students, groups of students, and content and resources” (Anderson, 2008, p. 345). In addition to the teaching presence, Anderson and colleagues (2001) described two types of presence: the *cognitive*, which focuses on the development of the content, and the *social*, which focuses on the collaborative and interactional learning setting.

The cognitive and social presence (Anderson et al., 2001) was also reflected in programs targeted to online mathematics teacher education.; however, when revisiting the literature on online teacher education, we found that most of the research has been conducted on pre-service teacher education (Anderson et al., 2001; Chieu & Herbst, 2016; Dede et al., 2009; Engelbrecht & Harding, 2005; Kastberg et al., 2014; Pape et al., 2015; Whitehouse et al., 2006). One example of in-service teacher education is the work reported by Pape et al. (2015), who described an asynchronous, one-year elementary teacher education program focusing on mathematical knowledge for teaching. Instructors in this program promoted social presence by establishing norms of participant-to-participant interactions. Cognitive presence was addressed through offering rich opportunities to do mathematics and engage in mathematical discourse.

The targeted audience for the *Contemporary, Emergent Mathematics for Teaching* program is teachers and educators as professionals; therefore, the program assumed certain experience and expertise that would be missing for other populations, such as student teachers. People enrolled in this program were expected to work at an educational institution, thus requiring flexibility to combine graduate studies with their jobs. Additionally, participants in the program were considered members of a professional community and thus able to contribute to the collective learning in their corresponding cohort, as we elaborate in the following sections.

Theoretical Framework

The theoretical framework for this case study comprises two components: online professional learning communities and knowledge for teaching mathematics. The former can be related to social presence, while the latter can be associated with cognitive presence.

We draw from the idea of learning communities to define *online professional learning communities* to refer to the participants in the online program offered to in-service teachers and educators. Kearns et al. (1999) defined learning communities as a "group of people, whether linked by geography or some other shared interest, which addresses the learning needs of its members through proactive partnerships. It explicitly uses learning as a way of promoting social cohesion, regeneration and economic development" (pp. 61–62). Regarding teacher education, Kilpatrick et al. (2003) argued that learning communities promote the creation and sharing of knowledge. This point of view is very close to Dewey's (1902) definition of society as "a number of people held together because they are working along common lines, in a common spirit, and with reference to common aims" (p. 14). This perspective can be extended also to the role of instructors in a course, which can be also a part of the learning community. In this study, an online professional learning community is similar to a learning community but removing the geographic component as members of the community interact online and adding the qualification of professional to emphasize that the members of the community hold a professional status in a specific field, in our case, teaching. Being a professional educator, which is the case of participants in the *Contemporary,* *Emergent Mathematics for Teaching* online program, also involves specialized knowledge for teaching mathematics, which is the other component of our theoretical framework for this study.

Teachers' knowledge for teaching mathematics has been a focus of educational research for decades. Early studies regarding teacher mathematical knowledge showed little to no correlation between teachers' college credits in mathematics and their students' performance on standardized tests (Begle, 1972; Monk, 1994) suggesting that the knowledge required for teaching mathematics should be specialized, going beyond mere mathematical content.

Shulman (1986) articulated this point distinguishing between content knowledge and pedagogical content knowledge; the former refers to the specific knowledge of a subject or profession, while the latter concerns specialized knowledge for teaching, such as key representations of concepts or ideas, metaphors, analogies, illustrations, and examples that can help to make the content accessible to others. Since then, several scholars, such as Ball et al. (2008), Davis and Simmt (2006), and Hill et al. (2005), have elaborated on the *specialized knowledge for teaching mathematics*. Particularly relevant for this paper is what Ball and colleagues (2008) proposed as knowledge at the mathematical horizon: "Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). Other authors, however, have included a broader perspective that goes beyond the curriculum. Jakobsen et al. (2012), for example, explained

> [Horizon content knowledge] is an orientation to and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory. (p. 4642)

This approach is also considered by Zazkis and Mamolo (2011), who posited that scholars had tended to focus on the learners' horizon and elaborated on how teachers' horizon also plays an important role in mathematical instruction. This perspective of knowledge at the mathematical horizon is consistent with the goals of the *Contemporary, Emergent Mathematics for Teaching* program that addresses mathematics not traditionally included in the program of studies.

