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Documenting the Multiplicative Reasoning of a High School Junior and Senior

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This study was guided by the question, how do we understand the multiplicative reasoning of upper high school students and use that to give insight to their performance on a standardized test? After administering a partial ACT assessment to a class of high school students, we identified students to make comparisons between low and high scoring students on the sample assessment. Through a written assessment targeted towards assessing students' number sequences, and through semi-structured interviews with two students, we documented a direct relationship between a student's number sequence and their performance on the partial assessment. The evidence that students showed of limited multiplicative reasoning skills help explain some of their challenges in responding to prompts on the ACT assessment. This study reflects the need to give more focused attention to the multiplicative reasoning skills of secondary students and to design interventions that might develop these students' multiplicative reasoning and number sequences.

Multiplicative reasoning (MR) is essential for many middle school and secondary mathematics topics including division (Hackenberg, 2010; Steffe, 1992), fractional reasoning (Hackenberg, 2007; Hackenberg & Tillema, 2009; Norton & Hackenberg, 2010; Steffe & Olive, 2010), area and volume (Mulligan, 2002), and ratio and proportions (Siemon et al., 2005; Tjoe & de la Torre, 2014). These topics, in turn, are foundational for secondary topics such as rate of change (H. L. Johnson, 2015) and algebraic reasoning (Boyce & Norton, 2017; Hackenberg & Lee, 2015; Hulbert et al., 2017; Ketterlin-Geller

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et al., 2015; Lesh et al., 1988; Lobato et al., 2010; Norton et al., 2015; Norton & Hackenberg, 2010; Russell et al., 2011; Tillema, 2013). Since MR is necessary to be successful in secondary mathematics topics, understanding the MR of students in high school who struggle with grade level standards is important as a pathway to support their learning.

MR refers to how students understand, conceptualize, and operate with numbers in multiplicative and divisional situations (Hackenberg, 2010; Hulbert et al., 2017; Kosko, 2019; Ulrich & Wilkins, 2017). Measuring a student's MR is complicated because students without fully developed MR can solve multiplicative tasks using standard multiplication algorithms or in an additive way without a sophisticated multiplicative understanding (Hulbert et al., 2017; Kosko & Singh, 2018; Ulrich & Wilkins, 2017). Standardized tests have been used to quantify students' abilities in areas such as literacy and math, including many of the middle and secondary concepts that support MR. Two commonly used standardized tests are the ACT and SAT. These assessments are used in 33 states as a pathway to graduation (Gewertz, 2019). However, one weakness with standardized tests is that students' scores can be affected by skills and knowledges the tests are not intended to measure. For example, studies have shown scores on standardized mathematics assessments are correlated with working memory (Gathercole et al., 2004; Howard et al., 2017), and targeting working memory can help students on mathematics assessments (Kroesbergen et al., 2014). The specific connections between students' MR and their performance on standardized tests are relatively understudied. One study demonstrated that on one standardized test, the National Assessment of Educational Progress, multiplication and division questions were a better indicator of procedural knowledge than of conceptual understanding of multiplication and division (Kosko & Singh, 2018). Because assessments such as the ACT and SAT are influential gatekeepers (Park & Becks, 2015), it is necessary to better understand how students' mathematical reasoning might shape their performances on these tests.

This study was guided by the research question, How does the multiplicative reasoning of upper high school students give insight to their performance on a standardized mathematics test? Our hypothesis was that upper high school students who struggled with standardized mathematics tasks were limited by their MR. We will discuss the MR of two focal students with lower scores on a standardized test along with a contrasting example of a student with a high score on the same standardized test. We will use data from a MR assessment and diagnostic interviews and will demonstrate how the two students' struggles on the standardized test can be explained, in part, by their challenges with MR.

Theoretical Framework

The theoretical framework for this research is scheme theory, which is based in the radical constructivist framework interpreted by von Glasersfeld (1995). Scheme theory explains that number sequences are the foundation for the mathematical operations students can implement (Hackenberg, 2010; Steffe, 1992). A number sequence is a hierarchical scheme that characterizes how students conceive of numbers which facilitates greater sophistication of operations. The different mental actions or operations afforded to students depending on their number sequence development include but are not limited to unitizing, iterating, units-coordinating, partitioning, and disembedding (Hackenberg, 2010; Kosko, 2018).

Unitizing refers to the ability to mentally create a single conceptual unit based on multiple individual entities (Steffe, 1992). For example, students who can unitize are able to mentally conceptualize a number such as 5 simultaneously as both a single composite unit and as a unit that represents five individual items. Iterating refers to the mental ability of repeating either a composite unit or an individual unit, such as adding 5 and 5 (Steffe & Olive, 2010). Units-coordinating is the operation that allows students to iterate a unit or composite unit on a sequential coordinate system (Steffe, 1992). For example, a student with this operation can skip count by coordinating a composite unit of 5 across the sequentially coordinated mental construct of 1, 2, and 3 as demonstrated in Figure 1 (Ulrich, 2015). The number of units students can coordinate is an essential characterization of a students' number sequence. Partitioning refers to the operation of mentally segmenting a whole into smaller parts (Hackenberg, 2010), which will be discussed further with the ability to disembed.