The specialized mathematical knowledge for teaching includes critical features necessary for students to discern in learning (Marton, 2014); such features may be addressed through curricular resources. In this sense, mathematical knowledge for teaching can be considered as distributed across people and resources (Davis & Renert, 2014; Preciado Babb et al., 2020). While Preciado Babb and colleagues (2020) described teaching as a partnership between the teacher and the resources used in the classroom, teachers still need to be aware of how mathematical concepts develop through a lesson, a unit, a grade level, and through the program of studies. For this reason, in this case study, we were also interested in how teachers identified the critical features for teaching the new mathematical content included in the online program.

The Contemporary, Emergent Mathematics for Teaching Program

This section describes the *Contemporary, Emergent Mathematics for Teaching* program as it was offered when writing this paper. The program's design was informed by Shulman's (2005) idea of signature pedagogies. Such pedagogies refer to the specific ways professionals are educated in a corresponding field, such as medicine, law, and education. In this case, while the program's content was mathematical, the learning objectives and the tasks were related to teaching as a profession, and participants in the program were expected to address the implementation of mathematical content in their classroom. In order to build on the professional expertise from both education and mathematics, the program was designed by a team of mathematicians and teacher educators from the University of Calgary. The program also considered Parker et al.'s (2016) discrete signature pedagogies with the potential to enhance teachers' learning and growth, namely "critical dialogue (process of acquiring knowledge through communicative interactions), public sharing of work (testing out practices in classrooms and share ideas with larger audiences), and communities of learners (collective learning around a shared concern or a passion)" (p. 137). The program included the following foci: (i) continuous discussions and collaborations, (ii) practice and exploration of the targeted mathematical content, and (iii) the design of learning tasks for K to 12 settings. While the content of the course was mathematical, people enrolled in the program (teachers) were expected to design—through constant interaction within their cohorts as online professional learning community—learning tasks for their students.

The program comprised four courses. In the first course, *Analytical and Algorithmic Reasoning*, students were expected to examine deeply the process of mathematical problem solving and characteristics of analytical reasoning; to develop deeper mathematical knowledge for analytical and algorithmic reasoning and computational thinking; to apply analytical and algorithmic reasoning and computational thinking to understand real-world problems and technological advancement; to analyze various decision-making systems and relevant mathematical representations including graphs; and to develop deeper mathematical knowledge for teaching and design mathematics lessons for modeling, analyzing, and problem solving.

In the second course, *Data: What it Means and How to Handle it*, students were expected to develop quantitative skills and critical thinking to explore real-world questions; to understand historical data well enough to make predictions in the future; to develop an appreciation for data and inferential conclusions; to design learning tasks that address issues related to sustainability, and to develop skills for managing data, modelling, analyzing, visualizing data to solve problems.

In the third course, *Geometry in Nature, Art and Computer Graphics*, students were expected to develop an understanding of the mathematical principles involved in different applications of geometry to arts, modeling nature, and computer graphics; to apply geometric concepts in the generation of images using diverse software; to write recursive codes to generate fractal images using specific computer languages; and to design learning activities for K – 12 students related to applications of geometry.

Finally, in the fourth course, *Mathematics for Sustainability*, students were expected to develop an understanding of key concepts related to sustainability, to develop an understanding of mathematical tools and techniques needed to deal with problems in the area of sustainability, to develop mathematical and writing skills required to craft persuasive pieces—any appropriate mode of expression—within which quantitative reasoning is employed to make cases for or against some courses of action in the sustainability arena, and to design learning activities for K-12 students related to mathematical techniques and concepts needed to respond to problems related to sustainability.

The courses followed a consistent structure in terms of learning outcomes and assignments. Each course was co-taught by a mathematician and an educator. The learning outcomes included explorations of the mathematical content in each course, as well as designs for implementation in the classroom. The course Mathematics for Sustainability followed a specific book (Roe et al., 2018). For the other three courses, instructors decided on the specific mathematical content and corresponding resources.

Due to the nature of the mathematical content in the program, the learning outcomes also included real-life applications. The assignments in each course comprised: (i) continuous online discussions and collaborations (40% of the final mark), (ii) a laboratory for practicing and exploring the targeted mathematical content (20% of the final mark), and (iii) the design of learning tasks for K to 12 settings (40% of the final mark).