Disembedding

Disembedding is an operation that allows students to mentally remove a partition from a whole while simultaneously keeping the whole intact (Hackenberg, 2010; Kosko, 2018). For example, if a student is asked to divide 12 marbles equally into three cups, a student who can disembed can mentally create a composite unit of 3 marbles, then iterate their composite unit of 3 to determine if the possible iterate 9 is too low based on comparing it to their retained mental conception of 12. A student can then modify their guess to 4 marbles to determine that iterating this composite unit three times will yield the desired amount of 12 marbles (Steffe, 1992). Students who reason additively can also solve this whole number marble problem in other ways without relying on the disembedding operation, which is why observing students' interactions with fractions can provide insight into student's whole number action schemes (Steffe, 2002; Ulrich & Wilkins, 2017). For example, if a student is asked to partition a whole unit into 7 equal partitions, the whole number of 12 marbles from the previous can be conceived as the whole unit, and each partition of the whole is a fractional partition, which is conceptualized as a composite unit. Disembedding allows the student to maintain the whole while iterating the fractional partition as separate yet identical units (Hackenberg, 2013; Ulrich, 2016b). Disembedding allows students to develop the first fractional scheme because students can conceive each equal fractional partition as an identical and equivalent fractional partition of the whole (Hackenberg, 2013; Steffe & Olive, 2010; Ulrich, 2016b). When observing student behavior in partitioning a whole into equal partitions, students who can disembed can accurately and visually estimate partition sizes, while students who cannot disembed usually estimate unequal partitions and require a physical representation to estimate accurately (Ulrich, 2016b). Disembedding is a powerful operation that facilitates a stronger understanding of multiplication, division (Hackenberg, 2010; Steffe, 1992, 1994), and fractions (Norton & Hackenberg, 2010; Steffe, 2010). Additionally, disembedding also facilitates strategic reasoning, which is similar to what has been referred to as number sense (Boaler, 2016; Humphreys & Parker, 2015). Strategic reasoning allows students to use related benchmark numbers to solve for differences and sums. For example, if finding the difference between 58 and 75, a student can find the difference of 75 and 60 and mentally retain that 58 is 2 less than 60 and then increase the difference by 2.

Number Sequences

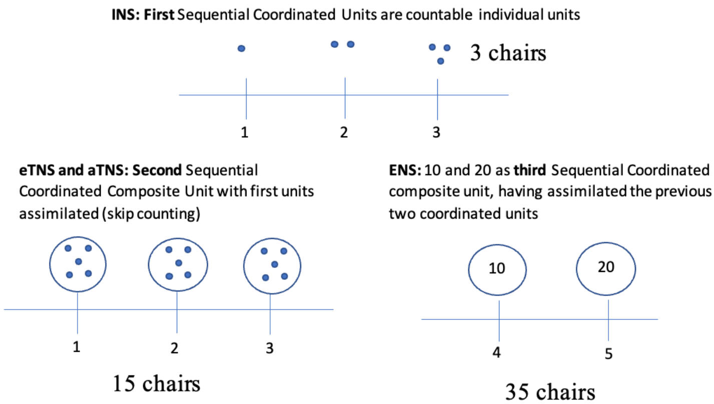
Since researchers cannot know the exact mental operations students use, number sequences characterize the mental framework students operate with based on the way students approach mathematical situations (Steffe, 1992; von Glasersfeld, 1995). A student's number sequence is characterized by how many units a student can coordinate simultaneously, beginning with a single unitized composite unit and then two or three composite units.

Students operating with the first counting scheme, Initial Nested Sequence (INS), understand numbers as sequential and can create a composite unit. Students operating with INS are limited to sequentially coordinating individual units instead of composite units (Figure 1). Students operating with INS can solve multiplicative problems in an additive way by drawing out individual representations of a situation and counting individual units.

A student who has developed early tacitly nested number sequence (eTNS), the next stage in the hierarchy and the first multiplicative number sequence, would be able to coordinate two units because their initial counting sequence is tacitly nested and taken as a given (Hackenberg & Tillema, 2009; Steffe, 1992). This allows students to skip count by conceptualizing a first sequential unit (e.g., 1, 2, and 3) each of which corresponds

Figure 1

A Visualization for One, Two, and Three Levels of Units-Coordinating



to a coordinated composite number (e.g., 5, 10, 15; see Figure 1). Students operating with eTNS have tacitly conceptualized their initial counting number sequence of 1, 2, and 3 but cannot assimilate the result of coordinating composite units. Ulrich (2016b) added the designation advanced TNS (aTNS) after observing students who had developed some attributes of ENS but had not developed the disembedding operation. For example, one distinguishing feature of students operating with aTNS is their ability to partition fractions into equal partitions by simultaneously partitioning, as opposed to students operating with eTNS who sequentially partition (Ulrich, 2016b; Ulrich & Wilkins, 2017). This means students operating with aTNS tend to create marks to determine estimates of a fair share to decide simultaneously if each piece is equal. Students operating with eTNS tend to either create inaccurate estimates or struggle to partition a fractional whole since each partition is created sequentially and does not necessarily need to be the same size as a previous or future partition (Wilkins & Ulrich, 2017).

Students operating with aTNS can partition a whole unit into equal partitions of fifths composed of three coordinated units: the whole unit, fifths partition, and their initial number sequence. However, since students operating with aTNS have not assimilated their composite unit structure, and need to create this structure in activity, students operating with aTNS cannot disembed partitions to use them in further operations. Students operating with aTNS do not conceive of each fifth as an identical and equivalent fractional partition but rather as separate parts of a whole (Hackenberg, 2013; Steffe, 1992; Steffe & Olive, 2010; Ulrich, 2016b).

Students operating with the second multiplicative concept, explicitly nested number sequence (ENS), can interiorize the result of their coordinated unit structure and coordinate a second composite unit, which allows them to disembed (Ulrich, 2016a). As shown in Figure 1, students operating with ENS can retain the structure of a composite unit of 15 composed of five composite units of three and can iterate one of the composite units or can add on a second coordinated composite unit of 10 two times to the original 15 composite unit. ENS students could mentally combine these composite units without the aid of writing numbers down.

Implications of operating with ENS can be observed with students' interactions with fractions. Students operating with ENS conceptualize each fractional partition as an equivalent and identical partition of the whole fraction and have therefore constructed the first true fractional scheme, the partitive unit fraction scheme (Hackenberg, 2013; Steffe & Olive, 2010; Ulrich, 2016b). Disembedding affords students the ability to mentally retain the fractional whole as a composite unit composed of fractional units, while at the same time iterating a second composite unit of a fractional piece. When students operating with ENS partition a fraction into equal partitions, they are equi-partitioning. This means they know beforehand each partition needs to be equivalent to the others and tend to use visual marks such as dots or tick marks to ensure each partition is equal by mentally assessing each partition (Ulrich & Wilkins, 2017). Equi-partitioning leads to the development of the splitting operation, which means a student can both equi-partition and iterate simultaneously, and it is important in helping students understand improper fractions (Steffe & Olive, 2010; Wilkins & Norton, 2011).

Secondary Mathematics Impact

Since disembedding has been “theorized as necessary for developing robust understanding of not only fraction operations, but also integer addition and subtraction and the use of algebraic notation” (Ulrich, 2016b, p. 18), and is helpful for more sophisticated understanding of multiplication and division (Steffe, 1992, 1994), identifying students who have not developed the disembedding operation (aTNS or eTNS) can have significant implications in targeting instruction for algebra, fractions, and more advanced mathematics concepts (Boyce et al., 2021; Hackenberg et al., 2017; Kerrigan, 2021; Zwanch, 2019). Zwanch (2019) demonstrated that only three out of 10 sixth-grade students operating with aTNS were able to write out an algebraic equation correctly, whereas all six students operating with ENS were able to do so successfully. Hackenberg (2013) demonstrated that two out of six (she did not distinguish between early or advanced TNS) seventh and eighth graders with TNS were able to successfully write an equation to represent a multiplicative situation with coaching, while the other four were not.

Methods

Two public, midwestern schools participated in a Youth Participatory Action Research (YPAR) study, one rural and one urban, each with one daily class dedicated to the YPAR project. One aim of the five-year YPAR project was to build foundational mathematical, scientific, and research skills to motivate students to get into STEM careers. The focal class of this study came from the second cohort of students from the urban school participating in the project and was comprised of 24 juniors and seniors (16-18 years old). The school had 49% of students eligible for the free and reduced lunch program. All students applied to the class through a teacher nomination and an essay application.

ACT

To determine students' mathematics performance on a standardized test, all students (*n* = 24) were administered a partial ACT assessment with 15 items that were determined by the first author to be the easiest questions on an ACT. These questions aligned with middle school and early high school Common Core State Standards: four on proportions, three on algebra, four on area and perimeter, two on arithmetic, one on graphing, and one on middle school statistics. Although the ACT has been shown to have issues with racial and economic bias (Gilmore, 2015; Johnson, 2003), this partial standardized test was useful in identifying students who struggled with typical middle and high school mathematics content. The ACT was selected instead of other standardized mathematics tests because students in the state where the research was conducted pays for all juniors to take the ACT.

Based on the results of the partial ACT assessment, the class was sorted into three scoring groups: students answering nine or fewer questions correctly (*n* = 8); students answering between 10 and 12 questions correctly (*n* = 7); and students answering 13 or more questions correctly (*n* = 7). Five students across the three groups were selected to complete the 25 question units-coordination assessment (UCA). One student was in the "above 13" group (Jeff, a junior; all names are pseudonyms); two students were in the "11 or 12" group; and Sunil (a senior) and Ariana (a junior) were in the "9 or below" group. This research centers on how Sunil and Ariana's number sequences may have impacted their performance on the standardized test questions. Since Jeff had evidence of having developed ENS, his work is included as a contrasting example of how he approached the problems on the standardized assessment.

Units Coordination Assessment

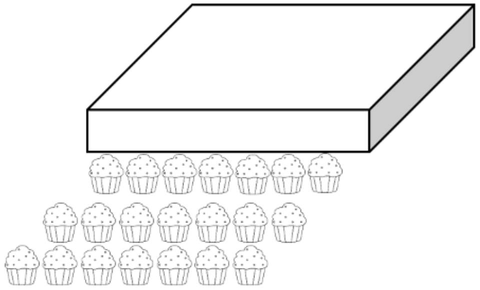
The UCA is part of a larger project of multiple scholars developing written assessments to determine the number sequences of students through dichotomous answers or analyzing written work without the need for lengthy clinical interviews (Kosko, 2019; Kosko & Singh, 2018; Norton et al., 2015; Ulrich & Wilkins, 2017). We chose the UCA by Ulrich and Wilkins (2017) because their assessment requires analyzing and coding student written work as indications or contraindications of their number sequence (INS, eTNS, aTNS or ENS). The UCA does not distinguish between ENS and the Generalized Number Sequence (GNS), which is a more sophisticated number sequence; therefore, students identified as ENS are operating with at least the ENS. Since the UCA is currently not published in its entirety, we were given the grading scoring codes, the rubric, instruction, and permission by the author (C. Ulrich, personal communication, December 23, 2019). We followed up with a diagnostic interview for the two focal students to further confirm evidence of their number sequences. Ulrich and Wilkins (2017) are continuing to work to refine the assessment by modifying questions and interviewing more students since they only initially validated the result by interviewing nine of the 93 students who took the assessment. Their inter-rater reliability for the written rubric kappa statistic, κ, was .95.

The UCA has 25 tasks to differentiate between number sequences based on written work without the use of a calculator. Six tasks used discrete multiplicative problems to distinguish between eTNS and aTNS (Figure 2A). Ten tasks in total pertained to the students' fractions schemes and operations to differentiate between eTNS, aTNS and ENS: two for the partitioning operation (Figure 2B), four for the partitive unit fraction scheme (Figure 2C), four for the splitting operation (Figure 2D; Hackenberg, 2007; Steffe, 2010). Tasks highlighted in Figure 2 are in order of increasing complexity. Ulrich and Wilkins (2017) also included six bar tasks increasing with complexity. Accurate estimations for more sophisticated bar tasks were associated with more sophisticated number sequences.

Figure 2
Sample UCA Tasks for a Some of the Concepts Assessed

A) Discrete Quantity Example (6 tasks)

18. Below there are 3 rows of 7 cupcakes unboxed. In the box there are 4 more rows of 7 cupcakes. How many cupcakes are there all together?