The online collaborations and discussions included virtual synchronous sessions two to four times per course. The sessions lasted up to two hours and were used to provide general information about the course, introduce content, and promote discussions among course participants. The sessions were also used to collect participants' feedback. The other component of the online collaboration consisted of weekly discussion topics, which comprised an initial post and peer responses for each student. The discussions usually addressed ways in which the course content could be incorporated into K to 12 classrooms. However, the discussion topics also included mathematical explorations and students' creations, such as pieces of code to generate images.

The laboratory served two purposes. On the one hand, it was used as a formative assessment in which participants in the course were able to assess their learning of the mathematical content. Some instructors used the quiz tool on the program's online platform (Desire2Learn) for this purpose. Participants could repeat the quizzes as many times as they needed. On the other hand, the laboratory was also to introduce course content. For instance, the quiz tool included not only questions for assessment but also introductory material, videos, and links to other resources. The laboratory was assessed by completion; that is, the grade was assigned as completed regardless of whether the responses were correct. This was particularly relevant as the program was targeted to teachers from K to 12, and therefore there was a significant gap in teachers' mathematical background that could set some participants in a disadvantaged position compared with peers who may have a stronger background in mathematics.

The third assignment of the courses required participants to create a task for the classroom (K to 12) incorporating the mathematical content of the course. This assignment varied from course to course. For instance, the Geometry in Nature, Art and Computer Graphics course required three tasks, while the course Mathematics for Sustainability required the design of only one task in the form of a class project. Instructors were encouraged to offer opportunities for formative feedback on drafts of these designs before final submission.

In contrast to the lecture-based traditional format of delivering course content, this online program followed a format similar to the flipped classroom (O'Flaherty & Philips, 2015; Kim et al., 2014). To a greater extent, participants were required to review and use different resources by themselves; this would inform their work in the course assignments. Such resources included books, websites, and free software selected specifically for each course. Participants in the program had to learn the corresponding mathematical content to discuss its potential integration into teaching at K to 12 education and the design of corresponding learning tasks.

The program was designed to build on the experience and expertise of its participants as professionals in education. The cohort, as an online professional learning community, addressed the mathematical content and its implementation in K to 12 education through the discussion topics and the synchronous sessions. The posts in the discussion topics, drafts of task designs, and responses in the laboratories provided useful feedback that could be used to adjust the courses, supporting participants in future offerings of the program. An example of how teachers may implement the mathematical ideas in their corresponding classroom was reported by Preciado Babb (2020). Based on the idea of rotating an agent for creating virtual images, teachers designed an activity in the gym in which students had to perform certain turns to pass the ball, such as 90-degree left pass. This way of introducing angle contrasts with the way it was described in the curriculum in Alberta as something related to the intersection of two lines.

While the course included a mathematician and an educator as instructors, we acknowledged the expertise and contributions of the participants as professionals in education. In this sense, instructors also learned with and from teachers and educators enrolled in the course.

Method of Inquiry

The purpose of this exploratory, qualitative case study was to gain insights into features of the graduate program for teachers that could inform both the redesign of this program and the design of other programs for practicing teachers. The case includes participants enrolled during the program’s first two cohorts (2019 and 2020). Ten people agreed to participate in the study: six out of eight persons enrolled in the first cohort, and four out of 15 persons enrolled in the second cohort. Eight participants were located in Alberta, one in Ontario, and one in British Columbia. Six participants were elementary teachers and four were high school teachers. There was one male teacher, and the rest were female teachers.

The study involved different sources of data: student course evaluations (collected anonymously by the university), student feedback on the courses and the program (collected by instructors and the coordinator of the program); course assignments from ten people who agreed to participate in the research; and notes from debriefing sessions with instructors and the coordinator of the program that took place after the end of each course. We conducted a thematic analysis (Braun & Clarke, 2006) on these data and identified three themes elaborated in the following section.

Findings

Two themes resulting from the analysis correspond to the theoretical orientation discussed in the previous section, namely "Specialized Knowledge for Teaching Mathematics" and "Mathematics Teacher Professional Learning Community". The third theme addresses other features of online teacher education that could inform a revision of the program or the implementation of other programs.