B) Partitioning Example (2 tasks)

15. Suppose the stick shown below is a piece of candy. Show how someone could share it equally among 7 friends.

C) Partitive Unit Fraction Scheme Example (4 tasks)

17. What fraction is the smaller stick out of the longer stick?

D) Splitting Operation Example (4 tasks)

22. The stick shown below is 6 times as long as another stick. Draw the other stick.

Scoring Procedure

Based on scoring codes that were validated by Ulrich and Wilkins (2017), student responses provided weak, strong, or decisive indications or contraindications for evidence of having constructed different number sequences. The scores were aggregated for evidence of having constructed each number sequence with positive values given to indications and negative values to contraindications. Since students do not always use the most sophisticated scheme afforded to them, indications of a number sequence always correspond to having evidence of at least operating with that number sequence.

Sunil and Ariana both answered nine or fewer questions correctly on the ACT assessment and both scored with contraindications for having constructed ENS with indications of operating with aTNS. Therefore, we interviewed Sunil and Ariana to further understand how their reasoning aligned with aTNS and to determine if they shared similar struggles with mathematics concepts that could be attributed to their number sequences. The underlying potential implication is that if number sequences can limit students who score poorly on a standardized mathematics test, then it is important to attend to building more sophisticated number sequences for these students. Jeff's work was used as a contrasting example since he had the highest score on the ACT and had strong evidence of operating with ENS.

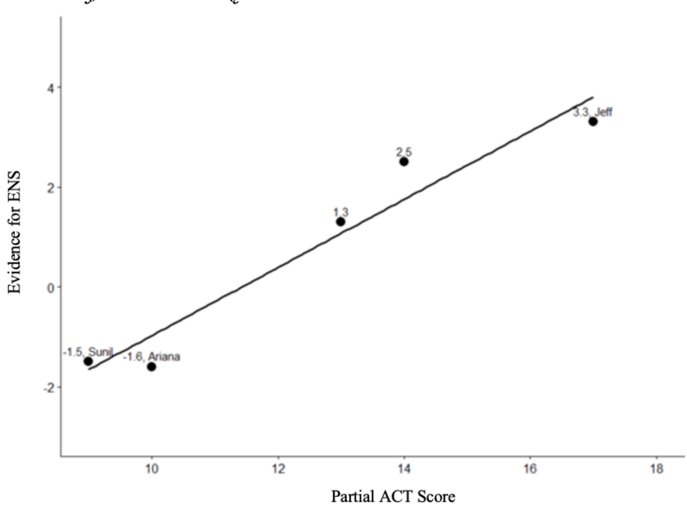
Interviews

Ariana's and Sunil's UCA responses demonstrated contraindications for having developed ENS due to incorrect responses on the partitive unit fraction tasks (Figure 2C) and the splitting operation tasks (Figure 2D). We designed a semi-structured, three-task interview reusing one question they answered previously from each of these two categories. We also added a new third task of a discrete quantity real world multiplicative task similar to Hackenberg's (2010) reversible multiplicative (divisional) problems to explore how Ariana's and Sunil's number sequence impacted their understanding of division with whole numbers. Both students were provided the same three tasks (see Appendix A) during a 30-minute interview. The interviews were transcribed and analyzed to document indications and contraindications for eTNS, aTNS and ENS alongside their written work.

Results

For an overview of the results, Figure 3 shows a scatterplot of each of the five students' aggregate score for ENS indications and their corresponding partial ACT score. There seems to be a direct relationship between students' scores on the UCA and ACT, indicating that it might be interesting to further explore how performance on the ACT might broadly relate to MR. We will next review Ariana and Sunil's indications for operating with aTNS and contraindications for having developed ENS.

Figure 3
Overview of Indications for ENS and Partial ACT Score



Ariana's Indications on the Written UCA

Ariana's work on the written UCA overall indicated that she was operating with aTNS with contraindications of having developed ENS. She also had indications of being constrained by eTNS in some fractional situations. Her strongest indication for operating with aTNS was demonstrated by her accurate estimation of shading one-fifth (Figure 4A) for one partitive-unit-fraction-task. The red markings are Ariana's while the blue region is an overlay to show exactly one-fifth. Besides the accurate estimation in Figure 4A, Ariana also estimated a second out of the four partitive unit fraction tasks (Figure 2C) accurately. Based on Ulrich and Wilkins' (2017) research, students who had constructed ENS consistently and accurately estimated all four partitive unit fraction tasks, while students with aTNS inconsistently estimated unit fractions partitions like Ariana. Another strong indication for operating with aTNS was her ability to determine the length of the bar in Figure 4B accurately.

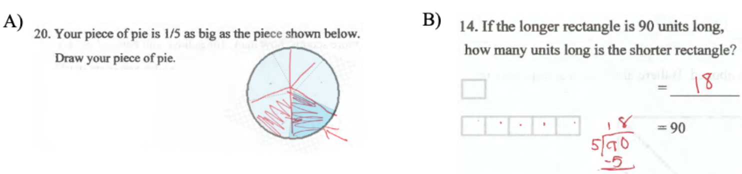


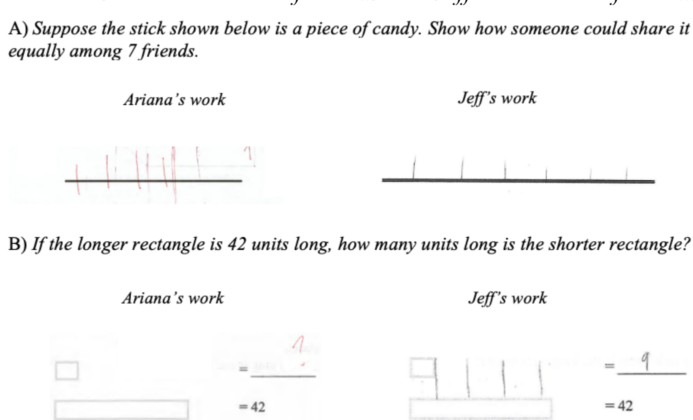
Figure 4
Ariana's Strong Indications for aTNS

The strongest contraindication of ENS for Ariana came from the partition task (Figure 5A). Ariana did not partition the bar into seven partitions, and the left-most line, which we presume was her first marking, was an inaccurate initial estimate. We infer that this means Ariana was constrained to creating "the composite unit of 7 in activity before strategizing" (Ulrich & Wilkins, 2017, p. 11) across the whole, due to not being able to mentally visualize the seven partitions simultaneously as would be expected by a student with aTNS or ENS (Steffe & Olive, 2010). This suggests that Ariana was constructing each segment independent of the others as indications of being constrained by eTNS and a contraindication of having constructed either ENS or aTNS. Students operating with ENS, like Jeff, are theoretically able to disembed a partition mentally and iterate the partition over the whole to estimate their partitions accurately on their first or second attempt.

Another contraindication that Ariana had developed ENS was her question mark on the most sophisticated bar task on the UCA (Figure 5B). She was able to solve the bar task in Figure 4B, but when the partitions were removed this added a barrier in her understanding. Jeff with strong indications of having constructed ENS was able to create his own partitions to estimate the smaller bar. Ariana left an additional four questions unanswered, three of which were fraction tasks which is a contraindication of having ENS since Ulrich and Wilkins (2017) mentioned that we may infer that blank responses may mean the tasks were cognitively challenging for a student, which would not be expected for students with ENS. One fraction task was explored further in the follow-up interview to understand what the blank questions may reveal about her number sequences.

Figure 5

Ariana's Contraindications for ENS and Jeff's Indication for ENS



Ariana's Interview

In tasks presented in the interview, Ariana gave mixed indications of both being limited by eTNS with fractional tasks, while also having indications of operating with ENS with a whole number division task. Since Ariana left the partitive unit fraction question shown in Figure 2C blank, before presenting Ariana with this question again, I gave her two bars, a beige bar (1 cm) and a dark green bar (6 cm) and asked her what fraction of the dark green bar was the beige bar. Immediately she iterated the beige bar 8 times across the dark green bar and estimated one-eighth. Then I proceeded to show her the task in Figure 2C and asked if they were similar tasks. She agreed they were similar tasks but mentioned she could not solve the question from the UCA because “these [referring to one of the physical bars I had provided] you can move over to see how big it is, you can't do that [on the UCA question] so I'm just not going to do it.” I then asked her if there was a way to find out the fractional size without moving a bar. Ariana created a length between her fingers and iterated it along the larger bar five times to determine that it was one-fifth of the larger bar. Ariana’s need to use a physical representation to estimate the partition was an indication of being limited to eTNS. A student with ENS or aTNS would be aware of the possibility to mentally estimate a partition by simultaneously partitioning the whole without requiring a physical representation to estimate (Ulrich, 2016b).

Contraindications of having developed ENS were further demonstrated by Ariana not completing the splitting interview task (Figure 2D), which she also left blank on the UCA. I gave Ariana an orange bar (10 cm) and asked her to imagine it was five times as large as a purple bar. I then gave her a purple crayon and asked her to draw the purple bar. Initially she did not draw the purple bar because she said the bar would be too long for the paper. After prompting her to consider the question again, she began to draw a large bar and said, “it’s still not going to fit on this paper.” Although I asked Ariana the question two more times in an attempt for her to see the situation differently, she did not change her response. Students who have not developed ENS tend to interpret the word “times” as meaning “more than” instead of implying a partitioning operation (Ulrich & Wilkins, 2017; Wilkins & Norton, 2011).

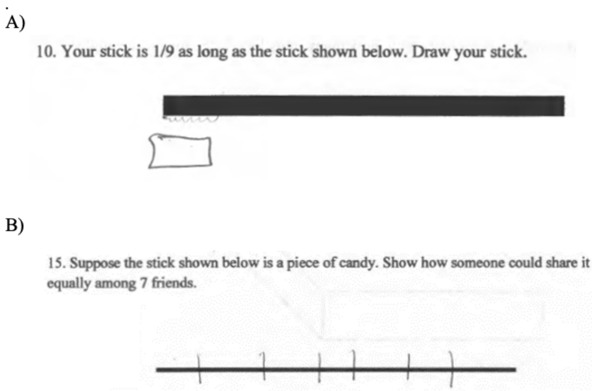
Ariana demonstrated evidence for having developed ENS with her fluent ability to reverse a multiplicative relationship in interview task. For this task, I asked Ariana to calculate how many weeks she would need to work to save up enough money to buy a car. After determining a hypothetical situation where she earned $70 per week, I asked how many weeks she would need to work to save $6,000. Ariana used a calculator and immediately divided and said it would take her around 86 weeks. Using her calculator, she also quickly determined that 86 weeks was one year, eight months and a week. She knew she needed to reverse the multiplicative process and use division to determine how many times $70 "fits" into $6,000. She completed this process fluently with the support of a calculator. Ariana's ability to reverse multiplicative relationships gave indication of coordinating three units (Hackenberg, 2010) and operating with ENS (Steffe, 1992). However, her work on the fraction tasks suggested she was limited by eTNS or aTNS. This could indicate that Ariana has operated with TNS for many years, which has afforded her the ability to be fluent with whole numbers (Ulrich & Wilkins, 2017) but has not yet fully reorganized her whole number reasoning to her fractional reasoning (Steffe, 2002).