Specialized Knowledge for Teaching Mathematics

Specialized knowledge for teaching the mathematical content of the program can be related to totally new mathematics, such as the use of networks for flow and stock but can also refer to ways of conceiving mathematical concepts from a different perspective. For instance, the notion of angle is introduced in many programs of studies as something static related to the intersection of two lines. However, in agent-based programming, angle is something dynamic related to the rotations of the turtle in LOGO. Although related, these two perspectives of angle are very different in nature and have important implications. For instance, it does not make sense to talk about an angle of 760 degrees as the intersection of two lines. However, this angle is meaningful when considering the rotations of an object. Preciado Babb (2020) already reported this example from the program's first cohort showing how teachers build on this idea of angle to design a task related to rotations in the school gymnasium. While this particular task did not involve coding, the dynamic perspective on angle is relevant for agent-based programming.

Another element of specialized knowledge for teaching in the program was the focus on problem-solving, which is often considered as a means of applying what has been learned in the classroom. Other perspectives include students' development of mathematical concepts through engagement in problem-solving. However, algorithmic reasoning in the program's first course had very specific features. In particular, it related to computational thinking, in the sense that the result of applying algorithmic thinking is a procedure that could be performed by another person or by a computer.

The conversations in the weekly discussions and the design of learning tasks provided some insights on the specialized knowledge required to teach the mathematical content of this program to K to 12 students. This was particularly challenging for teachers at the elementary level. A focus on critical features of the involved mathematical concepts was key to addressing this challenge. For instance, in the following excerpt, one teacher was responding to another participant regarding a way to introduce fractals to the elementary level:

> I like that you are thinking of how to bring fractals into elementary grades and think you have it right there, shrinking or growing patterns are the beginning of it. (Grade 3/4 teacher, Cohort 1).

Here, a critical feature of fractals is addressed: scale-invariance. With this feature in mind, it makes sense to connect fractals with shrinking and growing patterns. Teachers' curricular knowledge interacted with this new mathematical knowledge offered in the course to incorporate the content at the elementary-level. An important element of specialized mathematics for teaching in the program involved transdisciplinary. In the case of the Mathematics for Sustainability course, there was a need to include science content, particularly for studying the mathematical models for the greenhouse effect. While the flow and stock diagrams were critical features, in this case, it was important to build on scientific knowledge in order to make sense of how this mathematical content is used for studying sustainability issues.

A topic that was particularly difficult for participants in the program was recursion. The students' data prompted specialized knowledge for teaching this specific content. This topic was addressed in the first course, Algorithmic and Analytical Reasoning, and then used for generating fractals in the course Geometry in Nature, Art and Computer Graphics. When this course was taught for the first time, there was a need to review this topic as students performed poorly in the laboratory and reported struggling with the discussion topic. The instructors' response to this problem involved the development of further examples of recursion of number sequences without coding. After this, a fractal was generated step-by-step in LOGO showing each step's code without recursion. Then, a code using recursion was analyzed, showing the corresponding images in each case. The posterior success of this strategy in the second cohort, resulting from a comparison of the discussions in the two cohorts, suggests that such an approach is directly connected to the specialized knowledge to teach recursion in programming.

Mathematics Teacher Professional Learning Community

The courses in the program were designed to foster an online learning community through discussion topics and synchronous sessions. Participants were also involved in peer feedback through the online platform. While this approach to community learning has been identified and reported for online courses in general, we identified specific elements of the online communities related to participants' learning of mathematics and mathematics for teaching. A common question throughout the courses was related to how teachers integrate the mathematical content of the course in their classroom. Participants — teachers and educators — tended to engage in the conversations of the discussion topics beyond the expectations for the course. While the common expectation for participants was to post one initial comment and respond to one or two peers, many participants consistently responded to more than three peers each week. In this sense, the weekly discussion was a key component connecting the content of the course, based on mathematics, to the professional background of participants, in-service teachers.

In addition to the implementation of the content of the program in K to 12 education, participants also shared their work and supported each other in their learning. This was particularly evident in the Geometry in Nature, Arts and Computer Graphics course, in which participants were required to share generated images using LOGO and Geogebra, including the corresponding coding process.

An analysis of the weekly discussion topics entries showed different ways in which people benefited from the community. One way was to use the discussion topics as a means for quick consultations. Sometimes, participants worked with a peer for a part of the content. It was common to find peer responses advising those who reported struggling in the course. We present an example of this conversation:

> It is frustrating when things don’t work! I feel like a good coding community cafe is important where people can hang out and chat. It is so easy to feel isolated and frustrated when coding but [a peer] and I experienced great success this week (with help from [the instructor]) just chatting with each other and saying, hey this won't work what have you tried. (Grade 6 teacher, Cohort 1).