Sunil's Indications on the Written UCA

Sunil's work on the written UCA indicated that he was operating with aTNS and had strong contraindications of having developed ENS. The strongest indication of aTNS was that he accurately estimated the partition of one-fifth in the partitive unit fraction task as shown in Figure 6A. Like Ariana, Sunil also accurately estimated the same two partitive unit fraction tasks (Figure 4A and Figure 6A) and incorrectly estimated the two other partitive unit fraction tasks, one of which will be discussed with his contraindications for having developed ENS. Another indication for Sunil having developed aTNS was his ability to relatively accurately partition both partitioning tasks (one shown in Figure 6B). This suggests he was able to simultaneously partition and project an iteration throughout the rest of the whole to make relatively accurate partitions. Additionally, for the discrete problem in Figure 2A, Sunil solved for the number of cupcakes without creating individual representations for each cupcake and used only an intermediate step of showing 28 plus

21, which is consistent with students who have developed aTNS (Ulrich & Wilkins, 2017).

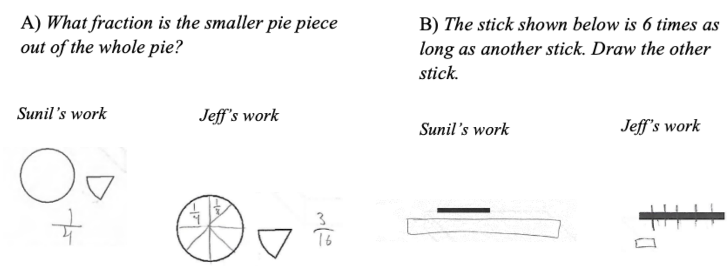
Figure 6
Sunil's Indications for aTNS



Contraindications for having developed ENS were Sunil's inaccuracies in estimating the size of a partition in the partitive unit fraction task (Figure 7A) and inaccuracies in estimating all four splitting operations tasks (one shown in figure 7B). His estimate of the size of the piece of pie in Figure 7A suggested that he did not visually determine his estimation was inaccurate because he was limited by the ability to iterate the partition mentally to assess if it would complete the whole. Jeff, who had strong indications for ENS, was able to determine the fractional size through the use of sophisticated fractional partitioning. Jeff's work suggests he has developed ENS because he was able to iterate a unit of one-sixteenth three times to estimate that this was close to the fractional partition, and he used accurate fractional notation (Ulrich & Wilkins, 2017). Additionally, Jeff had indications for having developed the partitive fraction scheme because he demonstrated an understanding that iterating a unit fraction of one-sixth three times yielded the partitive fraction of three-sixteenths.

Sunil did not successfully draw a stick one-sixth of a given whole (Figure 7B), a task that theoretically relies on using the splitting operation. As mentioned previously, students who have not developed ENS have been observed to understand "times" in splitting operation questions as larger than (Ulrich & Wilkins, 2017). Jeff, a student with strong indications of ENS, drew a stick approximately one-sixth the size of the given stick by understanding this task required him to partition the bar into sixths.

Figure 7
Sunil's Contraindications for ENS; Jeff's Indication for ENS



Sunil's Interview

Sunil's work in his interview further demonstrated contraindications for having developed ENS. In a partitive unit fraction task (Figure 8), he did not see the smaller bar as a separate partition related to the larger bar without guidance. During the interview I asked him to explain a question from the written UCA where he had estimated the size of the smaller bar as one-ninth of the larger bar. Sunil said he counted the little

Figure 8
Sunil's Partitive Unit Fraction Scheme Question Revisited During the Interview

spaces he had drawn on the diagram, and since he drew nine spaces it was one-ninth. When I asked why, he mentioned this fraction had no relation to the larger stick. I then asked him why he counted the small spaces and he said he did not know.

After I rephrased the question by asking him how many times the smaller bar would fit into the larger bar, Sunil drew the same little spaces with his fingers on the larger bar that he had drawn on the smaller bar and then counted around 30 spaces. He then said the smaller bar would fit four times into the larger bar.

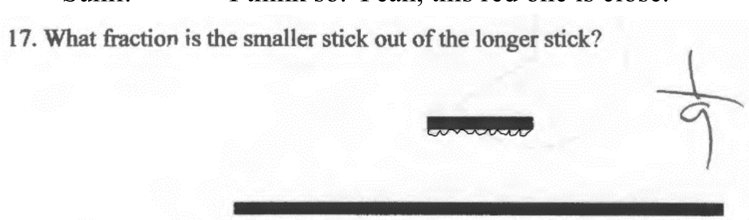
Sunil: (Counting the little spaces on the larger bar) So I go like 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18…(silently moving his finger along the bar)..29 or 29 or something like that, or something around like that, it should be like 30ish, it could be simplified if you counted right.

Researcher: So how many times do you think this [smaller] stick can fit into [the larger stick], would you say?

Sunil: Nine…. Four four four, about four-ish times, if I'm not wrong.

Researcher: (I placed all the Cuisenaire rods in front of him to select from) Can you use one of [these bars] to prove it? Which one is closest, maybe this red one?

Sunil: I think so. Yeah, this red one is close.

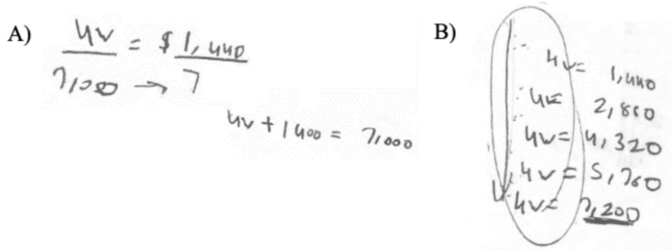


Researcher:	So what do you have?
Sunil:	(Counting how many times the red bar fits into the larger bar) Three, four. You need like one more [to make it 5]. Okay, I was close.

At first, Sunil did not see the connection between the two given bars on the written UCA and drew out many physical marks. Sunil's work was similar to how a seventh grader operating with TNS approached a fractional situation Hackenberg's (2013) study (p. 548), which was a contraindication of having developed ENS. Re-examining his written work in Figure 6A, the seven markings on the left of the fraction seem identical to the markings in Figure 8, which may weaken the evidence for having constructed aTNS, although we did not confirm this connection in understanding during the interview.

Figure 9

Sunil's Proportional Equation and Repeated Addition for Interview Task 3



The following excerpt demonstrates that Sunil struggled to use division as a reversible operation for his counting up operation, which could be a consequence of having not developed ENS. For Task 3, we established a hypothetical situation where Sunil earned \$360 a week, and then I asked how much he earned in a month. At first, he said around \$10,000 because he coordinated \$360 over 30 days, but Sunil checked his work, and realizing his mistake, he multiplied \$360 by 4 to calculate \$1,440. To understand how he would solve a problem that required division, I asked him how many weeks he would have to work to save up for a car that costs \$7,000. He initially came up with an equation and set up a proportion that he determined would not work (Figure 9A), so I asked him to estimate.

Researcher:	Can you start with an estimate?
Sunil:	*(Drawing Figure 9B)* 4 weeks is 1 thousand, eight weeks is equal to 2000. Plus 1440, that's 2000 (*pointing to eight weeks*). Okay, there's another simple way to do this. I know this.
Researcher:	You're doing great, that's ok. There's no one way to solve it in the real world. No one in the real world will tell you to solve it one way.
Sunil:	I know there's a better way to solve this. 1-2-3-4-5, 20 weeks.
Researcher:	So how much is this?
Sunil:	7,200

Sunil knew intuitively that there was a more efficient way to solve the task, but he struggled to reverse this multiplicative situation with division. I posed one more question to Sunil to see whether I could provoke him to see the usefulness of division.

Researcher:	One more challenge, can we get more exact?
Sunil:	You think so?
Researcher:	I'm asking you.
Sunil:	Okay. 1440. Divide that by 2 [to get] 720. 5,760 *(using his 16-week value from Figure 9B)* plus 720 [is] 6,480.
Researcher:	How did you get that?
Sunil:	So I subtract it by 2 because that would be two weeks. That's what I was doing, but I guess it doesn't work like that. So no, you cannot get more exact.
Researcher:	What about division? Can you use division to solve for how many weeks?

Sunil: Did I do this wrong? With these I do this like this, is, what was this completely off? I was on the right track, right?

Sunil did not use division for this problem even with my suggestion. He was disappointed in himself for not “knowing” the answer and understood that there was a more efficient way. This contrasts with Ariana, who was able to fluently operate with discrete numbers. Although Ariana and Sunil had indications of operating with aTNS and contraindications of ENS, both students had developed different strategies for managing division problems.

ACT Work

The students’ answers to four ACT questions demonstrates how the students’ number schemes can influence their work on standardized test questions. Figure 10 shows Ariana’s, Sunil’s, and Jeff’s respective responses on a question requiring division of fractions. Ariana and Sunil wrote down very little work and provided incorrect answers. Their written work on the UCA and our conversations during their interviews suggest they struggled to reason through fractional situations. On the ACT prompt, Ariana first wrote down fourteen-thirds and subsequently erased it (the erased work was reinserted by the researchers), which could suggest she knew procedurally how to turn a mixed number into an improper fraction but did not know how to use this knowledge in this situation. Operating with aTNS could explain the difficulty of this task for Ariana and Sunil. Jeff’s written work on the UCA suggests a more sophisticated understanding of fractions, which may be why he understood four and one-half is a more important fraction to use than four and two-thirds.

Figure 10
Student Work on Question 11 from the Partial ACT

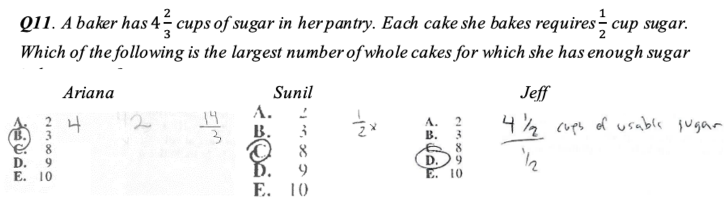


Figure 11 highlights a question Sunil answered incorrectly and Jeff and Ariana both answered correctly. Sunil's work on question two suggests that when he read the word "more" in the rate word problem, he attributed this to an additive problem as indicative by the equation *19 + p*. Using the evidence from his interview, this is consistent with Sunil not applying division in a contextual situation, which could be attributed to operating with aTNS. Ariana, who demonstrated an ability to solve a division problem during the interview, solved this problem fluently in the same way as Jeff.

Figure 11
Student Work on Question 2 from the Partial ACT

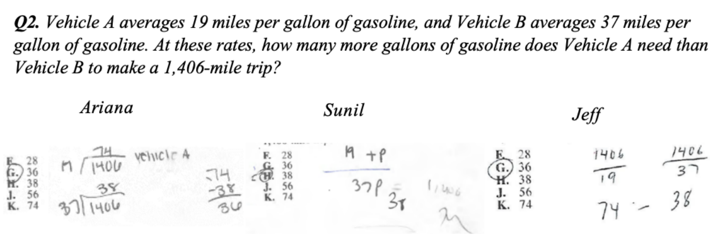
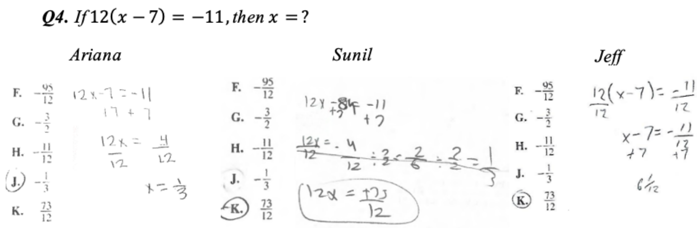


Figure 12 shows indications that Ariana, Sunil and Jeff all knew the basic procedural process in solving a linear equation. Although Ariana showed evidence of understanding division in application in her interview, she arrived at the incorrect answer due a procedural mistake (missing the distributive property). However, Sunil, who showed evidence of not understanding division in application, was able to use division correctly in this context presumably due to having enough experience with the standardized procedure for solving equations. Jeff's work demonstrates procedural fluency.