Another way in which participants benefited from the learning community was a result of sharing their work. In the discussions, some participants acknowledged building on others’ ideas or even being inspired by the work of their peers. We present the following excerpt of a conversation related to finding images with different types of symmetries (including scale-invariant symmetry for fractals) to exemplify this point:

> I was inspired by others and went through some of my pictures to find symmetry in my travels. I wasn’t yet able to find scale invariance in my own photos but I’m still looking.

> I had a lot of fun doing this. I'd like to present the students in my class with a challenge to photograph symmetry they find in their lives and create a photo wall. I love the idea of bringing math alive and think that my students would be able to understand this quite well. (Grade 6 teacher, Cohort 1).

Finally, participants also benefited from the programming scripts posted by their peers. Sometimes, they built on the code shared by a peer to make their own creations, as is evident in the following excerpt:

> Thank you to those of you who were done earlier in the week and allowed some of us to use your questions and your programs as our starting points. It was a saving grace for me this week! (Grade 3/4 teacher, Cohort 1).

In this particular course, participants also acknowledged the role of the instructors' support in the discussion topics in the form of formative assessment, as evidenced in the previous excerpt mentioning that two students received help from the instructor. The acknowledgement of such support was also indicated in both student course evaluations, the course feedback, and the excerpts.

Other features of Online Teaching and Learning

Other relevant aspects identified in this online program for teachers relate to the resources used in each course, the course structure, and student engagement during synchronous sessions. Regarding resources, we noticed a difference corresponding to the use of a specific textbook in a course. Two of the courses consistently followed textbooks: Data: What it Means and How to Handle it and Mathematics for Sustainability. The course Analytical and Algorithmic Reasoning did not follow a textbook but drew from books and other specific online resources. In contrast, the course Geometry in Nature, Art and Computer Graphics included content created by the instructors and utilized several resources. When contrasting instructors' experiences in the debriefings of the courses involving instructors and the coordinator of the program, it was noticed that having one specific resource, when there was one available that fit the course, had advantages for both instructors and students: It provided consistency and coherence to the course. Usually, the book also included problems in practice sections that could be used for formative feedback; it also provided a consistent conceptual framework and language, which was not possible when relying on diverse resources.

A particular issue regarding the use of resources was the diverse operating systems and browser programs when accessing online recourses, which was evident in student feedback and notes from the debriefing meetings. Software, or websites, may not work the same from one computer to another. For instance, in the first year of the program being offered, participants were allowed to use whatever version of LOGO they wanted. This proved challenging as the versions were incompatible with each other, and instructors had to support students individually depending on their devices and software. The next year, a single online version of the programming language was used—one that worked in any browser independently of the operating system—thus solving this problem.

All the courses followed a similar structure which provided flexibility to participants in the program. When analyzing student evaluations for each course and the feedback collected from each cohort at the end of the program, we identified comments welcoming the common structure of the courses. We present, as an example, the following quote taken from student course evaluations:

> Formative evaluation was provided through feedback on discussion posts … The Learning Tasks, although challenging, made sense and were applicable to teaching and learning. Detailed formative feedback was provided every 2-3 weeks … There was feedback given throughout the course so that changes could be made to your work before the next assessment. [Anonymous response from course evaluation, Cohort 2]

Students' feedback also referred to the flexibility of the online program allowing people with different schedules, including large time zone differences, to engage in the activities of the courses. With the exception of the synchronous sessions, all activities could be performed at times most convenient to participants. Participants with different time zones appreciated accommodations when they were not able to attend the sessions, including video recordings of the sessions and the ability to send contributions in advance of sessions.

The weekly activities in some courses were scheduled so that the weekends fell in between the beginning and the end of the course week. For example, course weeks may start on Wednesday and end on Tuesday the following week. This format was appreciated by some participants who indicated they worked on the courses mostly during the weekends. This format also supported the online community allowing people to initiate their posts before or during the weekend so others could respond to their posts during or after the weekend.

We conclude this theme with an observation regarding students' engagement during synchronous sessions. Instructors were encouraged to design interactive synchronous sessions for the course. During debriefing sessions between instructors and the program coordinator, we identified differences in how people engaged in these interactive sessions regarding participants' engagement in the corresponding activities. One of the authors noticed that students did not engage too much when asked a question during the sessions; there was silence after a question, and very few teachers participated. This was a contrasting experience compared to the instructors for a different course in which there was notably much more participation when the group asked a question in the synchronous sessions. After contrasting the format of the sessions, based on the notes from the debriefings of each course, we hypothesize two features that contributed to the difference in engagement in the sessions. The first one was the deliberate intention for interaction through the use of a platform for interactive presentations Mentimeter (https://www.mentimeter.com). This platform includes diverse tools facilitating interactive participation for online presentations. The other feature was the use of the Breakthrough

Rooms tool in the platform used for the session for quick discussions when asking a question. Participants in the course with more engagement would split into small groups to discuss a question for three minutes; they then had to report their conclusions to the whole group during the presentation. We believe that this format allowed time for participants to collectively think about the question and prepare a response, which contrasted with the format in which the instructor asked a question directly to the whole class and waited for responses during the synchronous session.

Discussion and Conclusion

The design of the *Contemporary, Emergent Mathematics for Teaching* program reflects several of the suggestions in the literature concerning online teacher education. The structure of each of its four courses included extensive opportunities for dialogue (Anderson, 2008; Anderson et al., 2001; Larkin & Jamieson-Proctor, 2015) through the weekly discussions, synchronous sessions, and feedback provided by both peers and instructors. As the audience for this program were professionals—teachers and educators—and the discussions were focused on teaching and learning, each cohort formed a community of learners engaged in critical dialogue, consistent with the discrete pedagogies proposed by Parker et al. (2016). Following Shulman's (2005) signature pedagogies, all the courses in the program included an assignment related to the design of a task for the classroom, representing an authentic assessment connecting the courses' activities with the learning outcomes of the courses and the profession, as suggested by Dyment and colleagues (2019). The program was also designed with flexibility in mind (Dyment et al., 2019), considering an audience busy with their professional duties as educators.

Most of the interventions we found in the literature focus on pre-service teacher education (Dyment et al., 2019; Larkin & Jamieson-Proctor, 2015). This paper, therefore, contributes to the literature by reporting the case of this program targeted to in-service teachers with its specific settings that differ from pre-service teacher education. This was reflected in the role of the online professional learning community, which was a key feature of the program. Participants' experience and expertise as teachers and educators were key for designing learning tasks based on the mathematical content offered in each course. While the content of the program focused on mathematics beyond the program of studies, the courses neither focused on any pedagogical approach nor included specialized mathematics for teaching—beyond the knowledge at the horizon (Jakobsen et al., 2012; Zazkis & Mamolo, 2011). Rather, the professional community brought ideas for the implementation of the mathematical content into the classroom through participants' interactions in the courses. The structure of the courses, reflected in the learning activities and tools for online interaction, supported a constant dialogue which not only helped participants to engage in the mathematical content but also promoted a professional conversation on how to bring the mathematical into the classroom. In this sense, the online professional learning community was key to addressing specialized knowledge for teaching mathematics, which was not explicitly addressed in the content offered by instructors in the program.

The third theme resulting from the analysis of this study provides some advice for online teacher education programs in general, beyond the specialized knowledge for teaching mathematics and the professional learning community. Regarding accessibility to resources, our findings suggest that the course design should consider not only open-access resources but also resources that can be used on different platforms, such as the online version of LOGO used by the second cohort of the program. Also, giving participants time to discuss questions before requiring them to participate during synchronous sessions seemed to increase student engagement.

Finally, we stress two future venues of research related to the findings from this study. The first venue corresponds to the specialized knowledge for teaching mathematics, which has predominantly focused on topics already included in programs of studies. Understandably, mathematical topics beyond current curricula are less studied. Considering the relevance of mathematical knowledge at the horizon, there is a need for more research on the teaching and learning of such topics, including their connections to current programs of study. The second venue of research relates to teacher education. As most research on teacher education has focused on pre-service teachers, there is a need for more research on programs for in-service teacher professional learning. Given the work commitments and geographic constraints of such professionals, they could benefit from the flexibility of online models. In this sense, a focus on online professional learning communities, as proposed by this manuscript, can inform future research regarding not only the tools and modes for online learning but also the characteristics of the intended audience as professionals in education (as opposed to pre-service teachers).

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