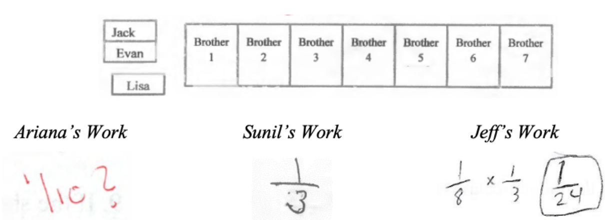
Figure 12
Student Work on Question 4 from the Partial ACT



In addition to how these three students performed on questions on the ACT, comparing their work on a fraction question (Figure 13) on the UCA gives further evidence of how students' number sequences can affect their performance on standardized test questions. The question required students to share one eighth of a candy bar among three people, calculating one third of one eighth. Arianna wrote one-tenth along with a question mark acknowledging her confusion, while Sunil wrote one-third with no other explanation. Jeff knew the correct procedure to apply in this situation to determine the correct portion of the chocolate bar.

Figure 13
Student Work on a Fraction Question from the UCA

4. Jack and his seven brothers shared a candy bar. Jack took his eighth to school and shared it with two of his friends, Lisa and Evan. What fraction was Lisa's piece of the original candy bar?



Discussion

This research described the MR of two students who had contraindications for having developed ENS (Ariana and Sunil) and one student with strong evidence of ENS (Jeff). Their work on a sample of the ACT questions demonstrates some of the ways students operating with aTNS and ENS may approach mathematical situations with grade level mathematics content. The questions in Figures 10 through 13 show examples of inconsistency in solving problems for Ariana and Sunil. Jeff's number sequence provided him with a stronger grasp of fractions, as demonstrated in Figure 7A and 7B, which offers a reason for his approach to his work in Figure 10 and 13. This suggests that using reasoning-based assessments in conjunction with standardized tests may help to understand how students reason with the numbers involved in mathematical situations.

While both Ariana and Sunil struggled with fractional reasoning, Ariana was more fluent than Sunil with whole number division. Their number sequence limitation could be a primary reason they struggled with solving the fraction question on the partial ACT assessment (e.g., Figure 10). Focusing on developing their number sequence and therefore their fractional reasoning could help target instruction. Sunil demonstrated he struggled with division in application, yet he was able to correctly solve a three-step linear equation, presumably through procedural knowledge.

Conclusion

Steffe (2017, p. 45) estimated that approximately 60% of 6th graders had not constructed ENS. The ways students struggle with proportional reasoning in middle and secondary mathematics (Ellis, 2007; Lobato, et al., 2003) and how students who have not built the necessary reasoning during the expected grade level get left behind (Tzur et al., 2010) suggests that there are large populations of high school students who still have not developed ENS and are operating with varying levels of TNS (e.g., Steinke, 2015, 2017; Zwanch & Wilkins, 2021).

Ulrich and Wilkins (2017) mentioned that "students at the aTNS stage during middle school [are] a particularly interesting and important group to identify and study because they have multiplicative schemes and yet will face a serious obstacle in trying to engage in truly multiplicative thinking" (p. 18). Extending this idea to high school and college students presents an equally interesting group to study for further research. Studies have shown adult college students without fully developed MR (Shaver, 2023; Steinke, 2015, 2017; Stigler et al., 2010), which suggests identifying and targeting developing these students' number sequences is a way to support their learning. There is evidence that students' number sequences can be developed when a teacher is aware of a student's conceptual understanding and provides targeted cognitive mathematical interventions (Boyce & Norton, 2017, 2019; Olive & Vomvoridi, 2006). There are resources to support educators on how to build number sequences with students (Hackenberg et al., 2016; Hulbert et al., 2017), but these resources are targeted for K-6 teachers. This includes instructors including visual mathematics for students (Boaler, 2016) and helping students to have concrete examples for mathematical situations (Hackenberg et al., 2016). Resources and strategies need to be created for secondary and remedial college mathematics classes as well.

While developing ENS is an important milestone in developing an understanding of fractions (Hackenberg, 2007), writing algebraic equations (Zwanch, 2019), and reasoning algebraically (Olive & Çağlayan, 2008), students operating with ENS are not as adept in equation writing and understanding fractions as GNS students (Hackenberg et al., 2016; Hackenberg & Lee, 2015). Going forward, we need to re-emphasize the connection between number sequences and secondary mathematics which could include supplementing standardized tests with reasoning assessments to identify and build students' number sequences.

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Appendix A

Interview Questions.
Task 1 [Splitting Task] [Provide 1 copy of the orange (10 cm) bar and a purple crayon]

Q1: Suppose the orange bar is 5 times as long as a purple bar. Can you draw the purple bar?

Q2: I'd like to show you a problem you did on the earlier test (Q22*). (Read through the problem, show them what they did.)

22. The stick shown below is 6 times as long as another stick. Draw the other stick.

(a) First, does this problem seem similar or different than what we just did? Why?

(b) How did you decide what to do?

(c) Looking at work, would you solve this question differently right now? If yes, how?

Task 2 [Unit Fraction Task] [Place beige bar (1 cm) and dark green (6 cm) in front of the student]

Q3: What fraction is the beige bar of the dark green bar?

Q4: I'd like to show you a problem you did on the earlier test (Q17*). (Read through the problem, show them what they did.)

17. What fraction is the smaller stick out of the longer stick?

(a) First, does this problem seem similar or different than what we just did? Why?

(b) How did you decide what to do?

(c) Looking at work, would you solve this question differently right now? If yes, how?

Task 3 [Real World Problem]
Q5: How much do get paid an hour? How many hours do you work in a week?

Q6: How many weeks would you need to work to save for a car that is $7,000 (student 1) or $6,000 (student 2)?